Maximal Partial Triple Systems of Order 13 With a Quadratic Leave: The Case of Nine Edges

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1. Introduction

A *partial triple system* (PTS) is a pair (V, B) where V is a v-set of *elements* and B is a set of 3-subsets of V called *triples* such that each 2-subset of V is contained in at most one triple of B. The *leave* of a PTS (V, B) is the graph (V, E) where E contains all pairs that do not appear in B. The leave is *quadratic* if each of its vertices has degree 0 or 2. A PTS (V, B) is *maximal* if its leave is triangle-free. A *Steiner triple system* (STS) is a PTS whose leave has no edges.

The existence of maximal partial triple systems (MPT) with a (triangle-free) quadratic leave Q was settled completely in [2]. Such an MPT exists if and only if certain obvious arithmetic conditions are satisfied (basically, the number of edges in the complement of Q has to be divisible by three), with exactly one exception: $v = 9, Q = C_4 \cup C_5$.

Maximal partial triple systems with a quadratic leave represent an important class of MPTs, in particular, because of their connections to totally symmetric quasigroups (see [4]) and to near-Steiner 1-factorizations of the complete graph (see [5]).

MPTs of order $v \leq 11$ were enumerated in [1]. MPTs of order 13 with hexagonal leave were enumerated in [3]. In this paper, the second in a series devoted to the enumeration of MPTs of order 13 with a quadratic leave, we deal with the case of leaves having nine edges. There are two such graphs: (i) C_9, and (ii) $C_4 \cup C_5$. The second case is interesting in that the leave $C_4 \cup C_5$ is the only "exceptional"

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Fig. 1

leave (for $v = 9$), and $v = 13$ is the first order after $v = 9$ for which an MPT with this leave exists. The number of nonisomorphic MPTs of order 13 with leave $C_4 \cup C_5$ turns out to be quite large: it equals 438. The number of nonisomorphic MPTs with leave C_9 is even larger: there are 2060 such systems.

2. Computational results

Unlike for MPTs with a hexagonal leave (cf.[3]), there is no obvious way to obtain MPTs with leave C_9 or $C_4 \cup C_5$, respectively, from Steiner triple systems. Call our MPTs briefly C_9-MPT, and $C_{4,5}$-MPT, respectively. The method to generate all C_9-MPTs and all $C_{4,5}$-MPTs was therefore similar to that used to obtain all those MPT(13) with a hexagonal leave which cannot be obtained from an STS through a triangle replacement (cf.[3]).

2a. C_9-MPTs

Let the leave of a C_9-MPT be $C = (012345678)$, and let the remaining four elements be a, b, c, d. A solution will fall into one of two classes, depending on whether it contains a triple $T \subset \{a, b, c, d\}$ (type 1) or not (type 2). Let us call a *diagonal* any edge of the complete graph K_9 on $\{0, 1, \ldots, 8\}$ not in the leave C. If a solution of type 1 contains a triple, say, $\{a, b, c\}$ then it also contains a triple $\{a, d, z\}$ with $z \epsilon C$, say, $\{a, d, 0\}$, and exactly four triples $\{a, x_i, y_i\}$, $i = 1, 2, 3, 4$, with $\{x_i, y_i\}$ a diagonal, and $\{x_i, y_i\} \cap \{x_j, y_j\} = \emptyset$ for $i \neq j$, and $\{x_i, y_i : i = 1, 2, 3, 4\} = \{1, 2, \ldots, 8\}$. There are 24 nonequivalent possibilities (subcases) for the mutual position of these four diagonals; two such typical subcases are shown in Fig.1. A solution of type 2 does not contain a triple on $\{a, b, c, d\}$, thus it must contain triples $\{a, b, u\}, \{a, c, v\}, \{a, d, w\}$, $u, v, w \epsilon C$, and three triples $\{a, x_i, y_i\}$, $i = 1, 2, 3$, where $\{x_i, y_i\}$ is a diagonal, $\{x_i, y_i\} \cap \{x_j, y_j\} = \emptyset$ for $i \neq j$, and $\{x_i, y_i : i = 1, 2, 3\} = \{0, 1, \ldots, 8\} \backslash \{u, v, w\}$. Here there are 38 nonequivalent possibilities (subcases) for the mutual position of the 3 diagonals; this number is somewhat larger because there are 7 nonequivalent possibilities to choose u, v, w from $\{0, 1, \ldots, 8\}$. Three typical subcases are shown in Fig.2.

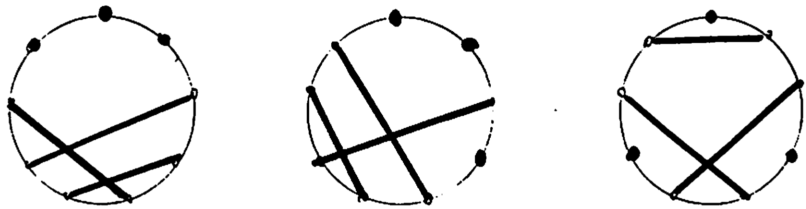

Fig. 2

In either case, the C_9-MPTs were generated by the computer hierarchically, according to a fixed order on subcases. Isomorph rejection was performed simultaneously. There are 632 nonisomorphic systems of type 1, and 1428 nonisomorphic systems of type 2, for a total of 2060 nonisomorphic C_9-MPTs. As an independent check, 50000 C_9-MPTs were generated randomly by hill-climbing (cf.,e.g.,[6]). There were 2059 nonisomorphic C_9-MPTs among these; just one C_9-MPT, with automorphism group of order 6, was missed.

For each of the solutions obtained, we computed the order of the automorphism group. The only orders that occur are 1,2,3 and 6 (the same as for C_6-MPTs, cf. [3]). The distribution of automorphism group sizes is as follows:

number of C_9-MPTs with automorphism group of order

	1	2	3	6
type 1	568	58	4	2
type 2	1398	20	10	-
total	1966	78	14	2

Other invariants computed included the number of almost parallel classes (APC), i.e. sets of four pairwise disjoint triples, and the chromatic index. Each system contains at least one APC. The number of APCs ranges from 1 to 13 while the chromatic index ranges from 6 to 8. This information is summarized below.

number of APCs

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
C_9-MPTs	0	8	52	146	269	411	347	314	216	153	106	25	11	2

chromatic index

	6	7	8
number of C_9-MPTs	15	2002	43

We list in Table 1 one C_9-MPT with automorphism group of order 1, 2, 3, respectively, of each of the two types, both C_9-MPTs with automorphism group of order 6, as well as one C_9-MPT with chromatic index 6 (in the Table, APC denotes the number of almost parallel classes).

2b. $C_{4,5}$-MPTs

In many respects, the procedure to generate all $C_{4,5}$-MPTs was similar to that for C_9-MPTs but there were some differences. Let the leave of a $C_{4,5}$-MPT be $C = (0123)(45678)$, and let the remaining elements be a, b, c, d. Again we have two types of solutions depending on whether there is a triple on $\{a, b, c, d\}$ or not. These fall into 9 and 11 subcases, respectively. A typical start for a solution of type 1 would include triples $\{a, b, c\}, \{a, d, 0\}, \{a, 1, 3\}, \{a, 2, 4\}, \{a, 5, 7\}, \{a, 6, 8\}$, while a typical start for a solution of type 2 would take triples $\{a, b, 0\}, \{a, c, 1\}, \{a, d, 2\}, \{a, 3, 4\}, \{a, 5, 7\}, \{a, 6, 8\}$. The $C_{4,5}$-MPTs were again generated hierarchically, with isomorph rejection performed during the process. The total number of nonisomorphic $C_{4,5}$-MPTs obtained is 438, of which 152 are of type 1, and 286 of type 2. Again, as a means of an independent check, 20000 $C_{4,5}$-MPTs were generated randomly by hill-climbing, confirming our computation. Of the 438 systems, 42 have automorphism group of order 2, and 396 are automorphism-free. No automorphism group orders other than 1 and 2 occur. Unlike in the case of C_9-MPTs, there exist systems without an APC. The information about the number of APCs and the chromatic index is summarized below.

number of APCs

	0	1	2	3	4	5	6	7	8	9	10	11
$C_{4,5}$-MPTs	2	3	8	27	66	66	93	77	54	23	16	3

chromatic index

	6	7	8
$C_{4,5}$-MPTs	3	427	8

In Table 2, we list the three $C_{4,5}$-MPTs with chromatic index 6, as well as two $C_{4,5}$-MPTs (one of each type) with automorphism group of order 2.

3. Conclusion

The authors hope to complete the enumeration of MPT(13)'s with a quadratic leave in a subsequent paper. This will involve considering the five possible leaves with 12 edges each.

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Table 1

Some C_9-MPT(13)s

No. 1	type 1; |Aut| =1; chr. index =7; APC =5 abc ad0 a15 a26 a37 a48 bd1 cd3 b04 b27 b35 b68 c07 c16 c24 c58 d28 d46 d57 025 036 138 147
No. 2	type 1; |Aut| =2; chr. index =7; APC =4 abc ad0 a15 a26 a37 a48 bd3 cd6 b04 b17 b25 b68 c05 c13 c28 c47 d18 d24 d57 027 036 146 358
No. 3	type 1; |Aut| =3; chr. index =7;APC =9 abc ad0 a17 a26 a35 a48 bd3 cd6 b05 b14 b27 b68 c02 c15 c38 c47 d18 d24 d57 037 046 136 258
No. 4	type 1; |Aut| =6; chr. index =7; APC =7 abc ad0 a16 a25 a38 a47 bd3 cd6 b04 b17 b26 b58 c05 c14 c28 c37 d18 d24 d57 027 036 135 468
No. 5	type 1; |Aut| =6; chr. index =7; APC =4 abc ad0 a14 a26 a37 a58 bd3 cd6 b05 b16 b28 b47 c04 c17 c25 c38 d18 d24 d57 027 036 135 468
No. 6	type 2; |Aut| =1; chr. index =7; APC =3 ab0 ac1 ad2 a37 a46 a58 bc3 bd4 cd5 b18 b26 b57 c04 c27 c68 d07 d16 d38 025 036 135 147 248
No. 7	type 2; |Aut| =2; chr. index =7; APC =6 ab0 ac1 ad3 a26 a48 a57 bc3 bd6 cd5 b15 b28 b47 c07 c24 c68 d04 d18 d27 025 036 137 146 358
No. 8	type 2; |Aut| =3; chr. index =7; APC =11 ab0 ac1 ad3 a26 a48 a57 bc7 bd6 cd4 b15 b24 b38 c02 c35 c68 d05 d18 d27 036 047 137 146 258
No. 9	type 2; |Aut| =1 chr. index =6; APC =11 a27 bc5 d36 148 ad4 b38 c26 057 ac1 bd7 046 258 a35 b16 c47 d02 a68 b24 c03 d15 ab0 cd8 137

Table 2

Some $C_{4,5}$-MPT(13)s

No.1	type 1; |Aut| =1; chr. index =6; APC =7 abc d27 146 358 a13 b28 cd5 047 a24 b15 c08 d36 a57 b06 c34 d18 a68 bd4 c17 025 ad0 b37 c26

No. 2 type 2; |Aut| =1; chr. index =6; APC =11
a25 bd7 c08 134
ab0 c27 d16 358
a68 b24 cd3 157
a37 b18 c46 d05
ac1 b36 d28 047
ad4 bc5 026

No. 3 type 2; |Aut| =1; chr. index =6; APC =11
ac1 bd5 024 368
a26 b34 c08 d17
a58 b27 cd3 146
ab0 c47 d28 135
a37 b18 c25 d06
ad4 bc6 057

No.4 type 1; |Aut| =2; chr. index =8; APC =0
ab0 ac6 ad1 a24 a38 a57 bcd b17 b28 b35 b46 c08
c14 c25 c37 d05 d27 d34 d68 026 047 136 158

No. 5 type 2; |Aut| =2; chr. index =7; APC =6
ab7 ac2 ad8 bc3 bd2 cd4 a06 a14 a35 b04 b15 b68
c07 c16 c58 d05 d17 d36 028 138 246 257 347

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