Signing Balanced Incomplete Block Designs over the Group of Order Two

Warwick de Launey

Cryptomathematics Research Group
Communications Division, ERL, DSTO
c/o DVR2 Registry
Victoria Barracks
St Kilda Road
Australia 3004

Abstract. If the non-zero entries of an incidence matrix X of BIBD($v, b, r, k, 2$) have been signed to produce a $(0, 1, -1)$ matrix Y such that

$$YY^T = rI_v,$$

then we say it has been signed. The resulting matrix Y is said to be a Bhaskar Rao design BRD($v, k, 2$). We discuss the complexity of two signing problems, (i) given v, k and λ, decide whether there is a BRD $(v, k, 2\lambda)$, (ii) given a $BIBD$ $(v, k, 2\lambda)$ decide whether it is signable. The paper describes related optimisation problems. We show that the signing problems are equivalent to finding the real roots of certain multivariable polynomials. Then we describe some linear constraints which reduce the size of the second problem, we show certain special cases have polynomial complexity, and we indicate how in some cases the second problem can be decomposed into smaller independent problems. Finally, we characterise signable Steiner triple systems in terms of their block-intersection graphs, and show that the complexity of deciding whether a twofold triple system can be signed is linear in the number of blocks.

1. Introduction

In this paper, we study the complexity of two problems associated with the signing of balanced incomplete block designs (BIBD) with even index.

Notation and Definition 1.1: We let D denote a BIBD $(v, b, r, k, 2\lambda)$ with blocks $B_1, B_2, \ldots, B_b$ and treatments $u_1, u_2, \ldots, u_v$. D is said to have been *signed over* Z_2 if the non-zero entries of an incidence matrix X of D have been signed to produce a $(0, 1, -1)$ matrix Y such that

$$YY^T = rI_v.$$

The matrix Y is said to be a Bhaskar Rao design, BRD($v, b, r, k, 2\lambda$), and, in this paper, D will simply be said to be *signable*. ∎

The class of Bhaskar Rao designs is a subclass of the class of Generalised Bhaskar Rao designs, and Hadamard matrices and weighing matrices are special types of Bhaskar Rao designs. These designs have numerous interesting connections with coding theory, cryptography, finite geometry and experimental design. We investigate the complexity of the following problems.

(i) Given v, k and λ, decide whether there is a BRD($v, k, 2\lambda$).
(ii) Given a BIBD $(v, k, 2\lambda)$, decide whether it is signable.

JCMCC **11** (1992), pp. 73-82

P. B Gibbons and R. A. Mathon ([4] and [5]) have implemented back-tracking algorithms employing on-going isomorph rejection to reduce the work needed to complete a search for a design based on a given incidence matrix. These techniques have allowed them to enumerate the signings over a number of groups of a number of interesting designs. Indeed, they found the apparently "sporadic" BRD(19,9,4). In this paper, we focus on the complexity of the problems. We will also be interested in alternative formulations of these problems.

One of the more interesting results in this paper is a characterisation of the signable twofold Steiner triple systems in terms of their block intersection graphs.

Definition 1.2: Using the notation of 1.1, the *ith block inersection graph* $G_i(D)$ *of* D is the graph on b vertices $1, 2, \ldots, b$ where (s, t) is an edge if and only if the blocks B_s and B_t have precisely i treatments in common. ∎

P. B. Gibbons, R. A. Mathon, and D. G. Corneil [3] suggest that block intersection graphs might be a useful in distinguishing non-isomorphic BIBD's. Our result for BIBD(v, 3, 2) proves that signability will provide no additional power to distinguish between non-isomorphic designs. It is an open question whether, for $k > 3$ or $\lambda > 2$, the signability of a BIBD is completely determined by its block intersection graphs.

Finally, we show how graph theoretic arguments can be used to strengthen a well known non-existence theorem for BRD($v, b, r, k, 2\lambda$).

2. Related Optimisation Problems

In this section we show that signing a BIBD ($v, k, 2\lambda$) which can be signed over Z_2 is equivalent to solving an optimisation problem over the bk-dimensional real cube $[-1, 1]^{bk}$. We begin with a technical lemma.

Lemma 2.1. *Let* $\lambda \geq 1$ *be an integer and let* $f\colon [-1, 1]^{2\lambda} \to \mathbf{R}$ *be the map*

$$f(Y_1, Y_2, \ldots, Y_{2\lambda}) = \sum_{1 \leq i < j \leq 2\lambda} Y_i Y_j;$$

then f *is minimised when half the* Y_i*'s equal 1 and the remainder equal -1.*

Proof: Observe that

$$2f = \left(\sum_{i=1}^{2\lambda} Y_i\right)^2 - \sum_{i=1}^{2\lambda} Y_i^2,$$

and, hence, $f \geq -\lambda$, with equality if and only if, for all $i = 1, 2, \ldots, 2\lambda, Y_i = \pm 1$, and $\sum_{i=1}^{2\lambda} Y_i = 0$. ∎

Theorem (Related Problem) 2.2. *Let* D *be a BIBD* ($v, k, 2\lambda$) *which may be signed over* Z_2. *Then signing* D *to obtain a BRD* Y *is equivalent to solving the optimisation problem: minimise*

$$C(Y) = \sum_{1 \leq i < j \leq v} \; \sum_{1 \leq s < t \leq b} y_{is} y_{js} y_{it} y_{jt},$$

subject to the constraints

$$-1 \le y_{it} \le 1 \quad \textit{if } u_i \in B_t,$$
$$y_{it} = O \quad \textit{otherwise}.$$

Proof: The term $\sum_{1\le s<t\le b} y_{is}y_{js}y_{it}y_{jt}$ equals $f(y_{it_1}y_{jt_1}, y_{it_2}y_{jt_2}, \ldots, y_{it_{2\lambda}}y_{jt_{2\lambda}})$, where $B_{t_1}, B_{t_2}, \ldots, B_{t_{2\lambda}}$ are the blocks which contain the ith and jth treatments. Hence $C(X)$ is the sum of $v(v-1)/2$ terms of the form $f(Y_1, Y_2, \ldots, Y_{2\lambda})$, and, by Lemma 2.1, each of these terms is minimised precisely when D is signed. ∎

It is possible to rephrase this result and other similar results, relating the existence of combinatorial designs to optimisation problems, as results involving the roots of multivariate polynomials. In our case, we have the following.

Theorem (Related Problem) 2.3. *There exists a BRD $(v, k, 2\lambda)$ if and only if the polynomial*

$$F(Y) = \sum_{1\le i<j\le v}\left(\left(\sum_{1\le s<t\le b} y_{is}y_{js}y_{it}y_{jt}\right) + \lambda\right)^2 + \sum_{s=1}^{b}\left(\left(\sum_{i=1}^{v} y_{is}^2\right) - k\right)^2 + \sum_{s=1}^{b}\sum_{i=1}^{v}(y_{is}^2 - y_{is}^4)^2 \tag{2.1}$$

has roots in $\mathbf{R}^n$, and a BIBD$(v, k, 2\lambda)$ with incidence matrix $X = (x_{ij})$ is signable if and only if the polynomial

$$F(Y) = \sum_{1\le i<j\le v}\left(\left(\sum_{1\le s<t\le b} y_{is}y_{js}y_{it}y_{jt}\right) + \lambda\right)^2 + \sum_{s=1}^{b}\sum_{i=1}^{v}(y_{is}^2 - x_{is})^2 \tag{2.2}$$

has roots in $\mathbf{R}^n$.

Proof: We prove the first result only. If Y is a BRD $(v, k, 2\lambda)$, then all three major summations in (2.1) are zero, and, hence, $F(Y)$ is zero. Conversely, $F(Y)$ is zero if and only if each of the double summations is zero. The third term is zero if and only if all entries of Y are 0,1, or -1, and the second term is zero if and only if there are k non-zero entries per column of Y. Hence

$$\sum_{s=1}^{b}\sum_{1\le i<j\le v} y_{is}^2 y_{js}^2 = bk(k-1)/2 = \lambda v(v-1)/2. \tag{2.3}$$

Also,

$$\sum_{1 \le s < t \le b} y_{is} y_{js} y_{it} y_{jt} = \frac{1}{2} \left(\left(\sum_{s=1}^{b} y_{is} y_{js} \right)^2 - \sum_{s=1}^{b} y_{is}^2 y_{js}^2 \right). \qquad (2.4)$$

So, if, in addition, the first double summation is zero, then

$$\sum_{1 \le i < j \le v} \left(\sum_{s=1}^{b} y_{is} y_{js} \right)^2 = 0, \qquad (2.5)$$

and, by (2.4) and (2.5),

$$\sum_{s=1}^{b} y_{is}^2 y_{js}^2 = \lambda,$$

for all i and j such that $1 \le i < j \le v$. ∎

Theorem 2.3 may allow methods for counting roots of multivariable polynomials to be used to settle existence questions or to enumerate BRDs. Theorem 2.2 may allow general optimisation techniques such as simulated annealing to be applied in generating random designs. Moreover, such an approach may uncover some interesting approximate solutions even when no solution exists.

3. Some Linear Constraints and a Polynomial Time Algorithm for Counting BRD$(v, k, 2)$s

Since any row or column of a BRD may be multiplied by -1 without destroying the defining properies of a BRD, the space which we need to search to enumerate BRD$(v, k, 2\lambda)$s can contain no more than $2^{(b-1)(k-1)}$ elements. In this section we describe linear constraints which may be used to reduce the size of this space.

Let $Y = (y_{ij})$ be a BRD$(v, k, 2)$ which is based on a BIBD$(v, k, 2)$ D; then for all i, j, s and t, where $1 \le i < j \le v$ and $1 \le s < t \le b$,

$$y_{is} y_{it} y_{js} y_{is} = \begin{cases} -1 & \text{if } u_i, u_j \in B_s, B_t, \\ 0 & \text{otherwise.} \end{cases} \qquad (3.1)$$

Hence finding all the BRD$(v, k, 2)$s which are based on a design D is equivalent to creating and solving a system of $v(v-1)/2$ linear equations in $bk = v(v-1)/2(k-1)$ variables. Given the blocks of X, the system of equations can be listed in order bk^2 operations, and using Gaussian elimination, the system can be solved in order bkv^4 arithmetic operations. The system will be sparse, because each equation involves only four variables; so there may be faster ways of solving the system. In any case, if D can be signed, it can be signed in 2^m (where m is a non-negative integer) ways.

Theorem 3.1. *Counting the number of BRD$(v,k,2)$ which are based on a BIBD $(v,k,2)$ can be done in order bkv^4 operations.* ∎

In general, for any BRD$(v,k,2\lambda)$ $Y = (y_{is})$, and any pair i and j, where $1 \le i < j \le v$,

$$\prod_{i,j,s \text{ such that } B_s \supset \{u_i,u_j\}} y_{is}y_{js} = (-1)^{\lambda}; \qquad (3.2)$$

For certain designs D, these constraints alone may be enough to rule out the existence of a BRD based on D. For example [2], there is no BRD$(10,4,2)$. This is especially likely when the number of equations $v(v-1)/2$ is greater than the number of entries $(b-1)(k-1)$ which we are free to vary.

Proposition 3.2. *Let D be a BIBD$(v,k,2\lambda)$, and consider the inequality*

$$v(v-1)/2 \ge (b-1)(k-1). \qquad (3.3)$$

(i) If D is symmetric, (3.3) holds if and only if $k \le v/2+1$.
(ii) If D is not symmetric, (3.3) holds if and only if $4\lambda \le k$.

Proof:

(i) Note $b = v$, and simplify (3.3).

$$\frac{bk(k-1)}{4\lambda} = v(v-1)/2 \ge (b-1)(k-1).$$

Equivalently,

$$4\lambda \le k + \frac{k}{b-1},$$

and, since $b > v > k$ and λ is an integer, this is equivalent to $4\lambda \le k$. ∎

When D satisfies Proposition 3.2 we would expect the linear constraints to dramatically reduce the size of the space which would need to be searched to find all possible signings. Preliminary investigations using a computer show that the linear constraints reduce the search space when $(v,k,\lambda) = (40,13,4)$ from 2^{468} to 2^{43}, and when $(v,k,\lambda) = (19,9,4)$ from 2^{144} to 2^{21}.

The calculation of the space of solutions to these linear equations can be done in polynomial time. This space merits further investigation. For example, the size of the space may be a good polynomial-time equivalence discriminator. Also, it would be particularly interesting to see how the group of auto-morphisms of a design acts on the space, and an attempt to adapt Gibbons and Mathon's backtracking and isomorph rejection techniques to this new search space would also be worthwhile.

4. Related Graph Theoretic Problems

Definition 4.1:

(i) Let D be a BIBD($v, k, 2\lambda$), and let $G(D)$ denote the labelled multigraph on the vertices $1, 2, \ldots, b$, where $(s, t)_{(i,j)}$ is an edge (labelled by the *unordered* pair (i, j)) if $i \neq j$ and $B_s \cap B_t \supset \{u_i, u_j\}$.

(ii) Similarly, if X is an incidence matrix of D, define a labeled network $N(X) \approx N(D)$, on vertices $1, 2, \ldots, b$, where $(s, t)_{(i,j)}$ is an edge if $x_{is}x_{it}x_{js}x_{jt} = 1$.

(iii) Finally, if Y is a BRD($v, k, 2\lambda$), define an edge-coloured labelled network $N(Y)$, on vertices $1, 2, \ldots, b$, where $(s, t)_{(i,j)}$ is a black edge if $y_{is}y_{it}y_{js}y_{jt} = 1$ and a red edge if $y_{is}y_{it}y_{js}y_{jt} = -1$. ∎

Lemma 4.2.

(i) *$N(D)$ is regular with degree $(2\lambda - 1)k(k-1)/2$.*

(ii) *For all i and j, where $1 \leq i < j \leq v$, precisely $\lambda(2\lambda - 1)$ edges of $N(D)$ are labelled with (i, j). Indeed, these edges form a clique. Given the blocks of D, the vertices of each of these cliques may be listed in order bk^2 operations.*

(iii) *Y is a BRD($v, k, 2\lambda$) if and only if, for all i and j, where $1 \leq i < j \leq v$, there are precisely λ^2 red edges of $N(Y)$ which are labelled by (i, j) (while the black edges which are labelled with (i, j) form two disjoint cliques on λ vertices).*

(iv) *Ignoring labelling, edges in $N(D)$ with multiplicity $i(i-1)/2$, where $i = 1, 2, \ldots, k$, correspond to edges in $G_i(D)$.*

Proof:

(i) Fix s; then $(s, t)_{(i,j)}$ is in $N(D)$ if and only if $B_s \supset \{u_i, u_j\}$, $B_t \supset \{u_i, u_j\}$, $t \neq s$ and $i < j$. Since D is a BIBD($v, k, 2\lambda$), there are $(2\lambda - 1)k(k-1)/2$ triples (i, j, t) which satisfy these constraints.

(ii) An edge labelled with (i, j) corresponds to a pair of blocks which contain the ith and jth treatments. There are 2λ blocks which contain u_i and u_j; so there are $2\lambda(2\lambda - 1)/2$ pairs of such blocks, and the corresponding edges form a clique. Finally, to produce lists of the vertices in these cliques, it is sufficient to build an array by processing each block once as follows. For each pair of treatments u_i and u_j in B_s, add an entry containing s to the $((i-1)v + j)$th row of the array. At the end of this process, the $((i-1)v + j)$th row lists (in ascending order) the vertices of the clique formed by the edges labelled with (i, j).

(iii) Half the non-zero entries in the list $y_{is}y_{js}$, where $s = 1, 2, \ldots, b$, will equal 1 and the remainder of non-zero entries will equal -1.

(iv) An edge with multiplicity $i(i-1)/2$ corresponds to a pair of blocks whose intersection contains exactly i treatments. ∎

We now introduce the concept of an impression of a treatment-block incidence.

Definition 4.3:

(i) We say the edge $(s,t)_{(i,j)}$ of $N(D)$ is supported by the four *incidences* $u_i \in B_s, u_i \in B_t, u_j \in B_s, u_j \in B_t$ termed *supports*. (Each edge has precisely four supports.)

(ii) With each incidence $u_i \in B_s$, we associate a subnetwork of $N(D)$ denoted by F_{is} whose edges are those which are supported by $u_i \in B_s$. F_{is} is called the *impression* of $u_i \in B_s$.

(iii) If $u_i \in B_s$, let κ_{is} denote the operation where the colour of each edge in F_{is} is reversed.

∎

Lemma 4.4. *Let $Y = (y_{ij})$ be a BRD$(v,k,2\lambda)$ based on X; then the edge-coloured network $N(Y)$ may be obtained from $N(X)$ by applying, in any order, the operations κ_{is}, where $y_{is} = -1$.*

Proof: Let κ_Y denote the composition of the operations given in the lemma. The edge $e = (s,t)_{(i,j)}$ in $N(Y)$ is red if and only if $y_{is}y_{it}y_{js}y_{jt} = -1$. But the colour of e is reversed by κ_{mn} if and only if $m = i$ or j, $n = s$ or t, and $y_{mn} = -1$. Hence $\kappa_Y(e)1 = \kappa_{is}\kappa_{it}\kappa_{js}\kappa_{jt}(e)$ and the colour of e will be reversed by κ_Y if and only if $y_{is}y_{it}y_{js}y_{jt} = -1$. ∎

Theorem (Related Problem) 4.5. *Signing D is equivalent to finding a sequence of operations κ_{is} which, for each i and j, where $1 \leq i < j \leq v$, reverses the colour of precisely λ^2 edges which are labelled by (i,j).*

∎

A related problem would be to determine whether there is any vector of weight $2v(v-1)/2$ in the vector subspace of the edge space of $N(D)$ which is spanned by the impressions. The general problem of determining whether a vector space contains a vector of a given weight is NP-complete. When $k = 3$ and $\lambda = 2$, a complete solution to 4.5 is easily stated.

Theorem 4.6. *When $k = 3$ and $\lambda = 2$, D may be signed if and only if all the connected components of $N(D)$ have an even number of edges (ie. $G_3(D)$ is null and all the connected conponents of $G_2(D)$ have an even number of edges).*

Proof: By Lemma 4.2 and Definition 4.3, when $k = 3$ and $\lambda = 2$, the degree of each vertex of $N(D)$ is $(2\lambda - 1)k(k-1)/2 = 3$, and any pair of incident edges comprises an impression. So, when $k = 3$ and $\lambda = 2$, the Related Problem 4.5 reduces to the following: "given $N(D)$, find a sequence of impression colouring operations, each constituting the reversal of the colours of a pair of incident edges, whose net effect is the reversal of the colour of every edge in $N(D)$". For there to be a solution to this problem, it is clear no connected component of $N(D)$ can have an odd number of edges.

Conversely, if (omitting the labels) $(1,2),(2,3),\ldots,(n-2,n-1),(n-1,n)$ is a path, then reversing the colours of the edges in the subnetworks $\{(1,2),(2,3)\}$,$\{(2,3),(3,4)\}$, ... , $\{(n-2,n-1),(n-1,n)\}$ has the net effect of reversing the colours of $(1,2)$ and $(n-1,n)$ only. Hence, if each component has an even number of edges, the edges may have their colours reversed in pairs. ∎

The following result shows that the complexity of deciding whether a BIBD$(v,3,2)$ is signable is linear in the amount of input.

Corollary 4.7. *Deciding whether a BIBD$(v,3,2)$ D is signable can be done in order b (ie. order v^2) operations.*

Proof: Deriving $N(D)$ from an input listing the blocks of D is, by Lemma 4.2, of at most order b complexity (k is fixed), and the complexity of listing the vertices in the connected components of a network is proportional to the number of edges. $N(D)$ has $3b/2$ edges. ∎

5. A Non-Existence Theorem

In this section we prove a non-existence result which improves on a non-existence result which has been progressively developed in [1], [6] and [7]. The most interesting feature of our treatment is the method of proof which involves a simple counting argument applied to $N(D)$. First we prove a lemma.

Lemma 5.1. *Let D be a BIBD$(v,k,2\lambda)$, and let $E_s=\{e_j \mid j=1,2,\ldots,d\}$ be the set of edges which are incident with the vertex s in $N(D)$. Let $W=(w_{ij})$ be the $k\times d$ $(0,1)$-matrix where*

$$w_{ij}=\begin{cases}1 & \text{if } e_j \text{ is contained in } F_{is}\\ 0 & \text{otherwise}\end{cases}$$

Then W is the incidence matrix of a BIBD$(k,2,(2\lambda-1))$.

Proof: By Definition 4.3(ii), there are exactly k impressions centred on the vertex s. Moreover, by Definition 4.3(i) and (ii), any edge in E_s is contained in precisely four impressions, exactly two of which are centred on s. Finally, any pair of treatments u_i and u_j in B_s are contained in $2\lambda-1$ other blocks; so F_{is} and F_{js} have precisely $(2\lambda-1)$ edges in common. ∎

Theorem 5.2. *Let $b_1,b_2,\ldots,b_n$ be the the respective numbers of vertices in the connected components of $N(D)$; then for each $m=1,2,\ldots,n$, there exist nonnegative integers $x_{1m},x_{2m},\ldots,x_{k-1,m}$ such that*

$$\sum_{i=1}^{k-1} i(k-1)x_{im}=b_m k(k-1)/4 \qquad (5.1)$$

and

$$\sum_{i=1}^{k-1} x_{im} \leq b_m k. \tag{5.2}$$

Proof: Let $C_m(D)$ denote the mth connected component of $N(D)$, and $C_m(Y)$ denote the corresponding component of $N(Y)$. ($C_m(D)$ will correspond to a subset of blocks in D.) Also let x_{im} be the number of vertices in $C_m(D)$ which correspond to a column of Y which contains exactly i negative entries. (So inequality (5.2) follows immediately.

By Lemma 4.4, $C_m(Y)$ may be obtained from $C_m(D)$ by applying, in any order, the operations κ_{is} where $y_{is} = -1$. For each $s = 1, 2, \ldots, b$, let

$$\kappa_s = \prod_{i, y_{is}=-1} \kappa_{is},$$

where $\prod$ denotes map composition. Then

$$C_m(Y) = \left\{ \prod_{s \in C_m(D)} \kappa_s \right\} (C_m(D)) .$$

We now count, in two ways, the number of colour-reversals of edges which occur when the operations κ_s are applied in sequence. The (mod 2) sum of any i rows of W in Lemma 5.1 gives a vector of weight $(2\lambda - 1)i(k-i)$; so, for some i, the net effect of κ_s is to reverse the the colour of $(2\lambda - 1)i(k-i)$ edges, and the number of colour-reversals is given by $(2\lambda - 1)$ times the left-hand side of (5.1).

Note that an edge has its colour reversed by κ_s, if and only if it has supports of the form $u_i \in B_s$ and $u_j \in B_s$ where $y_{is}y_{js} = -1$. So an edge with supports $u_i \in B_s, u_j \in B_s, u_i \in B_t$, and $u_j \in B_t$ will have its colour reversed x times, where x is the number $-1\,s$ in the set $\{y_{is}y_{js}, y_{it}y_{jt}\}$. But, for any i and j, where $1 \leq i < j \leq v$, there are λ values of s for which $y_{is}y_{js} = -1$ and λ values of s for which $y_{is}y_{js} = 1$. So each pair of rows will correspond to λ^2 edges whose colour is changed once and to $\lambda(\lambda-1)/2$ edges whose colour is changed twice. Hence the number of colour-reversals is given by $(\lambda^2 + 2 \times \lambda(\lambda-1)/2) \times b_m k(k-1)/4\lambda = (2\lambda - 1)b_m k(k-1)/4$. ∎

It is quite easy to derive [6,theorem 1] from (5.1) and (5.2). The improvement over the earlier results stems from the application of the theorem to the sets of blocks which correspond to the connected components of $N(D)$ instead of the entire design.

Because each component of $N(D)$ can be examined separately, the amount of work done using exhaustive methods to decide whether a design is signable will in general decrease as the number of components of $N(D)$ increases.

6. Concluding Remarks

In this paper an attempt has been made to investigate the complexity of deciding whether, (i) given v, k and λ, whether a BRD($v, k, 2\lambda$) exists, and (ii) given design, whether it can be signed over Z_2.

Some reformulations of the original problems or easier related problems are described: partly in an attempt to gain information about the complexity of the original problems, and partly because of their intrinsic interest. In particular, we discussed linear constraints which may, in some cases, lead to a complete resolution of the second problem and its associated enumeration problem. Finally, certain graph-theoretical ideas gave insight into a known non-existence result, and allowed us to show that, when $\lambda = 1$ and $k = 3$, the complexity of the second problem is linear in b.

Aknowledgement

The author would like to thank Graham Farr for a number of stimulating and enjoyable discussions relating to this paper.

References

1. M. Bhaskar Rao, *Balanced orthogonal designs and their applications in the construction of some BIB and group divisible designs*, Sankhya Ser. A **32** (1970), 439–448.
2. W. de Launey and D. G. Sarvate, *Non-existence of certain GBRD's*, Ars Combinatoria **18** (1983), 5–20.
3. P. B. Gibbons, R. A. Mathon, and D. G. Corneil, *Computing techniques for the construction and analysis of block designs*, Uilitas Mathematica **11** (1977), 161–192.
4. P. B. Gibbons and R. A. Mathon, *Construction methods for Bhaskar Rao and related designs*, J. of the Australian Math. Soc. (Series A) **42** (1987), 5–30.
5. P. B. Gibbons and R. A. Mathon, *Group signings of symmetric balanced incomplete block designs*, Ars Combinatoria **23A** (1987), 123–134.
6. J. Seberry, *Regular group divisible designs and Bhaskar Rao designs with block size three*, J. Stat. Inf. and Plan. **10** (1984), 69–82.
7. D. J. Street and C. A. Rodger, *Some results on Bhaskar Rao Designs*, Combinatorial Mathematics VII, Lecture Notes in Mathematics No. **829** (1980), Springer, Berlin - Heidelberg- New York, 238–245.