Real Rank as a Bound for Nonnegative Integer Rank

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1. Introduction

The problem of finding the biclique partition number of various bipartite graphs (bigraphs) and digraphs has been investigated in several recent papers (for example, [1], [3], [4]). This problem is equivalent to finding the nonnegative integer rank of the corresponding adjacency matrix (see [4] or [5]). Unfortunately, in general these numbers are difficult to calculate, but real rank provides a lower bound. In this paper we show how bad this bound can be by examining a variety of digraphs and $(0,1)$-matrices.

Let X be an $n \times n$ matrix over Z^+, the semiring of nonnegative integers. The *nonnegative integer rank*, $r_Z + (X)$, is the least k for which there exists $n \times k$ and $k \times n$ matrices F and G over Z^+ providing the factorization $X = FG$. If $r(X)$ denotes the ordinary real rank, then it is easy to see that $r(X) \leq r_Z + (X)$.

Next we consider minimum partitions of bigraphs and digraphs. Our digraphs will have no loops or multiple arcs. A biclique of a bigraph is a complete bipartite subgraph. A directed biclique is a biclique with vertex partition (X, Y) whose edges have been oriented from X to Y. The *biclique partition number* $bp(B)$ of a bigraph B is the smallest number of bicliques which partition the edges of B. Similarly, we let $\overrightarrow{bp}(D)$ be the minimum number of directed bicliques which partition the arcs of the digraph D.

To see the relationship between the partition problem for digraphs and the partition problem for bigraphs, let $A(D)$ be the adjacency matrix for a digraph D on n vertices. $A(D)$ is a $(0,1)$-matrix with zeros on the diagonal. The bicliques of D are in one-to-one correspondence to those submatrices of $A(D)$ with all entries equal to one and for which the sets of row indices and column indices are disjoint.

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Furthermore, if B is the bipartite graph on $2n$ vertices determined by $A(D)$, then $\overrightarrow{bp}(D) = bp(B)$.

The following result follows immediately from the definitions (see Gregory et al. [4] for details).

Proposition 1.1. *Suppose X is an $n \times n\,(0,1)$-matrix.*

a) *If B is its bigraph, then $bp(B) = r_Z + (X)$.*

b) *If all diagonal entries in X are zero and D is the digraph of X, then $\overrightarrow{bp}(D) = r_Z + (X)$.*

Given this result, we will move interchangeably between bigraphs, digraphs, and matrices. Graph-theoretic and matrix-theoretic methods are used together in many of the proofs.

Since in general it is difficult to calculate $r_Z + (A)$, the bound $r(A) \leq r_Z + (A)$ is frequently used to approximate or calculate $r_Z + (A)$. For example, if $I_n^c = J_n - I_n$, then it is easy to show that $r(I_n^c) = n$, so $r_Z + (I_n^c) = n$. For D a digraph, $r_Z + (A(D)) = \overrightarrow{bp}(D)$, so the complete digraph has partition number n. Thus, if $r(A)$ is close to n, we get a good bound, but how bad can this bound be? For

$$A = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_m \end{bmatrix},$$

$r_Z + (A) \geq \sum_{i=1}^{m} r_Z + (A_i)$ by Lemma 2.1 of [5]. So we will consider the situation where A is irreducible or where the digraph D is strongly connected.

Theorem 2.3 of [6] gives an infinite family of matrices A that satisfy

$$\frac{r(A)}{r_Z + (A)} = \frac{2}{3}.$$

In section 2 of this paper we give examples of infinite families of matrices where this ratio approaches $\frac{1}{2}$ as n approaches infinity. In section 3 we show that there exists families of matrices where this ratio approaches zero.

2. Matrices Satisfying $\frac{r(A)}{r_Z+(A)} > \frac{1}{2}$

In Hefner et al. [6] an infinite family of matrices was given that satisfies

$$\frac{r(A)}{r_Z + (A)} = \frac{2}{3}.$$

At the time that was the smallest ratio given in the literature. In this section we will give some infinite families where the ratio approaches $\frac{1}{2}$.

Let A_{2k} be the k-regular $2k \times 2k$ circulent $\{0,1\}$-matrix with first row having $\lceil \frac{k}{2} \rceil$ zeroes followed by k ones followed by $\lfloor \frac{k}{2} \rfloor$ zeroes. For example,

$$A_{14} = \begin{bmatrix} 0&0&0&0&1&1&1&1&1&1&\underline{1}&0&0&0 \\ 0&0&0&0&0&1&1&1&1&1&1&\underline{1}&0&0 \\ 0&0&0&0&0&0&1&1&1&1&1&1&\underline{1}&0 \\ 0&0&0&0&0&0&0&1&1&1&1&1&1&\underline{1} \\ \underline{1}&0&0&0&0&0&0&0&1&1&1&1&1&1 \\ 1&\underline{1}&0&0&0&0&0&0&0&1&1&1&1&1 \\ 1&1&\underline{1}&0&0&0&0&0&0&0&1&1&1&1 \\ 1&1&1&\underline{1}&0&0&0&0&0&0&0&1&1&1 \\ 1&1&1&1&\underline{1}&0&0&0&0&0&0&0&1&1 \\ 1&1&1&1&1&\underline{1}&0&0&0&0&0&0&0&1 \\ 1&1&1&1&1&1&\underline{1}&0&0&0&0&0&0&0 \\ 0&1&1&1&1&1&1&\underline{1}&0&0&0&0&0&0 \\ 0&0&1&1&1&1&1&1&\underline{1}&0&0&0&0&0 \\ 0&0&0&1&1&1&1&1&1&\underline{1}&0&0&0&0 \end{bmatrix}$$

A set S of ones in a matrix A is said to be *independent* if no two occur in the same row or column of A. A set S of ones is *isolated* if S is independent and no two ones of S are in a 2×2 submatrix of A of the form

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

If we let S be the set of fourteen underlined ones in A_{14}, then clearly S is isolated. By Lemma 3.4 of [4], it follows that $r_Z+(A_{14}) = 14$. In general, the same argument shows that $r_Z+(A_{2k}) = 2k$.

Next, we show that $r(A_{2k}) = k+1$. Let $\bar{A}$ be the $(k+1) \times (k+1)$ submatrix of the first $k+1$ columns of A_{2k} and the $k+1$ consecutive rows starting with the first row that has a one in column one. For example, in A_{14}, $\bar{A}$ is the submatrix consisting of the first eight columns and rows 5–12. Since $\bar{A}$ is a lower triangular matrix with ones on its diagonal, $\det(\bar{A}) = 1$. Therefore, $r(A_{2k}) \geq k+1$.

Now, consider the $k-1$ k-tuples $\vec{X}^T_{k_1} = (1,-1,0,\ldots,0)^T$, $\vec{X}^T_{k_2} = (0,1,-1,0,\ldots,0) \ldots, \vec{X}z^T_{k_{k-1}} = (0,\ldots,0,1,-1)^T$. Clearly these vectors are linearly independent. Now, for $i = 1,\ldots,k-1$, let $\vec{\alpha}^T_{k_i} = (\vec{X}^T_{k_1}, \vec{X}^T_{k_1})$. If we let $N_k = \langle \vec{\alpha}^T_{k_i} \rangle$ be the subspace generated by $\vec{\alpha}^T_{k_i}$, then $\dim(N_k) = k-1$. Furthermore, if $NS(A_{2k})$ designates the nullspace of A_{2k}, then it is easy to check that $N_k \subseteq NS(A_{2k})$, so $\dim(NS(A_{2k})) \geq k-1$. But $r(A_{2k}) \geq k+1$, so $\dim(NS(A_{2k})) = k-1$ and $r(A_{2k}) = k+1$. We have proved the following theorem.

THEOREM 2.1. *Let A_{2k} be the k-regular $\{0,1\}$-circulent matrix described above.*

Then

$$\frac{r(A_{2k})}{r_Z+(A_{2k})} = \frac{k+1}{2k}.$$

So this gives us an infinite series of matrices where the ratio of ranks decreases to $\frac{1}{2}$. The next infinite series of matrices that we consider allows us to get a ratio equal to any rational number between $\frac{1}{2}$ and 1 except rationals of the form $\frac{\lceil q/2 \rceil}{q}$ where q is odd.

THEOREM 2.2. *Let*

$$M = \begin{bmatrix} K_n & Q \\ P & K_m \end{bmatrix}$$

where $K_n = J_n - I_n$, $K_m = J_m - I_m$, $n, m \geq 3$ and P and Q are (0,1) matrices with at most one 1 in each row and column. Then $r_Z + (M) = n + m$.

Proof: Let $\mathcal{R}$ be a collection of rectangles in M (i.e., rank 1 (0, 1)-submatrices of M) which partition the 1's of M (i.e., sum to M).

We wish to show that $|\mathcal{R}| \geq n + m$. Let $\mathcal{A}$ consist of those members of $\mathcal{R}$ which meet K_n but not K_m; $\mathcal{B}$ those which meet K_m but not K_n; and, $\mathcal{C}$ those which meet both K_n and K_m. Then $\mathcal{R} \supseteq \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. It is therefore sufficient to show that $|\mathcal{A}| \geq n - \frac{|\mathcal{C}|}{2}$ and $|\mathcal{B}| \geq m - \frac{|\mathcal{C}|}{2}$.

Note that each member of $\mathcal{C}$ is a $J_{2,2}$ and that those 1's of K_n that are in members of $\mathcal{C}$ are independent in the sense that no two of them share a row or column.

Let A and B be the submatrices of K_n and K_m obtained by deleting those 1's of K_n and K_m respectively, that are in the $\mathcal{C}$ matrices. Fom the paragraph above, we see that A and B are the adjacency matrices of directed graphs obtained by deleting the arcs of collections of vertex disjoint directed paths and cycles from the complete directed graphs on n and m vertices, respectively.

In particular, by ordering the vertices appropriately, we may assume that $\overline{A} = J - A$ (the "complement" of A) is a direct sum of $n_i \times n_i$ cycle matrices[2]

$$C_{n_i} = \begin{bmatrix} 1 & 1 & & 0 & \\ & 1 & 1 & & 0 \\ & & & \ddots & \\ 0 & & \ddots & & 1 \\ & \ddots & & \ddots & \\ 1 & & 0 & & 1 \end{bmatrix}, i = 1, 2, \ldots, k; n_i \geq 2$$

[2] In particular $C_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

and $m_j \times m_j$ path matrices[3]

$$P_{m_j} = \begin{bmatrix} 1 & 1 & & & 0 \\ & 1 & 1 & & \\ & & 1 & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 1 \end{bmatrix}, j = 1,2,\ldots,\ell; m_j \geq 2$$

and an $r \times r$ identity matrix

$$I_r = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

Since

$$r(C_{n_i}) = \begin{cases} n_i & \text{if } n_i \text{ is odd} \\ n_i - 1 & \text{if } n_i \text{ is even} \end{cases}$$

and since the P_{m_j} and I_r have full real rank, it follows that $r(\overline{A}) = n$ minus the number of even n_i, $i = 1,2,\ldots,k$.

Note that the all 1's column $\vec{u} = [1,1,\ldots,1]^T$ is in the column space of $\overline{A}$ (choose each C_{n_i} column with weight $\frac{1}{2}$; choose the last, third from last, fifth from last, ... columns of each P_{m_j} with weight 1, and all of the columns of I_r).

Consequently, $\vec{u} = \overline{A}\vec{x}$ where $\sum x_i > 1$ (since $n > 2$).

Since $\vec{u} \in$ column space $\overline{A} = J - A$, we have column space $A \subseteq$ column space $\overline{A}$.

Now, $\vec{u} \in$ column space A too because $A\vec{x} = (J - \overline{A})\vec{x} = (\sum x_i - 1)\vec{u}$ where $\sum x_i \neq 1$. Thus, column space $\overline{A} \subseteq$ column space A. Therefore,[4] $r(A) = r(\overline{A}) = n$ minus the number of even n_i, $i = 1,2,\ldots,k$. Now, the number of off-diagonal 0's in A is

$$\begin{aligned} |\mathcal{C}| &= \sum n_i + \sum (m_j - 1) + r \\ &\geq 2(\text{the number of } n_i) \\ &\geq 2(\text{the number of even } n_i). \end{aligned}$$

Therefore, $r_Z + (A) \geq r(A) \geq n - \frac{|\mathcal{C}|}{2}$. Similarly, $r_Z + (B) \geq m - \frac{|\mathcal{C}|}{2}$.

Now, the members of $\mathcal{A}$, when restricted to K_n, yield a partition of A. Thus, $|\mathcal{A}| \geq r_Z + (A)$. Similarly, $|\mathcal{B}| \geq r_Z + (B)$.

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[3] In particular $C_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

[4] These observations about real rank of A versus $J - A$ appear essentially in Brualdi, Manber, and Ross [2] and are attributed there to H. Ryser.

Therefore,

$$\begin{aligned} |\mathcal{R}| &\geq |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| \\ &\geq n - \frac{|\mathcal{C}|}{2} + m - \frac{|\mathcal{C}|}{2} + |\mathcal{C}| \\ &= n + m \end{aligned}$$

as required. ∎

Remark: It is interesting to note that although the nonnegative integer rank of M remains unchanged for all choices of subpermutation matrices P and Q, the real rank can vary (by at most 1 with the addition or deletion of a single 1 from or to P or Q). Two extreme cases are

(i) P and Q are all 0: $M = K_n \bigoplus K_m$, $r(M) = m + n$;

(ii) $P = Q = I_n$: $M = \begin{bmatrix} K_n & I_N \\ I_n & K_n \end{bmatrix}$, $r(M) = n + 1$.

To get the real rank we are after, assume $n \geq m$ and let

$$M = \begin{bmatrix} K_n & \begin{matrix} I_m \\ 0 \end{matrix} \\ I_m \quad 0 & K_m \end{bmatrix}.$$

Then it is easy to see that the m bottom rows are the complements of the first m rows, so that the sum of each of these complements gives the all 1's vector. It follows that $r(M) = n + 1$. Combining this fact with Theorem 2.2 we get the following result.

Theorem 2.3. *Let*

$$M = \begin{bmatrix} K_n & \begin{matrix} I_m \\ 0 \end{matrix} \\ I_m \quad 0 & K_m \end{bmatrix} \textit{ for } n \geq m \geq 3.$$

Then

$$\frac{r(M)}{r_Z + (M)} = \frac{n+1}{n+m}.$$

3. $\frac{r(A)}{r_Z+(A)}$ Arbitrarily Small

In order to find matrices satisfying $\frac{r(A)}{r_Z+(A)} < \frac{1}{2}$, we had to use tensor products.

We are then able to find infinite families of matrices A_n satisfying

$$\lim_{n \to \infty} \frac{r(A_n)}{r_Z + (A_n)} = 0.$$

Let $i(A)$ denote the maximum size of a set of isolated 1's in the $\{0, 1\}$-matrix A. Let $A \otimes B$ be the Kronecker product, i.e., the matrix obtained by replacing each entry of A by the matrix $a_{ij}B$. The following result is well-known in the literature.

Proposition 3.1. $r(A \otimes B) = r(A)r(B)$.

The next result gives a lower bound for the nonnegative integer rank.

Theorem 3.2. $r_{Z^+}(A \otimes B) \geq i(A) r_{Z^+}(B)$.

Proof: Choose a set of $i(A)$ isolated 1's in A. In $A \otimes B$ this gives a set $\{B_1, B_2, \dots\}$ of $i(A)$ copies of B that are "isolated" in the sense that no rectangle contained in $A \otimes B$ will have 1's in two of these $i(A)$ copies of B. Now let $\mathcal{R}$ be any rectangle partition of $A \otimes B$; let $\mathcal{R}_i$ be the rectangles of $\mathcal{R}$ that have a 1 in B_i. We have just observed that these $\mathcal{R}_i$'s are disjoint. Hence

$$|\mathcal{R}| \geq \sum_{i=1}^{i(A)} |\mathcal{R}_i| \geq i(A) r_{Z^+}(B),$$

and the result follows. ∎

We can now get a ratio of ranks to be arbitrarily small by applying this result to any of the matrices A_{2k} from the previous section. For example, $r(A_6) = 4$, $r_{Z^+}(A_6) = i(A_6) = 6$.

Let $A^{(k)} = A_6 \otimes A_6 \otimes \cdots \otimes A_6$ (k times). Applying Proposition 3.1 and Theorem 3.2 repeatedly, we get: $r(A^{(k)}) = 4^k$ and $r_{Z^+}(A^{(k)}) = 6^k$. Hence,

$$\lim_{k \to \infty} \frac{r(A^{(k)})}{r_{Z^+}(A^{(k)})} = \lim_{k \to \infty} \left(\frac{2}{3}\right)^k = 0.$$

We have been unable to find any examples where the ratio is exactly $\frac{1}{2}$. Using tensor products is the only way we could find ratios less than $\frac{1}{2}$.

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