On t-multi-set designs

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Abstract. A multi-set design of order v, $MB_t(v,k,\lambda)$, first defined by Assaf, Hartman and Mendelsohn, is an ordered pair (V,B), where V is a set of cardinality v and B is a collection of multi-subsets of V of size k (called blocks), with the property that every multi-subset of V of size t is contained a total of λ times in the blocks of B. (For example, the multi-set $\{a,b,b\}$ is contained $\binom{2}{1}\binom{4}{2} = 12$ times in the multi-set $\{a,a,b,b,b,b,c\}$ and not at all in the multi-set $\{a,a,b,c\}$.) Previously the first author had pointed out that any t-multi-set design is a 1-design. Here we show the pleasant yet not obvious fact that any t-multi-set design is a t'-multi-set design for any positive integer $t' \le t$.

A 'traditional' t-design is a pair (V,B) where V is a set of size v and B is a collection of k-sets, called blocks, chosen from the set V, so that each t-tuple occurs λ times altogether. Such t-designs are often denoted by t-(v,k,λ). (See Wallis [3] Section 2.3 for example.) It is well-known, and easy to verify, that a t-design is also a t'-design for any $t' < t$.

In recent years some attention has focussed on relaxing the concept of a block as a *subset* of V, to allow repetition of elements of V within blocks; thus blocks, of size k, may be *multi-subsets* of V. For a survey of different types of designs with such repetitions of elements in blocks, see [2]. The type we consider here, t-multi-set designs, were first described in [1], and are a direct generalisation of a t-design described above. So a multi-set design of order v, $MB_t(v,k,\lambda)$ (using notation of [1]) is a pair (V,B) where V is our set of elements of size v, and B is a collection of multi-subsets of V of size k (called blocks), with the property that every multi-subset of V of size t is contained altogether λ times in the blocks of B.

An example illustrates this: An $MB_3(6,4,3)$ is given by the pair (V,B) where $V = \{0,1,2,3,4,5\}$ and B is the collection of 42 blocks obtained from the following seven starter blocks. Here, for brevity, a block $\{w,x,y,z\}$ is denoted by $wxyz$.

0002, 0003, 0004, 0011, 0015, 0134, 0124.

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In this 3-multi-set design, each triple $\{x, y, z\}$ (where x, y and z may not be distinct) occurs precisely three times. For example, the triple $\{0, 0, 2\}$ occurs 3 times in the one block 0002, and the triple $\{0, 0, 1\}$ occurs twice in the block 0011 and once in the block 0015.

We shall now proceed to show that any t-multi-set design is also a t'-multi-set design with $t' < t$. The paper [1] in which these designs were introduced made no mention of this fact, and in the survey [2] the first author showed that a multi-set design is equi-replicate (that is, a 1-design) in that each element does occur (counting multiplicities) equally often. Here it will suffice to show that any t-multi-set design is a $(t-1)$-multi-set design; the required result for any $t' < t$ will then follow by induction.

Henceforth, to avoid confusion, we shall denote the parameter λ in an $MB_t(v, k, \lambda)$ by λ_t.

Firstly, if an $MB_t(v, k, \lambda_t)$ is a t'-multi-set design, with $t' < t$, what should $\lambda_{t'}$ be? Suppose there are b blocks of size k. Each block has $\binom{k}{t}$ t-multi-sets in it. There are $\binom{v+t-1}{t}$ possible t-multi-sets altogether, and each one occurs λ_t times. So we have

$$\binom{v+t-1}{t}\lambda_t = b\binom{k}{t}. \tag{1}$$

Now consider t'-multi-sets. There are $\binom{v+t'-1}{t'}$ possible t'-multi-sets that can be formed from a v-set. If each one occurs $\lambda_{t'}$ times, then

$$\binom{v+t'-1}{t'}\lambda_{t'} = b\binom{k}{t'}. \tag{2}$$

From (1) and (2) we see that we hope to have

$$\lambda_{t'} = \lambda_t \frac{(v+t-1)\dots(v+t')}{(k-t')\dots(k-t+1)}. \tag{3}$$

In particular, we hope to show that

$$\lambda_{t-1} = \lambda_t \frac{(v+t-1)}{(k-t+1)}. \tag{4}$$

Now we introduce some notation. A $(t-1)$-multi-set which contains x_i precisely j_i times, for $1 \le i \le s$, will be denoted by

$$\mathbf{x} = x_1^{j_1} x_2^{j_2} \dots x_s^{j_s}, \tag{5}$$

where of course

$$j_1 + j_2 + \dots + j_s = \sum_{\ell=1}^{s} j_\ell = t-1 \tag{6}$$

(so $s \leq t-1$). Here the elements $x_1, x_2, \dots, x_s$ are distinct (but of course the case $s = 1, \mathbf{x} = x_1^{j_1}$ and $j_1 = t-1$ is included here).

Now we concentrate on the $(t-1)$-multi-set $\mathbf{x}$ given in (5). Suppose we consider blocks containing the element x_ℓ precisely i_ℓ times, for $1 \leq \ell \leq s$. Denoting the ordered set $i_1 i_2 \dots i_s$ by $\mathbf{i}$, we let $\rho_{\mathbf{i}}^{\mathbf{x}}$ denote the number of blocks containing the element x_ℓ precisely i_ℓ times, for $1 \leq \ell \leq s$. (See Figure 1.)

Figure 1 $\quad \rho_{\mathbf{i}}^{\mathbf{x}}$ blocks of the form:

$$\overbrace{x_1 \dots x_1}^{i_1} \overbrace{x_2 \dots x_2}^{i_2} \dots \overbrace{x_s \dots x_s}^{i_s} \underbrace{\overbrace{* \cdots *}^{k - \sum_{\ell=1}^{s} i_\ell}}_{}$$

any of the $v - s$

other elements here

Let our $(t-1)$-multi-set $\mathbf{x}$ occur $\lambda_{t-1}^{\mathbf{x}}$ times altogether. Then we have

$$\lambda_{t-1}^{\mathbf{x}} = \sum_{\mathbf{i}} \binom{i_1}{j_1} \binom{i_2}{j_2} \cdots \binom{i_s}{j_s} \rho_{\mathbf{i}}^{\mathbf{x}}, \tag{7}$$

where of course $\binom{i_\ell}{j_\ell} = 0$ if $i_\ell < j_\ell$, and where the summation is over all possible $\mathbf{i} = i_1 i_2 \dots i_s$; in other words the summation is over all blocks that contain the elements $x_1, x_2, \dots, x_s$.

Now consider any t-multi-set which contains the $(t-1)$-multi-set $\mathbf{x}$. There are v possible elements that we can adjoin to x in order to make it a t-multi-set. Also we know that each t-multi-set occurs λ_t times altogether. Suppose we adjoin element z to the $(t-1)$-multi-set $\mathbf{x}$. If $z = x_\ell$ for some $\ell \in \{1, 2, \dots, s\}$ then

$$\sum_{\mathbf{i}} \rho_{\mathbf{i}}^{\mathbf{x}} \binom{i_1}{j_1} \cdots \binom{i_{\ell-1}}{j_{\ell-1}} \binom{i_\ell}{j_\ell + 1} \binom{i_{\ell+1}}{j_{\ell+1}} \cdots \binom{i_s}{j_s} = \lambda_t, \quad 1 \leq \ell \leq s. \tag{8}$$

On the other hand, if z is one of the $v - s$ elements distinct from $x_1, x_2, \dots, x_s$ which occur in the $(t-1)$-multi-set $\mathbf{x}$, then adjoining z to $\mathbf{x}$ to make a t-multi-set, and counting, yields:

$$\sum_{\mathbf{i}} \rho_{\mathbf{i}}^{\mathbf{x}} \binom{i_1}{j_1} \cdots \binom{i_s}{j_s} \left(k - \sum_{\ell=1}^{s} i_\ell \right) = \lambda_t (v - s). \tag{9}$$

The left-hand side of (9) can be written

$$k \sum_{\mathbf{i}} \rho_{\mathbf{i}}^{\mathbf{x}} \binom{i_1}{j_1} \cdots \binom{i_s}{j_s} - \sum_{\ell=1}^{s} \sum_{\mathbf{i}} \rho_{\mathbf{i}}^{\mathbf{x}} \binom{i_1}{j_1} \cdots \binom{i_{\ell-1}}{j_{\ell-1}} i_\ell \binom{i_\ell}{j_\ell} \binom{i_{\ell+1}}{j_{\ell+1}} \cdots \binom{i_s}{j_s}. \tag{10}$$

However it is easy to show that

$$i\binom{i}{j} = (j+1)\binom{i}{j+1} + j\binom{i}{j}, \tag{11}$$

and using (11) we find that (10) becomes

$$\begin{aligned}
&k\sum_{\mathbf{i}} \rho_{\mathbf{i}}^{\mathbf{x}} \binom{i_1}{j_1}\cdots\binom{i_s}{j_s} \\
&\quad - \sum_{\ell=1}^{s}\sum_{\mathbf{i}} \rho_{\mathbf{i}}^{\mathbf{x}} \left\{ (j_\ell+1)\left[\binom{i_1}{j_1}\cdots\binom{i_{\ell-1}}{j_{\ell-1}}\binom{i_\ell}{j_\ell+1}\binom{i_{\ell+1}}{j_{\ell+1}}\cdots\binom{i_s}{j_s}\right]\right. \\
&\qquad\qquad \left. + \left[j_\ell\binom{i_1}{j_1}\cdots\binom{i_\ell}{j_\ell}\cdots\binom{i_s}{j_s}\right]\right\} \\
&= k\sum_{\mathbf{i}} \rho_{\mathbf{i}}^{\mathbf{x}} \binom{i_1}{j_1}\cdots\binom{i_s}{j_s} - \sum_{\ell=1}^{s}(j_\ell+1)\lambda_t - \sum_{\ell=1}^{s}\sum_{\mathbf{i}} \rho_{\mathbf{i}}^{\mathbf{x}} j_\ell \binom{i_1}{j_1}\cdots\binom{i_s}{j_s} \\
&\qquad\qquad \text{(using (8))} \\
&= k\lambda_{t-1}^{\mathbf{x}} - (t-1)\lambda_t - s\lambda_t - \sum_{\ell=1}^{s} j_\ell \lambda_{t-1}^{\mathbf{x}} \ \text{(using (7) and (6))} \\
&= \lambda_{t-1}^{\mathbf{x}}(k-t+1) - (t+s-1)\lambda_t.
\end{aligned}$$

This was the left-hand side of (9). So (9) becomes

$$\lambda_{t-1}^{\mathbf{x}}(k-t+1) - (t+s-1)\lambda_t = \lambda_t(v-s),$$

so that

$$\lambda_{t-1}^{\mathbf{x}} = \lambda_t \frac{(v+t-1)}{(k-t+1)},$$

which is the result we predicted in (4) above; moreover this shows that $\lambda_{t-1}^{\mathbf{x}}$ is independent of the $(t-1)$-multi-set $\mathbf{x}$ under consideration. Thus any t-multi-set design is a $(t-1)$-multi-set design. Then (by induction) any t-multi-set design is also a t'-multi-set design for $t' < t$. This is precisely what we wanted to show.

References

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