K_2-Node Expansion Problems

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Abstract. In this paper we introduce the concept of node expansion. Node expansion is a generalization of edge subdivision and an inverse of subgraph contraction. A graph $G_1 = (V_1, E_1)$ is an H-node expansion of $G = (V, E)$ if and only if G_1 contains a subgraph $H = (V_H, E_H)$ such that $V = V_1 - V_H \cup \{v\}$ and $E = E_1 - E_H \cup \{vw \mid wh \in E_1$ and $h \in V_H\}$. The concept of node expansion is of considerable importance in modernization of networks.

We consider the node expansion problem of transforming a graph to a bipartite graph with a minimum number of node expansions using K_2. We show that the K_2-node expansion problem is NP-Complete for general graphs and remains so if the input graph has maximum degree 3. However, we present a $O(n^2 \log n + mn + p^3)$ algorithm for the case when the input graph is restricted to be planar 3-connected and output graph is required to be planar bipartite.

1. Introduction.

The modernization of networks involves improvement of nodes and/or edges. In the case of computer networks, improvement may imply replacement of a single processor node with a multiple processor node. In such a case, the interconnection of these new nodes to the remaining network is of critical concern, so that vital parameters of the network, such as, diameter, connectivity, planarity, maximum degree, are not adversely affected. However, no model has been proposed to study such operations on Networks(graphs).

In this paper, we introduce the concept of node expansion. Node *expansion* is a generalization of edge subdivision and an inverse of subgraph contraction. A graph $G_1 = (V_1, E_1)$ is an H-node expansion of $G = (V, E)$ if and only if G_1 contains a subgraph $H = (V_H, E_H)$ such that $V = [V_1 - V_H] \cup \{v\}$ and $E = \{E_1 - E_H\} \cup \{vw \mid wh \in E_1 \text{ and } h \in V_H\}$.

As stated earlier, we are motivated to study node expansion, as it is of importance in the modernization of networks when a node is replaced by a graph. Several interesting modifications can be made by node expansion. In particular, one might like to modify the network so that it satisfies a certain property using the minimum number of node expansions. Although any graph can be used for the expansion of nodes, we use simple graphs for this purpose. Specifically, in this paper, we concentrate on the node expansion problems with K_2 as the graph used for expansion. This is motivated by availability of many commercial two-processor systems.

In this paper, we show that the problem of finding the minimum number of K_2-node expansions of a graph to obtain a bipartite graph is NP-Hard. In fact, we

JCMCC 11 (1992), pp. 33-46

show that the problem remains NP-Hard even if the input graph has maximum degree three. However, we show that if the input graph is planar, 3-connected and the output graph is required to be planar bipartite, then the node expansion problem is polynomial and we give a $O(n^2 \log n + mn + p^3)$ algorithm for this case, where n is the number of vertices, m is the number of edges and p is the number of vertices with odd degree.

The rest of the paper is organized as follows. In Section 2 we present the relevant definitions and some preliminary results. In Section 3, we investigate the complexity status of Node expansion problems. In particular, we show the NP-Completeness proof of NE(arbitrary, bipartite, K_2). In Section 4, we present an algorithm for NE(planar 3-connected, planar bipartite, K_2) followed by a complete example. Section 5 presents the formal proof of the algorithm.

2. Preliminaries.

In this section, we present our notation and several definitions that are used in later sections. In this section, we present our notation and several definitions that are used in later sections.

NE(π_1, π_2, H):

Instance: Graph $G = (V, E)$ with property π_1, graph property π_2, connected graph $H = (V_H, E_H)$ and a positive integer K.

Question: Is there a graph $G' = (V', E')$ satisfying the property π_2, obtained by at most K, H-node expansions of G.

The node expansion is a generalization of edge subdivision. An edge $e = uv$ is said to be *subdivided* if edge e is removed and a new node x is added to the graph along with the edges ux and vx.

ES($\pi_1, \pi_2,$):

Instance: Graph $G = (V, E)$ with property π_1 and a positive integer K.

Question: Is there a subset $E' \subset E, |E'| \leq K$ such that subdividing the edges of E' produces a graph with property π_2.

A close relative of edge subdivision problem is the Max Cut problem. The Max Cut problem can be defined as follows:

SIMPLE MAX CUT:

Instance: Graph $G = (V, E)$ and a positive integer $K \leq |E|$.

Question: Is there a bipartite graph $G' = (V, E')$ obtained by deleting at most K edges of G.

A restricted but very useful version of max cut is called Simple Restricted Max Cut problem (SRMC). In fact, it is the max cut problem in which the graph has maximum degree 3.

Given a graph $G = (V, E)$ and a subset $S \subset V$, a collection of paths in G is an *S-vertex* cover if each vertex of S is an endpoint of exactly one of the paths and S is the union of the endpoints of the paths. An *n-wheel* is the join of a vertex and an n-cycle. The vertex and cycle will be called the *hub* and the *rim*, respectively. The edges not in the rim will be called the spokes. A graph which is homeomorphic to an n-wheel will also be called an n-wheel. If T is a planar embedding of G we define the *pseudo-dual* $G_T = (V_T, E_T)$ of G as the graph with vertices and edges

$$V_T = V \cup \{f \mid f \text{ is a face of the embedding } T(G)\}$$

and

$$E_T = \{vf \mid v \in V, \text{ and } f \text{ is a face of } T(G) \text{ with } v \text{ a vertex on the boundary of } f\}$$

respectively. Figure 4 shows a graph G along with its corresponding G_T. Note that the pseudo-dual of a planar graph is also a planar graph. We use F_T to refer to the set of faces, that is, $F_T = V_T - V$.

If a vertex v in a graph is the hub of a deg(v)-wheel then the rim has two orientations which induce cyclic orderings of the spokes and the vertices adjacent to v. If the graph is planar then the faces incident to v also have two cyclic orderings. We shall refer to these cyclic orderings as an orientation at v.

Lemma 1. *Every vertex v in a 3-connected planar graph is the hub of a deg(v)-wheel.*

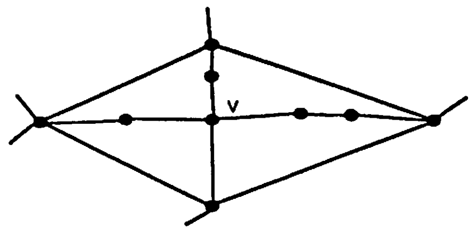
Proof: Let v be a vertex in a planar graph G and let T be a 2-sphere embedding of G. Since G does not contain cut vertices each face incident to v must be a closed 2-cell. If the intersection of two faces incident to v is not a path, that is, *not connected*, then G contains two vertices which together form a cut set. Since G is 3-connected this is not possible. Thus, the number of faces incident at v is equal to deg(v) and there union is a closed 2-cell whose boundary is a cycle. Thus, v is the hub of a deg(v)-wheel. ∎

Definition 1: Let v be the hub of a deg(v)-wheel in a graph G. A K_2-node expansion of G at v is a *simple* K_2-node expansion if the rim of the deg(v)-wheel at v can be partitioned into two edge disjoint paths P_1 and P_2 such that $\{e_i w_j \mid e_i$ is a new vertex in the node expansion for $i = 1,2,\ 1 \leq j \leq \deg(v)$, and $w_j \in P_k$ for $k = 1,2\}$ along with $e_1 e_2$ are the new edges in the node expansion.

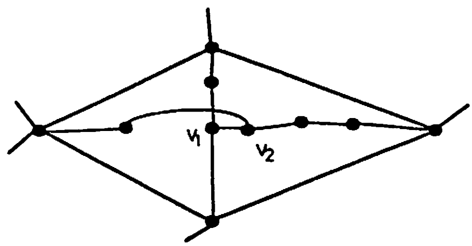
Lemma 2. *Let G be a planar graph with the property that every vertex v is the hub of a homeomorphic deg(v)-wheel. A K_2-node expansion of G is simple if and only if it is planar.*

Proof: Let G satisfy the hypothesis of the Lemma. Clearly, a simple K_2-node expansion is a planar K_2-node expansion. Conversely, suppose we have a non-simple K_2-node expansion of G at v. Since there are no non-simple nor non-planar

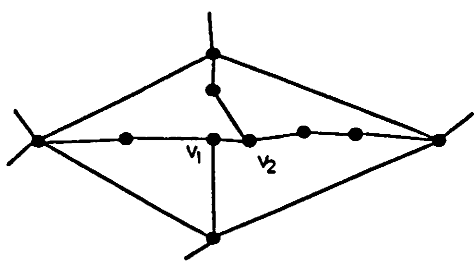
K_2-node expansions at vertices with degree less than four, the degree of v must be at least four. If the K_2-node expansion is non-simple there must exist vertices w_1, w_2, w_3, and w_4 adjacent to v in G with the induced order $w_1 w_2 w_3 w_4$ and the edges $v_1 w_1$, $v_1 w_3$, $v_2 w_2$, and $v_2 w_4$ in the K_2-node expansion. Thus, the graph obtained contains a homeomorphic $K_{3,3}$ and is not planar. In Figure 2 we show an example of a simple K_2-node expansion and a non-simple K_2-node expansion. ∎



(a) Homeomorphic 4-wheel

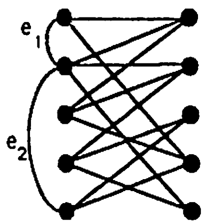


(b) Non-simple Node expansion

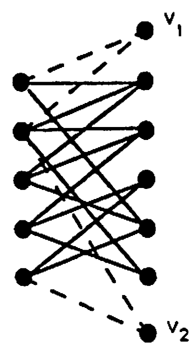


(c) Simple Node expansion

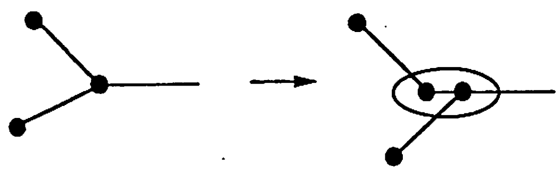
Simple and Non-simple K_2-node expansions
Figure 1



(a) Edge Removal



(b) Edge Subdivision



(c) Equivalence of Edge subdivision and Node expansion

NP-Completeness of node expansion and edge subdivision

Figure 2

Lemma 3. *The K_2-node expansion of a non-planar graph is non-planar.*

Proof: Edge contraction is a one sided inverse of K_2-node expansion. Since the edge contraction of a planar graph is obviously planar, we have the result. ∎

3. NP-Completeness of NE(ϕ, bipartite, K_2).

In this section we show that the NE(ϕ, bipartite, K_2) is NP-Complete, where ϕ refers to the class of all graphs.

Consider the operation of edge removal and edge subdivision. These operations are equivalent with respect to making the graph bipartite, since removing an edge as in SIMPLE MAX CUT has the same effect as subdividing it. An example of equivalence of edge removal and edge subdivision operation is shown in Figure 2. Figure 2(a) shows a solution of SIMPLE MAX CUT where edges e_1 and e_2 are the removed edges. For edge subdivision we create a vertex v_1 and remove edge $e_1 = xy$ and add edges v_1x, v_1y. Similarly, a vertex v_2 and edges are created for edge e_2.

Let γ be the class of graphs with maximum degree 3. SIMPLE MAX CUT for graphs with maximum degree 3 and no edge weights, that is, SRMC was shown to be NP-Complete by Yanakakis [5]. This implies that a solution of SRMC exists if and only if a solution for ES(ϕ, bipartite) exists, giving us the following result.

Lemma 4. *ES(γ, bipartite) is NP-Complete.*

Next, we show that NE(ϕ, bipartite, K_2) is NP-Complete by restriction to ES(γ, bipartite). Let us consider a graph from class γ. It is easy to see that a node expansion operation in such a graph is always same as subdivision of an edge. This is so because a node v with degree 3 can be expanded exactly three ways each corresponding to subdivision of one of the incident edges of v. An example of equivalence of node expansion and edge subdivision when vertex degree is three is shown in Figure 2(c).

This implies that a solution of ES(γ, bipartite) exists if and only if a solution for NE(γ, bipartite,K_2) exists. This implies that NE(γ, bipartite, K_2) is NP-Complete and gives us the following theorem.

Theorem 1. *NE(ϕ, bipartite, K_2) is NP-Complete and remains NP-Complete even for graphs with maximum degree 3.*

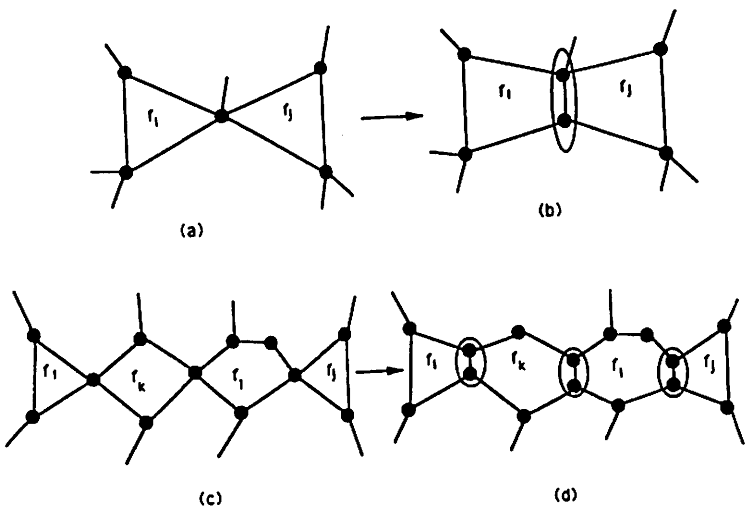
4. An algorithm for NE(Γ, bipartite, K_2).

As shown in the previous section, NE(ϕ, bipartite, K_2) is NP-Complete, so there is little hope of finding a polynomial algorithm for the general problem. In this section we present a polynomial algorithm for NE(Γ, bipartite, K_2), where a graph with the property Γ is a planar graph such that each vertex v of degree greater than 3 is a hub of a deg(v)-wheel. It should be noted that this class of graphs contains the planar 3-connected graphs. First we present the algorithm informally followed by a formal description and an example.

A face with an odd number of edges is referred to as an odd face. The bipartite planar graphs do not have odd faces. Moreover, for any graph the number of odd faces is even [4]. The main idea of the algorithm is to pair up two odd faces and make each face even by a series of node expansions. Depending upon the relative location of the faces in the embedding we have two cases:

CASE 1: If the faces share a common node then two faces can be made even by expansion of this node as shown in Figure 3(a). This operation does not affect the number of edges on any other face. The result of such an operation is shown in Figure 3(b).

CASE 2: If the faces do not share a node then we find a shortest path between the two faces and expand all nodes in the path as shown in Figure 3(c). This operation makes two faces in question even. In addition it does not change any even face to an odd face since it adds exactly two edges to every intermediate face as shown in Figure 3(d). If there is an odd intermediate face then it remains odd.



Two cases for node expansion
Figure 3

Using this idea of pairing odd faces reduces the problem to that of finding odd faces, and then finding a particular shortest path between paired odd faces. This is accomplished by using the pseudo-dual G_T of graph G.

The vertices with odd degree in F_T are the odd faces of G_T. We define *path* (f_i, f_j) to be a shortest path between two odd faces f_i and f_j in G_T. Let $len(f_i, f_j)$ be the length of $path(f_i, f_j)$. The algorithm finds a $path(f_i, f_j)$ for all pairs f_i, f_j in G_T using all-pair shortest paths algorithm.

Next, we define a weighted complete graph $G_c = (V_c, E_c)$. Where

$$V_c = \{v \mid v \in F_T \text{ and } deg(v) \text{ is odd}\}.$$

The weight on the edges is defined by the following function:

$$w(f_i, f_j) = len(f_i, f_j).$$

Then we find a minimum weight matching in G_c; this gives us the required odd vertex pairing required. Finally, the nodes are expanded along each path giving us the required bipartite graph.

Formally the algorithm is stated below.

ALGORITHM Planar Node Expand

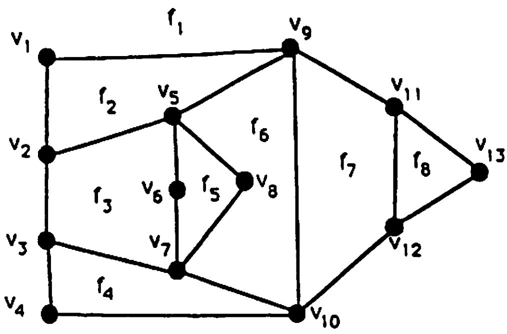
Input: A planar graph G with the property that every vertex v with degree $\deg(v) \geq 4$ is a hub of a subgraph homeomorphic to an n-wheel.

Output: A minimum number of simple K_2-node expansions of G which yields a bipartite graph.

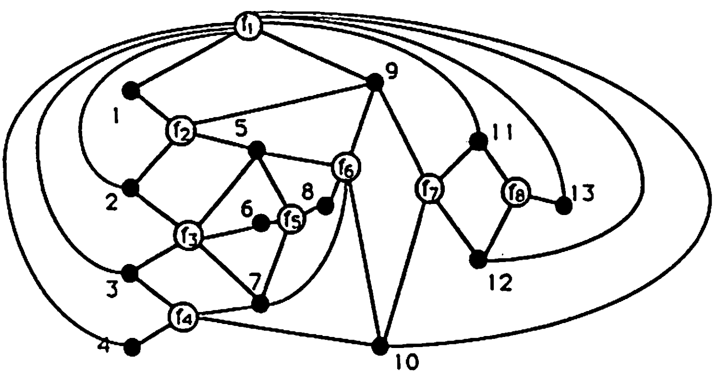
(1) While $\exists v \in V'$ such that $\deg(v) = 0$ or 1 do
If $\deg(v) = 0$ or 1, let $G = (V', E')$, $V' = V' - v$, $E' = E' - \{vw\}, w \in V'$
(2) Let T be a planar embedding of the graph G. Let $f_1, f_2, \ldots, f_k$ be the faces of the T.
(3) Using T construct G_T.
(4) Let R be the set of vertices of odd degree in F_T and let Q be the set of all pairs consisting of vertices in R.
(5) Construct the complete weighted graph $G_c, = (R, Q)$.
(6) Find a minimum weight matching M in G_c.
(7) Using M find a set of paths, one for each of the matched pairs in G_T, whose length is the distance between the pair.
(8) If some paths are crossing then use Lemma 5 to replace them with an equivalent set of non-crossing paths.
(9) Each path determines a set of simple K_2-node expansions of G, the total of which is the desired set.

We now present a complete example of finding a minimum number of K_2-node expansions for a graph G using the algorithm given above. The input graph is shown in Figure 4(a) and its pseudo-dual G_T is shown in Figure 4(b).

The graph G_T has 4 vertices of odd degree *viz* f_1, f_3, f_6, f_8. We then form a complete graph G_c on these vertices. The edge weights are then computed for the



(a) Planar Graph G



(b) The Graph G_T

Planar Graph and its pseudo-dual
Figure 4

graph G_c and are shown below.

	f_1	f_3	f_6	f_8
f_1		2	2	2
f_3			2	4
f_6				4
f_8				

The minimum weight matching can be done in the graph G_c by inspection and it is shown below.

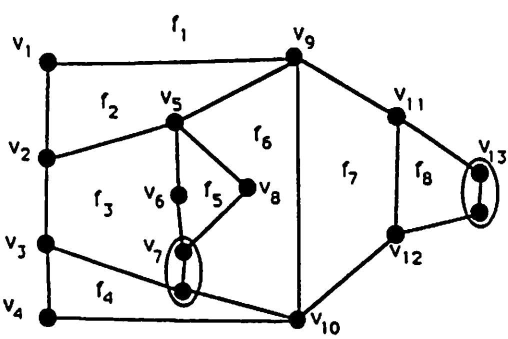
$$M = \{(f_1, f_8), (f_3, f_6)\}.$$

Now we must find $path(f_i, f_j)$ in G_T if (f_i, f_j) appears in M.

There are three paths between f_1 and f_8 of length 2, $f_1 v_{11} f_8$, $f_1 v_{12} f_8$ and $f_1 v_{13} f_8$, we break ties arbitrarily and choose $f_1 v_{13} v_8$. Similarly, there are two paths between f_3 and f_6 of length 2 *viz* $f_3 v_5 f_6$ and $f_3 v_7 f_6$, we chose $f_3 v_7 f_6$. It is important to note that this choice does not affect the optimality of the algorithm since the same number of nodes are expanded. The paths chosen are

$$P = \{f_1 v_{13} f_8, f_3 v_7 f_6\}.$$

This means the nodes v_7 and v_{13} are to be expanded and final solution is shown in Figure 5.



Minimum K_2-node expansions of graph G.
Figure 5

5. Analysis of algorithm Planar Node Expand.

In this section, we establish the time complexity of the algorithm Planar Node Expand and show that the algorithm produces an optimal solution. In particular, we show that crossing paths can be exchanged with non-crossing paths with the same total length. This leads to a proof that there exists a K_2-node expansion of G to obtain a bipartite graph. Finally, we prove that expanding of the V_T nodes of a minimal odd F_T-vertex cover leads to an optimal solution.

Theorem 2. *Algorithm Planar Node Expand runs in $O(n \log n + mn)$ time*

Proof: Each step in the algorithm is clearly polynomial. Steps 4-5 and 6 determine the overall time complexity of the algorithm. Steps 4-5 can be done using all-pairs shortest path algorithm with time complexity of $O(p^3)$, where p is the number of vertices with odd degree. The asymptotically fastest algorithms for minimum weight matching are due to Gabow [3] and runs in $O(n \log n + mn)$ time, where n is the number of vertices and m is the number of edges (see also [1, 2]). It should be noted that Step 8 can be easily accomplished in $O(p.\Delta)$, where Δ in the maximum degree of the graph. Thus, the overall time complexity of algorithm Planar Node Expand is $O(n \log n + mn + p^3)$. ∎

Let P_1 and P_2 be edge disjoint paths in G_T which have a vertex $v \in V$ in common, that is, $f_j v f_k$ is a subpath of P_1 and $f_r v f_s$ is a subpath of P_2. We say that P_1 and P_2 are *non-crossing* at v if either both f_r and f_s are between f_j and f_k in the orientation at v or neither of them is.

Lemma 5. *If $P_1, P_2, \ldots, P_n$ are edge disjoint paths in G_T which have $v \in V$ in common, then there exist paths $P_1', P_2', \ldots, P_n'$ on the same set of edges which are pair wise non-crossing at v.*

Proof: Since $P_1, P_2, \ldots, P_n$ are edge disjoint and contain v it follows that $f_{i1} v f_{i2}$ is a subpath of P_i for $i = 1, 2, \ldots, n$ and $f_{ik} \neq f_{jk}$ for $i \neq j$ and $k = 1, 2$. We can write each of the paths P_i as follows:

$$P_i = P_{i1} \cup (f_{i1} v f_{i2}) \cup P_{i2}.$$

Since there is an orientation at v we can order the f_{ik}'s to be consistent with this orientation. Let $g_1, g_2, \ldots, g_{2n}$ be such an ordering. Define

$$\psi: \{1, \ldots, 2n\} \rightarrow \{1, \ldots, n\} \times \{1, 2\}$$

as

$$\psi(j) = (i, k) \text{ if } f_{ik} = g_j.$$

Note that ψ is a bijection. If we define

$$P_i' = P_{\psi(2i-1)} \cup (g_{2i-1} v g_{2i}) \cup P_{\psi(2i)} \text{ for } i = 1, \ldots, n$$

then we have the desired paths. ∎

A simple induction argument gives us the following result:

Corollary 1. *If $P_1, P_2, \ldots, P_n$ are edge disjoint paths in G_T then there exist paths $P'_1, P'_2, \ldots, P'_n$ which are pair wise non-crossing at each vertex of V_T.*

Lemma 6. *Let $G = (V, E)$ a graph with a planar embedding T. If P is a path in G_T connecting two vertices of F which have odd degree then there is a K_2-node expansion of each of the vertices of G in G_T, which lie on P, such that the new graph is planar and has two fewer odd faces than G.*

Proof: Since both end points of P are in the same partite set of G_T its length must be even, that is, length of P is $2n$ for some positive integer n. We proceed by induction on n. If $n = 1$ then $P = f_i v f_j$, where f_i and f_j are in the orientation at v. We will assume that f_i precedes f_j in the orientation. The following K_2-node expansion of v gives the desired graph G_1: Replace v with the vertices v_1 and v_2 along with the edges $v_1 v_2$,

$$\{v_1 w_k \mid v w_k \in E \text{ and } v w_k \text{ is an edge of the face } f_k \text{ for } k < i \text{ or } k > j\}$$

and

$$\{v_2 w_k \mid v w_k \in E \text{ and } v w_k \text{ is an edge of the face } f_k \text{ for } i < k < j\}.$$

G_1 is planar and the faces corresponding to f_i and f_j have one additional edge. The other faces remain unchanged. Assume the lemma holds for paths with length $2(k-1)$. If P has length $2k$, that is, $P = f_1 v_1 f_2 v_2 \ldots v_k f_{k+1}$ then a K_2-node expansion at v_1 as above will change the parity of the number of edges of the faces corresponding to f_1 and f_2. By induction there are $k-1$ K_2-node expansions which change the parity of the number of edges of faces corresponding to f_2 and f_{k+1}. Since the parity of the number of edges of the face corresponding to f_2 was changed twice, in the final graph only the parity of the number of edges of the faces corresponding to f_1 and f_{k+1} are changed. ∎

Lemma 7. *If $P_1, P_2, \ldots, P_n$ is a minimal odd F-vertex cover of G_T then the paths are edge disjoint.*

Proof: Suppose that $P_i = w_{i1} w_{i2} \ldots w_{in_i}$ and $P_j = w_{j1} w_{j2} \ldots w_{jn_j}$ are two paths which have at least two vertices in common. Let $p_{ir} = p_{js}$ be the first vertices they have in common and $p_{it} = p_{ju}$ the last vertices they have in common. If we replace P_i and P_j with

$$P'_i = p_{i1} p_{i2} \ldots p_{ir} p_{js-1} \ldots p_{j1}$$

and

$$P'_j = p_{jn_j} p_{jn_j-1} \ldots p_{ju} p_{it+1} \ldots p_{in_i}$$

respectively, then we get an odd F_T-vertex cover with smaller total length. Thus, each pair of paths in a minimal odd F_T-vertex cover can share at most one vertex.

∎

Theorem 3. *N is the minimal number of simple K_2-node expansions of a planar graph G which yields a bipartite graph if and only if there exists a planar embedding T and a minimum F_T-odd vertex cover $P_1, P_2, \ldots, P_n$ such that*

$$N = \sum_{i=1}^{n} \frac{length(P_i)}{2}.$$

Proof: Let $G = (V, E)$ be a planar graph. Let T be a planer embedding of G and let N be the minimal number of simple K_2-*node* expansion of G required to obtain a bipartite graph G_1. Let G_T be the pseudo-dual of G associated with T. By definition each simple K_2-node expansion is defined by a triple $f_{i1} v_i f_{i2}$, $i = 1, 2, \ldots, n$, where $f_{ik} \in F_T$ and $v_i \in V$. Since G_1 is bipartite each of the vertices of F_T with odd degree must be included in the list $L = \{f_{ik} \mid i = 1, 2, \ldots, N$ and $k = 1, 2\}$ an odd number of times and the other vertices of F_T must be included in the list L an even number of times, possibly zero.

If $N = 1$ then f_{11} and f_{12} must both have odd degree and the path $P_1 = f_{11} v_1 f_{12}$ is the desired minimum odd F_T-vertex cover. Assume the theorem is true for all graphs which require $N - 1$ simple K_2-node expansions to obtain a bipartite graph. Since N is minimal at least one of the f_{ik}'s in the list L has odd degree, without loss of generality, we assume f_{11} has odd degree. Consider the graph G' obtained from G by a single simple K_2-node expansion at v_1 determined by f_{11} and f_{12}. G' requires only $N - 1$ simple K_2-node expansions to obtain a bipartite graph. By the induction assumption there exists an odd F'_T-vertex cover of G'_T: $P'_1, P'_2, \ldots, P'_m$ such that

$$(N - 1) = \sum_{i=1}^{m} \frac{length(P'_i)}{2}.$$

There are two cases to consider:

Case 1: f_{12} is a vertex of F_T with odd degree.
In this case we add the path $P = f_{11} v_1 f_{12}$ to the set of paths $P'_1, P'_2, \ldots, P'_m$ to obtain the desired odd F_T-vertex cover of G_T.

Case 2: f_{12} is a vertex of F_T with even degree.
In this case f_{12} is an F'_T vertex of odd degree in G'_T. Thus, one of the paths $P'_1, P'_2, \ldots, P'_m$ must have f_{12} as an endpoint, say P'_j. If the union of the paths $f_{11} v_1 f_{12}$ and P'_j contains a loop then the K_2-node expansion at each V_T vertex of the loop is unnecessary to obtain a bipartite graph. Since N is minimal, $f_{11} v_1 f_{12} \cup P_j$ must be a path. We obtain the desired odd F_T-vertex cover by replacing P_i with P'_i, $i \neq j$ and P'_j with $f_{12} v_1 f_{12} \cup P'_j$ this shows that $N = \sum_{i=1}^{m} \frac{length(P_i)}{2}$.

Conversely, suppose that $P_1, P_2, \ldots, P_n$ is an odd F_T-vertex cover in G_T with minimal total length. By Lemma 7 and Lemma 5 we can assume that $P_1, P_2, \ldots, P_n$

are pair wise non-crossing at each vertex of V in G_T. By repeatedly applying Lemma 6 we obtain a graph which is planar and bipartite. The total number of simple K_2-node expansions required for each path P_i is $\frac{length(P_i)}{2}$, since we get a simple K_2-node expansion for each V_T vertex on P_i. Thus, the total number of simple K_2-node expansions is

$$\sum_{i=1}^{m} \frac{length(P_i)}{2}.$$

If N is not minimal then by the first part of the proof there exists an odd F_T-vertex cover with fewer edges contrary to the hypothesis. ∎

6. Conclusions.

In this paper, we have shown that the NE(arbitrary, bipartite, K_2) problem is NP-Complete. However, this problem can be solved in polynomial time if the input graphs are restricted to be planar graphs where every vertex v with degree greater than 3 is the hub of a deg(v)-wheel. These graphs include planar 3-connected graphs. We conjecture that the NE(planar, bipartite, K_2) problem is NP-Complete even if the output graph is to be planer.

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