Decompositions of Various Complete Graphs into Isomorphic Copies of 4-cycles with Three Pendant Edges

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Abstract

An H-decomposition of a graph G is a partition of the edges of G into copies isomorphic to H. When the decomposition is not feasible, one looks for the best possible by minimizing; the number of unused edges (leave of a packing), or the number of reused edges (padding of a covering). We consider the H-decomposition, packing, and covering of the complete graphs and complete bipartite graphs, where H is a 4-cycle with three pendant edges.

1 Introduction

An *H-decomposition* or *H-design* of the complete graph K_m is a set of subgraphs of K_m, $\{H_1, H_2, \ldots, H_n\}$, where all the H_i's are isomorphic to H for $1 \leq i \leq n$, $E(H_i) \cap E(H_j) = \emptyset$ for $i \neq j$, and $\cup_{i=1}^{n} E(H_i) = E(K_m)$. The subgraphs H_i are called the blocks of the design or the decomposition. There are several studies on H-decomposition of the complete graph into a given graph H. See [8] and [10] for more about the history of the graph decomposition problem.

A variation of the decomposition problems is to consider a pair of non-isomorphic graphs, say G and H, as the blocks of the decomposition. There are several articles discussing these ideas. See [1, 2, 3, 4, 5, 11] for more.

In [9], the authors studied the decompositions of various complete graphs into isomorphic copies of the 4-cycle with a single pendant edge. In this paper, we study the G-decomposition of the complete graph K_m, the complete bipartite graph $K_{m,n}$, when H is the 4-cycle with three pendant

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JCMCC 111 (2019), pp.53-64

edges. We denote this graph as $H = \{a, b, c, d; (e, f, g)\}$. That is, $V(H) = \{a, b, c, d, e, f, g\}$ and $E(H) = \{(a, b), (b, c), (c, d), (d, a), (a, e), (b, f), (c, g)\}$. We also study the packing and the covering of K_m and $K_{m,n}$ with H.

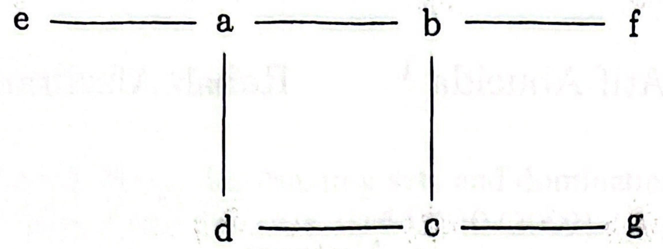

Figure 1: The graph H

2 H-Decomposition of $K_{m,n}$

We assume the partite sets of the complete bipartite graph $K_{m,n}$ are $V_m = \{0_1, 1_1, ..., (m-1)_1\}$ and $V_n = \{0_2, 1_2, ..., (n-1)_2\}$.

Example 2.1. The set $\{[0_1, 1_2, 1_1, 3_2; (0_2, 2_1, 4_2)], [0_1, 2_2, 2_1, 4_2; (6_2, 1_1, 0_2)], [1_1, 5_2, 2_1, 6_2; (0_2, 0_1, 3_2)]\}$ is an H-decomposition of $K_{3,7}$.

Example 2.2. The set $\{[0_1, 1_2, 1_1, 0_2; (2_2, 2_1, 5_2)], [1_1, 4_2, 2_1, 2_2; (6_2, 3_1, 5_2)], [0_1, 3_2, 3_1, 5_2; (4_2, 1_1, 1_2)], [2_1, 6_2, 3_1, 0_2; (3_2, 0_1, 2_2)]\}$ is an H-decomposition of $K_{4,7}$.

Example 2.3. The set $\{[0_1, 5_2, 2_1, 0_2; (1_2, 3_1, 3_2)], [0_1, 6_2, 3_1, 3_2; (4_2, 1_1, 0_2)], [1_1, 4_2, 4_1, 0_2; (5_2, 2_1, 3_2)], [2_1, 2_2, 4_1, 6_2; (1_2, 0_1, 5_2)], [1_1, 1_2, 3_1, 2_2; (3_2, 4_1, 4_2)]\}$ is an H-decomposition of $K_{5,7}$.

The previous examples give us the following theorem:

Theorem 2.1. *There is an H-decomposition of $K_{m,n}$ if and only if $mn \equiv 0 (mod\ 7)$ $m \geq 3$ and $n \geq 7$.*

Proof. As H is a bipartite graph on 7 vertices, one part of $K_{m,n}$ should have at least 3 vertices while the other should have at least 4 vertices. Since $mn \equiv 0 \pmod 7$, either m or n is a multiple 7. Without loss of generality, let $n \equiv 0 \pmod 7$.

Case 1. Suppose $m \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 7$, there is an H-decomposition of $K_{3t,7s}$ given by:
$\{[(3i)_1, (1+7j)_2, (1+3i)_1, (3+7j)_2; ((7j)_2; (2+3i)_1, (4+7j)_2)], [(3i)_1, (2+7j)_2, (2+3i)_1, (4+7j)_2; ((6+7j)_2, (1+3i)_1, (7j)_2)], [(1+3i)_1, (5+7j)_2, (2+$

$3i)_1, (6+7j)_2; ((7j)_2, (3i)_1, (3+7j)_2)]\}$; where $i = 0, 1, ..., (t-1); j = 0, 1, ..., (s-1)$.

Case 2. Suppose $m \equiv 0 \pmod 4$ and $n \equiv 0 \pmod 7$, then there is an H-decomposition of $K_{4t,7s}$ given by: $\{[(4i)_1, (1+7j)_2, (1+4i)_1, (7j)_2; (2+7j)_2, (2+4i)_1, (5+7j)_2], [(1+4i)_1, (4+7j)_2, (2+4i)_1, (2+7j)_2; ((6+7j)_2, (3+4i)_1, (5+7j)_2), [(4i)_1, (3+7j)_2, (3+4i)_1, (7j)_2; ((4+7j)_2, (1+4i)_1, (1+7j)_2)], [(2+4i)_1, (6+7j)_2, (3+4i)_1, (7j)_2; ((3+7j)_2, (4i)_1, (2+7j)_2)]\}$; where $i = 0, 1, ...(t-1); j = 0, 1, ...(s-1)$.

Case 3. Suppose $m = 0 \pmod 5$ and $n \equiv 0 \pmod 7$, then there is an H-decomposition of $K_{5t,7s}$ given by: $\{[(5i)_1, (5+7j)_2, (2+5i)_1, (7j)_2; ((1+7j)_2, (3+5i)_1, (3+7j)_2], [(5j)_1, (6+7j)_2, (3+5i)_1, (3+7j)_2; ((4+7j)_2, (1+5i)_1, (7j)_2)], [(1+5i)_1, (4+7j)_2, (4+5i)_1, (7j)_2; ((5+7j)_2, (2+5i)_1, (3+7j)_2)], [(2+5i)_1, (2+7j)_2, (4+5i)_1, (6+7j)_2; ((1+7j)_2, (5i)_1, (5+7j)_2)], [(1+5i)_1, (1+7j_2), (3+5i), (2+7j)_2; ((3+7j)_2, (4+5i)_1, (4+7j)_2)]\}$; where $i = 0, 1, ...(t-1); j = 0, 1, ...(s-1)$.

Note that for any $m \geq 6$, $m = 3t + 4r$, $n \equiv 0 \pmod 7$, it follows that $K_{3t+4r,7s} = (ts)K_{3,7} \cup (rs)K_{4,7}$. Thus, there is an H-decomposition of $K_{3t+4r,7s}$.

□

3 H-Decomposition of K_m

We start with the following examples:

Example 3.1. Let $V(K_{14}) = \{\infty, 0, 1, \cdots, 12\}$. An H-decomposition of K_{14} is given by the set $\{[i, i+1, i+3, i+6; (\infty, i+5, i+8)] : 0 \leq i \leq 12\}$.

Example 3.2. Let $V(K_{15}) = \{0, 1, \cdots, 14\}$. An H-decomposition of K_{15} is given by the set $\{[i, i+1, i+3, i+10; (i+6, i+4, i+7)] : 0 \leq i \leq 14\}$.

We denote with $K(m, n)$, the complete graph K_m with the set of edges between a specific n vertices is removed.

Example 3.3. The set {[9,5,13,2; (3,11,1)], [7,8,6,12; (11,1,9)], [1,9,7,14; (11,4,13)], [3,14,13,12; (11,2,4)], [11,10,4,12; (13,3,8)], [10,9,8,5; (7,13,11)], [14,9,12,5; (4,11,1)], [12,14,8,10; (2,6,3)], [13,8,2,10; (3,12,11)], [10,6,11,14; (1,13,4)]} is an H-decomposition of $K(14, 7)$.

Example 3.4. An H-decomposition of $K(15, 7)$ is given by
$\{[15, 13, 11, 4; (9, 10, 5)], [10, 14, 13, 12; (5, 9, 1)], [2, 10, 9, 8; (13, 7, 4)],$
$[11, 15, 8, 10; (3, 2, 13)], [12, 3, 8, 11; (6, 14, 1)], [14, 15, 6, 8; (7, 5, 9)],$

$[15, 1, 12, 7; (10, 14, 5)]$, $[5, 14, 12, 8; (9, 4, 2)]$, $[7, 13, 4, 8; (9, 5, 10)]$, $[14, 11, 9, 2; (6, 7, 13)]$, $[12, 9, 3, 15; (4, 1, 13)]$, $[11, 6, 10, 1; (2, 13, 3)]\}$.

We are now ready to introduce the main result of this section

Theorem 3.1. *An H-decomposition of K_m exists if and only if $m \equiv 0, 1 \pmod 7$, $m \geq 14$.*

Proof. As the graph H consists of 7 edges, $|K_m|$ must be divisible by 7.
Case 1. Suppose $m \equiv 0 \pmod 7$ and let $m = 14t + r$ such that $r \in \{0, 7\}$ and $t \geq 1$. Let $V(K_m) = \{0_1, \cdots, 13_1, 0_2, \cdots, 13_2, \cdots, 0_t, \cdots, 13_t\} \cup \{0_0, \cdots, 6_0\}$. On each set in $\{\{0_i, \cdots, 13_i\}, 1 \leq i \leq t\}$, place an H-decomposition of K_{14} as in example 3.1. Use Theorem 2.1, to find an H-decomposition, on each of the $\binom{t}{2}$ complete bipartite graphs $K_{14_i,14_j}$, $1 \leq i < j \leq t$. If $r = 0$, then we are done. Otherwise $r = 7$. Place an H-decomposition of $K(14, 7)$ on the set $\{0_0, \cdots, 6_0, 0_1, \cdots, 6_1\}$ with the hole to be the vertices in $\{0_1, \cdots, 6_1\}$. Use Theorem 2.1 to find an H-decomposition on the complete bipartite graphs $K_{7,7}$ (and the $\binom{t-1}{2}$ copies of $K_{7,14}$) on the vertices $\{0_0, \cdots, 6_0, 7_1, \cdots, 13_1\}$ (on the vertices $\{0_0, \cdots, 6_0, 0_2, \cdots, 13_t\}$).

Case 2. Suppose $m \equiv 1 \pmod 7$ and let $m = 14t + r$ such that $r \in \{1, 8\}$ and $t \geq 1$.
Let $V(K_m) = \{0_1, \cdots, 13_1, 0_2, \cdots, 13_2, \cdots, 0_t, \cdots, 13_t\} \cup \{0_0, \cdots, 7_0\}$. On each set in $\{\{0_i, \cdots, 13_i\}, 2 \leq i \leq t\}$, place an H-decomposition of K_{14} as in example 3.1. Place an H-decomposition of K_{15} on the vertices $\{0_0, 0_1, \cdots, 13_1\}$ as in example 3.2. If $r = 8$, then place an H-decomposition of $K(14, 7)$ on the vertices $\{1_0, \cdots, 7_0, 0_1, \cdots, 6_1\}$ such that the hole consists of the vertices in $\{1_0, \cdots, 7_0\}$. Place an H-decomposition of $K_{7,7}$ on the complete bipartite graph on the vertices $\{1_0, \cdots, 7_0, 7_1, \cdots 13_1\}$. Now, place an H-decomposition of $K_{14,14}$ (or $K_{14+r,14}$) on the vertices complete bipartite graphs $K_{14_i,14_j}$, $2 \leq i < j \leq t$ (on the vertices complete bipartite graphs $K_{14+r,14_j}$, $2 \leq j \leq t$). □

4 Packing and Covering K_m

Since H has 7 vertices, we consider the complete graphs K_m such that $m \geq 7$.

Theorem 4.1. *A maximal H- Packing of $K_m, m \geq 7$ has leave L, where $|E(K_m)| \equiv |E(L)| \pmod 7$, except when $m \in \{7, 8\}$, in which case $|E(L)| = 7$.*

Proof. Since $|E(H)| = 7$, then it is necessary that in any packing of K_m with leave L, to consider $|E(K_m)| \pmod 7$. Therefore, such a packing with $|E(K_m)| \equiv |E(L)| \pmod 7$ would be maximal. If $m \in \{7,8\}$, then $|E(K_m)| \equiv 0 \pmod 7$, but there is not an H-decomposition of K_m. So for $m \in \{7,8\}$ there is an H-packing of K_m with leave L, where $|E(L)| = 7$ would be maximal.

Case 1. Suppose $m = 7$, the set $\{[2,5,3,6;(4,1,0)],[0,6,1,4;(2,5,3)]\}$ is a maximal packing of K_7 with leave L, where $E(L) = \{\{0,1\},\{1,2\},\{2,3\},$ $\{3,4\},\{5,4\},\{0,5\},\{4,6\}\}$. So, $|E(L)| = 7$.

Case 2. Suppose $m = 8$, the set $\{[7,1,5,2;(0,4,3)],[4,2,6,5;(3,0,1)],$ $[7,6,0,4;(5,3,1)]\}$ is a maximal packing of K_8 with leave L, where $E(L) =$ $\{\{1,2\},\{2,3\},\{1,3\},\{3,7\},\{3,0\},\{4,6\},\{0,5\}\}$. So, $|E(L)| = 7$.

Case 3. Suppose $m \equiv 2$ or $6 \pmod 7$, since $|E(K_m)| \equiv 1 \pmod 7$, $|E(L)| =$ 1 would be optimal. First, if $m = 2 \pmod 7$, we consider packing of K_{7t+2}. For $t = 1$, $K_9 = K(9,2) \cup K_2$. For $t \geq 2$, $K_{7t+2} = tK(9,2) \cup \binom{t}{2} K_{7,7} \cup K_2$. From Theorem 2.1, $K_{7,7}$ has an H-decomposition. The graph $K(9,2)$ has an H-decomposition given by the following set $\{[5_1,1_2,3_1,4_1;(2_1,6_1,1_1)],$ $[1_1,0_1,2_1,0_2;(1_2,6_1,4_1)]$, $[6_1,4_1,0_2,3_1;(1_1,1_2,0_1)]$, $[3_1,2_1,6_1,5_1;(0_1,1_2,0_2)]$, $[1_1,5_1,0_1,4_1;(2_1,0_2,1_2)]\}$. So, $|E(L)| = 1$ given by K_2. Second, if $m = 6 \pmod 7$, we consider packing of K_{7t+6}. For $t = 1$, $K_{13} = K(13,2) \cup K_2$, where $K(13,2)$ has an H-decomposition given by the following set: $\{[8_1,0_1,7_1,1_1;(3_1,6_1,9_1)]$, $[9_1,2_1,6_1,1_2;(1_1,10_1,8_1)]$, $[9_1,10_1,0_2,8_1;(4_1,5_1,2_1)]$, $[7_1,3_1,5_1,6_1;(2_1,9_1,8_1)]$, $[1_1,2_1,4_1,5_1;(0_2,8_1,6_1)]$, $[4_1,3_1,2_1,0_1;(0_2,6_1,1_2)]$, $[5_1,9_1,0_1,1_2;(2_1,6_1,3_1)]$, $[1_1,6_1,0_2,0_1;(1_2,10_1,9_1)]$, $[1_2,8_1,7_1,4_1;(3_1,10_1,0_2)]$, $[4_1,10_1,3_1,1_1;(8_1,1_2,0_3)]$, $[5_1,7_1,10_1,0_1;(0_2,1_2,1_1)]\}$. For $t \geq 2$, we consider $k_{7t+6} = K(13,2) \cup (t-1)K(9,2) \cup \binom{t-2}{2} K_{7,7} \cup K_2 \cup (t-1)K_{7,11}$, where all of $K(13,2)$, $K(9,2)$, $K_{7,7}$ and $K_{7,11}$ has an H-decomposition and $|E(L)| = 1$ given by K_2.

Case 4. Suppose $m \equiv 3$ or $5 \pmod 7$. Since $|E(K_m)| = 3 \pmod 7$, $|E(L)| = 3$ would be optimal. First, if $m = 3 \pmod 7$, we consider packing of K_{7t+3}. For $t = 1$, K_{10} has an H-packing given by the following set: $\{[7_1,2_1,3_1,5_1;(4_1,8_1,9_1)]$, $[5_1,6_1,7_1,0_1;(1_1,2_1,8_1)]$, $[9_1,1_1,4_1,5_1;(2_1,7_1,6_1)]$, $[3_1,8_1,0_1,4_1;(6_1,5_1,1_1)]$, $[0_1,2_1,1_1,3_1;(6_1,5_1,8_1)]$, $[4_1,9_1,6_1,8_1;(2_1,0_1,1_1)]\}$ with a leave L given by the P_4 edges $\{\{3_1,7_1\},\{7_1,9_1\},$ $\{8_1,9_1\}\}$. For $t = 2$, $K_{17} = K(10,3) \cup K_{7,7} \cup K_{10}$. It is sufficient to construct an H-decomposition of $K(10,3)$ as follows $\{[0_2,2_1,3_1,5_1;(4_1,1_2,2_2)]$, $[5_1,6_1,0_2,0_1;(1_1,2_1,3_1)]$, $[2_2,1_1,4_1,5_1;(2_1,0_2,6_1)]$, $[3_1,1_2,0_1,4_1;(6_1,5_1,1_1)]$, $[0_1,2_1,1_1,3_1;(6_1,5_1,1_2)]$, $[4_1,2_2,6_1,1_2;(2_1,0_1,1_1)]\}$. For $t \geq 3$, $K_{7t+3} = K_{7(t-1)} \cup (t-1)K_{7,10} \cup k_{10}$.

It is obvious that K_{7t+3} has an H-packing with a leave isomorphic to P_4.

Second, if $m \equiv 5 \pmod 7$, we consider packing of K_{7t+5}. For $t = 1$, K_{12} has an H-packing given by the following set $\{[6_1, 1_1, 5_1, 4_1; (8_1, 11_1, 9_1)]$, $[6_1, 7_1, 10_1, 3_1; (2_1, 9_1, 4_1)]$, $[2_1, 8_1, 9_1, 3_1; (10_1, 5_1, 6_1)]$, $[1_1, 2_1, 4_1, 3_1; (7_1, 9_1, 8_1)]$, $[5_1, 0_1, 1_1, 6_1; (2_1, 4_1, 8_1)]$, $[0_1, 7_1, 11_1, 6_1; (10_1, 5_1, 2_1)]$, $[0_1, 9_1, 1_1, 11_1; (2_1, 4_1, 5_1)]$, $[8_1, 10_1, 3_1, 0_1; (11_1, 1_1, 5_1)]$, $[7_1, 4_1, 11_1, 3_1; (2_1, 1_1, 5_1)]\}$. So, $|E(L)| = 3$ is given by the P_4 edges in $\{\{7_1, 10_1\}, \{10_1, 9_1\}, \{9_1, 11_1\}\}$. For $t = 2$, $K_{19} = K(14, 7) \cup K_{7,5} \cup K_{12}$. Since there is an H-decomposition of each of $K(14, 7)$ and $K_{7,5}$, and an H-packing of K_{12}, we get an H-packing of K_{19} with a leave consisting of P_4. For $t \geq 3$, $K_{7t+5} = K_{7(t-1)} \cup K_{7(t-1),12} \cup K_{12}$. It is obvious that K_{7t+5} has an H-packing with a leave isomorphic to P_4.

Case 5. Suppose $m \equiv 4 \pmod 7$, since $|E(K_v)| = 6 \pmod 7$, $|E(L)| = 6$ would be optimal. The following is an H-packing of K_{11}: $\{[3, 0, 1, 2; (7, 4, 6)]$, $[3, 4, 5, 6; (10, 9, 0)]$, $[6, 7, 8, 9; (2, 10, 1)]$, $[9, 10, 0, 2; (1, 4, 8)]$, $[3, 5, 7, 1; (9, 10, 4)]$, $[8, 10, 1, 4; (3, 2, 5)]$, $[6, 8, 2, 4; (10, 0, 7)]$ with a leave consisting of the following 6 edges in $\{\{10, 5\}, \{2, 5\}, \{5, 9\}, \{9, 7\}, \{9, 0\}, \{0, 6\}\}$.

Since $K_{18} = K(14, 7) \cup K_{11} \cup K_{7,4}$, there is an H-packing of K_{18} with a leave consisting of 6 edges. For $t \geq 3$, $K_{7t+4} = K_{7(t-1)} \cup K_{11} \cup K_{7(t-1),11}$. It is obvious that K_{7t+4} has an H-packing with a leave consisting of 6 edges. □

Theorem 4.2. *A minimal H- covering of K_m, $m \geq 7$ has padding P, where $|E(K_m)| \equiv -|E(P)| \pmod 7$, except when $m \in \{7, 8\}$ in which case $|E(L)| = 7$.*

Proof. Since $|E(H)| = 7$, then it is necessary that in any H-covering of K_m with padding P, we have $|E(K_m)| + |E(P)| \equiv 0 \pmod 7$ or that $|E(K_m)| \equiv -|E(P)| \pmod 7$. So, if $|E(K_m)| \equiv -|E(P)| \pmod 7$, then the covering is minimal. If $m \in \{7, 8\}$, then $|E(K_m)| = 0 \pmod 7$, but there is no H-decomposition of K_m. So, for $m \in \{7, 8\}$, an H-covering of K_m with padding P, where $|E(P)| = 7$ would be minimal.

Case 1. Suppose $m = 7$ the set $\{[6, 3, 0, 4; (5, 2, 1)]$, $[1, 3, 4, 5; (2, 0, 6)]$, $[2, 6, 1, 4; (5, 0, 3)]$, $[5, 0, 2, 3; (1, 4, 6)]\}$ is a minimal covering of K_7 with padding P, where $E(P) = \{\{3, 2\}, \{3, 0\}, \{4, 6\}, \{1, 3\}, \{0, 4\}, \{2, 6\}, \{5, 1\}\}$. So, $|E(P)| = 7$.

Case 2. Suppose $m = 8$. The set $\{[1, 5, 3, 7; (0, 4, 6)]$, $[1, 2, 4, 6; (3, 0, 7)]$, $[7, 2, 6, 5; (0, 1, 4)]$, $[5, 2, 3, 4; (0, 6, 7)]$, $[7, 4, 0, 6; (2, 1, 3)]\}$ is a minimal covering of K_8 with padding P, where $E(P)$=$\{\{4, 5\}$, $\{4, 7\}$, $\{2, 1\}$, $\{6, 4\}$,

$\{2,6\}$, $\{3,7\}$, $\{7,2\}\}$. So, $|E(P)| = 7$.

Case 3. Suppose $m \equiv 2$ or $6 \pmod 7$, $m \geq 9$. Adding a copy of H that contains the leave edge in the packing of K_m will result in a covering of K_m with a padding of 6 edges, i.e. $|E(P)| = 6 = -|E(K_m)| \pmod 7$.

Case 4. Suppose $m \equiv 3$ or $5 \pmod 7$. Adding a copy of H that contains the leave edges in the packing of K_m will result in a covering of K_m with a padding of 4 edges, i.e. $|E(P)| = 4 = -|E(K_m)| \pmod 7$.

Case 5. Suppose $m \equiv 4 \pmod 7$, $m \geq 11$. Adding a copy of H that contains the leave edges in the packing of K_m will result in a covering of K_m with a padding consisting of a single edge, i.e. $|E(P)| = 1 = -|E(K_m)|$ (mod 5). □

5 Packing and Covering the Complete Bipartite Graph $K_{m,n}$

In this section, we consider the H-packing and H-covering of the complete bipartite graph $K_{m,n}$. We assume the partite sets of $K_{m,n}$ are $\{1_1, 2_1, ..., (m)_1\}$ and $\{1_2, 2_2, ..., (n)_2\}$.

Theorem 5.1. *A maximal H-packing of $K_{M,N}$ has leave L, where $|E(L)| = MN$ if $M, N \in \{1,2,3\}$, and $|E(K_{M,N})| \equiv |E(L)|$ (mod 7), otherwise, except possibly $K_{3,9}$.*

Proof. If $M, N \in \{1,2,3\}$, it is obvious that H is not a subgraph of $K_{m,n}$, and the leave will have mn edges. Let $M = 7t_1 + m$ and $N = 7t_2 + n$. For $m \geq 3$, and $n \geq 4$ An H-decomposition of $K_{M,N}$ with leave L, $|E(L)| = |E(K_{m,n})| \pmod 7$ would be minimal. For $M \geq 3$, $N \geq 4$, since $K_{m+7i,n+7j} = K_{m,n} \cup K_{m,7j} \cup K_{7i,7j} \cup K_{7i,n}$, there is an H-packing of $K_{m+7i,n+7j}$ with leave L for all $i, j \in \mathbb{N}$ if there is an H-packing of $K_{m,n}$ with leave L. The maximal H-packing of $K_{m,n}$ with leave L is described in the following cases:

Case 1. Suppose $m \equiv 1 \pmod 7$ and $n \equiv 1 \pmod 7$ then $K_{m,n} = K_{8,8} \cup (t_1 - 1)(t_2 - 1)K_{7,7} \cup (t_1 - 1)K_{7,8} \cup (t_2 - 1)K_{8,7}$. $K_{8,8}$ has a maximal packing given by the following set $\{[2_2, 1_1, 1_2, 2_1; (7_1, 5_2, 6_1)]$, $[3_1, 2_2, 6_1, 6_2; (5_2, 5_1, 3_2)]$, $[3_1, 3_2, 4_1, 4_2; (7_2, 5_1, 8_2)]$, $[4_2, 5_1, 7_2, 2_1; (6_1, 8_2, 4_1)]$, $[4_2, 1_1, 3_2, 8_1; (7_1, 6_2, 2_1)]$, $[1_2, 7_1, 8_2, 3_1; (4_1, 7_2, 1_1)]$, $[8_1, 6_2, 7_1, 5_2; (1_2, 2_1, 3_2)]$, $[8_1, 8_2, 6_1, 7_2; (2_2, 2_1, 5_2)]$, $[4_1, 5_2, 5_1, 6_2; (2_2, 2_1, 1_2)]\}$ with leave $L = \{(1_1, 7_2)\}$; $|E(L)| = 1$. Therefore, there is a maximal packing of $K_{m,n}$ with leave L where $|E(L)| = 1 =$

$|E(K_{m,n})|$ (mod 7).

Case 2. Suppose $m \equiv 1 \pmod 7$ and $n \equiv 2 \pmod 7$ then $K_{m,n} = K_{8,9} \cup (t_1-1)(t_2-1)K_{7,7} \cup (t_2-1)K_{8,7} \cup (t_1-1)K_{9,7}$. $K_{8,9}$ has a maximal packing given by the following $\{[1_1, 1_2, 2_1, 2_2; (9_2, 5_1, 6_2)], [4_1, 5_2, 3_1, 3_2; (9_2, 8_1, 7_2)],$
$[5_1, 4_2, 2_1, 3_2; (2_2, 8_1, 5_2)], [6_1, 7_2, 5_1, 6_2; (1_2, 2_1, 9_2),$
$[5_2, 7_1, 8_2, 5_1; (6_1, 2_2, 4_1)], [9_2, 7_1, 7_2, 8_1; (2_1, 6_2, 4_1)],$
$[6_1, 4_2, 1_1, 3_2; (2_2, 4_1, 7_2)], [3_1, 2_2, 4_1, 6_2; (9_2, 8_1, 1_2)],$
$[3_1, 1_2, 7_1, 4_2; (8_2, 8_1, 3_2)], [8_1, 8_2, 1_1, 6_2; (3_2, 6_1, 5_2)]\}$
with leave $L = \{\{2_1, 8_2\}, \{6_1, 9_2\}\}$; $|E(L)| = 2$. Therefore, there is a maximal packing of $K_{m,n}$ with leave L, where $|E(L)| = 2 = |E(K_{m,n})| \pmod 7$.

Case 3. Suppose $m \equiv 1 \pmod 7$ and $n \equiv 3 \pmod 7$ then $K_{m,n} = K_{8,3} \cup (t_1-1)(t_2-1)K_{7,7} \cup (t_2-1)K_{8,7} \cup (t_1-1)K_{10,7}$. $K_{8,3}$ has a maximal packing given by $\{[2_2, 1_1, 1_2, 2_1; (5_1, 3_2, 3_1)], [2_2, 4_1, 3_2, 3_1; (7_1, 1_2, 5_1)],$
$[3_2, 6_1, 1_2, 8_1; (2_1, 2_2, 7_1)]\}$ with leave $L = \{\{5_1, 1_2\}, \{7_1, 3_2\}, \{8_1, 2_2\}\}$,
$|E(L)| = 3$. Therefore, there is a maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 3 = |E(K_{m,n})| \pmod 7$.

Case 4. Suppose $m \equiv 1 \pmod 7$ and $n \equiv 4 \pmod 7$ then $K_{m,n} = K_{8,4} \cup (t_2)K_{8,7} \cup (t_1-1)(t_2-1)K_{7,7} \cup (t_1-1)K_{11,7}$. $K_{8,4}$ has a maximal packing given by $\{[1_2, 1_1, 2_2, 2_1; (8_1, 3_2, 3_1)], [4_2, 3_1, 3_2, 4_1; (1_1, 1_2, 2_1)],$
$[6_1, 3_2, 8_1, 4_2; (1_2, 5_1, 2_2)], [5_1, 1_2, 7_1, 2_2; (4_2, 4_1, 3_2)]\}$ with leave
$L = \{\{4_1, 2_2\}, \{6_1, 2_2\}, \{7_1, 4_2\}, \{2_1, 4_2\}\}$. Therefore, there is a maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 4 = |E(K_{m,n})| \pmod 7$.

Case 5. Suppose $m \equiv 1 \pmod 7$ and $n \equiv 5 \pmod 7$ then $K_{m,n} = K_{8,5} \cup (t_2)K_{8,7} \cup (t_1-1)(t_2-1)K_{7,7} \cup (t_1-1)K_{7,12}$. $K_{8,5}$ has a maximal packing given by the following $\{[1_1, 1_2, 2_1, 2_2; (3_2, 8_1, 5_2)], [4_2, 3_1, 3_2, 4_1; (8_1, 5_2, 2_1)],$
$[5_1, 4_2, 6_1, 5_2; (1_2, 7_1, 3_2)], [5_2, 7_1, 3_2, 8_1; (1_1, 2_2, 5_1)],$
$[1_2, 4_1, 2_2, 3_1; (7_1, 5_2, 8_1)]\}$ with leave $L = \{\{1_2, 6_1\}, \{2_2, 5_1\}, \{2_2, 6_1\},$
$\{4_2, 1_1\}, \{4_2, 2_1\}\}$, $|E(L)| = 5$. Therefore, there is a maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 5 = |E(K_{m,n})| \pmod 7$.

Case 6. Suppose $m \equiv 1 \pmod 7$ and $n \equiv 6 \pmod 7$ then $K_{m,n} = K_{8,6} \cup (t_2)K_{8,7} \cup (t_1-1)(t_2-1)K_{7,7} \cup (t_1-1)K_{7,13}$. $K_{8,6}$ has a maximal packing given by the following $\{[1_2, 4_1, 2_2, 3_1; (6_1, 4_2, 5_1)], [1_1, 1_2, 2_1, 2_2; (3_2, 7_1, 5_2)],$
$[5_2, 4_1, 6_2, 3_1; (7_1, 3_2, 1_1)], [4_2, 2_1, 3_2, 3_1; (1_1, 6_2, 6_1)],$
$[6_1, 5_2, 5_1, 6_2; (4_2, 1_1, 3_2)], [8_1, 2_2, 7_1, 6_2; (5_2, 6_1, 3_2)]\}$ with leave
$L = \{\{5_1, 1_2\}, \{5_1, 4_2\}, \{7_1, 4_2\}, \{8_1, 3_2\}; \{8_1, 4_2\}; \{8_1, 1_2\}\}$, $|E(L)| = 6$.
Thus, there is a maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 6 = |E(K_{m,n})| \pmod 7$.

Case 7. Suppose $m \equiv 2 \pmod 7$ and $n \equiv 2 \pmod 7$ then $K_{m,n} = K_{9,9} \cup (t_2-1)K_{9,7} \cup (t_1-1)(t_2-1)K_{7,7} \cup (t_1-1)K_{9,7}$. $K_{9,9}$ has a maximal packing given by the following $\{[1_1, 1_2, 2_1, 2_2; (5_2, 9_1, 6_2)], [3_1, 3_2, 4_1, 4_2; (6_2, 5_1, 2_2)],$
$[5_1, 5_2, 6_1, 6_2; (2_2, 2_1, 3_2)], [7_1, 7_2, 8_1, 8_2; (1_2, 1_1, 5_2)],$
$[8_1, 6_2, 9_1, 9_2; (2_2, 4_1, 5_2)], [1_1, 3_2, 2_1, 4_2; (8_2, 9_1, 9_2)],$
$[3_1, 2_2, 7_1, 5_2; (7_2, 9_1, 3_2)], [5_1, 7_2, 4_1, 1_2; (8_2, 2_1, 5_2)],$
$[6_1, 8_2, 9_1, 7_2; (2_2, 4_1, 4_2)], [4_2, 7_1, 9_2, 5_1; (6_1, 6_2, 4_1)],$
$[1_2, 3_1, 9_2, 6_1; (8_1, 8_2, 1_1)]\}$ with leave $L = \{\{1_1, 6_2\}, \{2_1, 8_2\}, \{8_1, 4_2\},$
$\{8_1, 3_2\}\}$. Therefore, there is an maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 4 = |E(K_{m,n})| \pmod 7$.

Case 8. Suppose $m \equiv 2 \pmod 7$ and $n \equiv 3 \pmod 7$ then $K_{m,n} = K_{10,9} \cup (t-2)K_{9,7} \cup (t_1-1)(t_2-1)K_{7,7} \cup (t_1-1)K_{7,10}$. $K_{10,9}$ has a maximal packing given by $\{[1_1, 2_2, 2_1, 1_2; (9_2, 8_1, 6_2)], [3_1, 3_2, 4_1, 2_2; (10_2, 8_1, 1_2)],$
$[5_1, 4_2, 6_1, 3_2; (2_2, 9_1, 6_2)], [7_1, 6_2, 8_1, 5_2; (2_2, 9_1, 1_2)],$
$[9_1, 10_2, 8_1, 8_2; (2_1, 1_1, 7_2)], [1_1, 4_2, 2_1, 3_2; (6_2, 4_1, 5_2)],$
$[6_1, 1_2, 3_1, 5_2; (2_2, 7_1, 9_2)], [7_1, 8_2, 1_1, 7_2; (3_2, 2_1, 5_2)],$
$[7_2, 8_1, 9_2, 5_1; (4_1, 4_2, 2_1)], [4_1, 6_2, 3_1, 7_2; (10_2, 5_1, 8_2)],$
$[5_1, 5_2, 9_1, 1_2; (8_2, 4_1, 3_2)], [10_2, 7_1, 9_2, 6_1; (2_1, 4_2, 8_1)]\}$. with leave
$L = \{\{9_1, 7_2\}, \{7_2, 6_1\}, \{6_1, 8_2\}, \{8_2, 4_1\}, \{9_1, 9_2\}, \{7_2, 2_1\}\}$. Therefore, there is an maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 6 = |E(K_{m,n})| \pmod 7$.

Case 9. Suppose $m \equiv 2 \pmod 7$ and $n \equiv 4 \pmod 7$ then $K_{m,n} = K_{9,4} \cup (t_2)K_{9,7} \cup (t_1-1)(t_2-1)K_{7,7} \cup (t_1-1)K_{7,11}$. $K_{9,4}$ has a maximal packing given by the following $\{[2_1, 2_2, 1_1, 1_2; (3_2, 5_1, 4_2)], [3_2, 3_1, 2_2, 4_1; (1_1, 4_2, 8_1)],$
$[4_2, 5_1, 3_2, 6_1; (2_1, 1_2, 9_1)], [1_2, 7_1, 4_2, 8_1; (3_1, 3_2, 4_1)],$
$[1_2, 9_1, 2_2, 6_1; (4_1, 4_2, 7_1)]\}$. with leave $L = \{\{8_1, 3_2\}\}$. Thus, there is an maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 1 = |E(K_{m,n})| \pmod 7$.

Case 10. Suppose $m \equiv 2 \pmod 7$ and $n \equiv 5 \pmod 7$ then $K_{m,n} = K_{9,5} \cup (t_2)K_{9,7} \cup (t_1-1)(t_2-1)K_{7,7} \cup (t_1-1)K_{7,12}$. $K_{9,5}$ has a maximal packing given by $\{[2_2, 1_1, 1_2, 2_1; (9_1, 3_2, 6_1)], [3_2, 3_1, 4_2, 2_1; (4_1, 2_2, 5_1)],$
$[5_2, 4_1, 1_2, 3_1; (5_1, 2_2, 8_1)], [3_2, 5_1, 2_2, 6_1; (8_1, 1_2, 7_1)],$
$[4_2, 7_1, 5_2, 6_1; (1_1, 3_2, 2_1)], [9_1, 5_2, 8_1, 4_2; (3_2, 1_1, 2_2)]\}$. with leave
$L = \{\{4_1, 4_2\}, \{7_1, 1_2\}, \{9_1, 1_2\}\}$. Therefore, there is an maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 3 = |E(K_{m,n})| \pmod 7$.

Case 11. Suppose $m \equiv 2 \pmod 7$ and $n \equiv 6 \pmod 7$ then $K_{m,n} = K_{9,6} \cup (t_2)K_{9,7} \cup (t_1-1)(t_2-1)K_{7,7} \cup (t_1-1)K_{7,13}$. $K_{9,6}$ has a maximal packing given by $\{[2_2, 1_1, 1_2, 2_1; (4_1, 5_2, 3_1)], [2_1, 3_2, 3_1, 4_2; (5_2, 5_1, 6_2)],$
$[5_2, 4_1, 6_2, 3_1; (6_1, 1_2, 7_1)], [5_1, 1_2, 6_1, 6_2; (4_2, 7_1, 3_2)],$

$[7_1, 4_2, 8_1, 5_2; (2_2, 9_1, 1_2)]$, $[8_1, 2_2, 9_1, 6_2; (3_2, 5_1, 1_2)]$, $[4_2, 1_1, 3_2, 4_1; (6_1, 6_2, 7_1)]\}$. with leave $L = \{\{3_1, 2_2\}, \{5_1, 5_2\}, \{6_1, 2_2\}, \{9_1, 3_2\}, \{9_1, 5_2\}\}$. Therefore, there is an maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 5 = |E(K_{m,n})| \pmod 7$.

Case 12. Suppose $m \equiv 3 \pmod 7$ and $n \equiv 3 \pmod 7$ then $K_{m,n} = K_{10,3} \cup (t_2)K_{10,7} \cup (t_1 - 1)(t_2 - 1)K_{7,7} \cup (t_1 - 1)K_{10,7}$. $K_{10,3}$ has a maximal packing given by $\{[2_2, 1_1, 1_2, 2_1; (8_1, 3_2, 4_1)], [3_2, 3_1, 2_2, 4_1; (2_1, 1_2, 7_1)], [1_2, 6_1, 3_2, 7_1; (5_1, 2_2, 10_1)], [2_2, 9_1, 3_2, 5_1; (10_1, 1_2, 8_1)]\}$. with leave $L = \{\{8_1, 1_2\}, \{10_1, 1_2\}\}$. Thus, there is an maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 2 = |E(K_{m,n})| \pmod 7$.

Case 13. Suppose $m \equiv 3 \pmod 7$ and $n \equiv 4 \pmod 7$ then $K_{m,n} = K_{3,4} \cup (t_2)K_{3,7} \cup (t_1)(t_2)K_{7,7} \cup (t_1)K_{7,4}$. $K_{3,4}$ has a maximal packing given by the following $\{[1_1, 2_2, 2_1, 1_2; (4_2, 3_1, 3_2)]\}$. with leave $L = \{\{3_1, 1_2\}, \{3_1, 3_2\}, \{3_1, 4_2\}, \{2_1, 4_2\}, \{1_1, 3_2\}\}$, $|E(L)| = 5$. Therefore, there is an maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 5 = |E(K_{m,n})| \pmod 7$.

Case 14. Suppose $m \equiv 3 \pmod 7$ and $n \equiv 5 \pmod 7$ then $K_{m,n} = K_{10,5} \cup (t_2 - 1)K_{10,7} \cup (t_1 - 1)(t_2 - 1)K_{7,7} \cup (t_1 - 1)K_{7,12}$. $K_{10,5}$ has a maximal packing given by $\{[1_2, 2_1, 2_2, 1_1; (5_1, 5_2, 4_1)], [3_2, 3_1, 4_2, 2_1; (9_1, 2_2, 1_1)], [1_2, 4_1, 5_2, 3_1; (6_1, 4_2, 7_1)], [5_2, 6_1, 4_2, 5_1; (8_1, 3_2, 7_1)], [2_2, 7_1, 3_2, 5_1; (6_1, 1_2, 4_1)], [4_2, 9_1, 5_2, 10_1; (8_1, 2_2, 1_1)], [1_2, 10_1, 3_2, 8_1; (9_1, 2_2, 1_1)]\}$. with leave $L = \{\{8_1, 2_2\}\}$. Therefore, there is an maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 1 = |E(K_{m,n})| \pmod 7$.

Case 15. Suppose $m \equiv 3 \pmod 7$ and $n \equiv 6 \pmod 7$ then $K_{m,n} = K_{10,6} \cup (t_2)K_{10,7} \cup (t_1 - 1)(t_2 - 1)K_{7,7} \cup (t_1 - 1)K_{13,7}$. $K_{10,6}$ has a maximal packing given by $\{[2_2, 1_1, 1_2, 2_1; (8_1, 3_2, 4_1)], [3_2, 3_1, 2_2, 4_1; (2_1, 1_2, 7_1)], [1_2, 6_1, 3_2, 7_1; (5_1, 2_2, 10_1)], [2_2, 9_1, 3_2, 5_1; (10_1, 1_2, 8_1)], [5_2, 1_1, 4_2, 2_1; (8_1, 6_2, 4_1)], [6_2, 3_1, 5_2, 4_1; (2_1, 4_2, 7_1)], [4_2, 6_1, 6_2, 7_1; (5_1, 5_2, 10_1)], [5_2, 9_1, 6_2, 5_1; (10_1, 4_2, 8_1)]\}$ with $L = \{\{8_1, 1_2\}, \{8_1, 4_2\}, \{10_1, 1_2\}, \{10_1, 4_2\}\}$; where $|E(L)| = 4 = |E(K_{m,n})| \pmod 7$.

Case 16. Suppose $m \equiv 4 \pmod 7$ and $n \equiv 4 \pmod 7$ then $K_{m,n} = K_{4,4} \cup (t_1)K_{7,4} \cup (t_2)K_{4,7} \cup (t_1)(t_2)K_{7,7}$. $K_{4,4}$ has a maximal packing given by $\{[2_1, 2_2, 1_1, 1_2; (4_2, 3_1, 3_2)], [4_1, 4_2, 3_1, 3_2; (2_2, 1_1, 1_2)]\}$. with leave $L = \{\{2_1, 3_2\}, \{4_1, 1_2\}\}$; $E(L) = 2 = |E(K_{m,n})|$ with leave L; where $|E(L)| = 2 = |E(K_{m,n})| \pmod 7$.

Case 17. Suppose $m \equiv 4 \pmod 7$ and $n \equiv 5 \pmod 7$ then $K_{m,n} = K_{4,5} \cup (t_2)K_{4,7} \cup (t_1)K_{5,7} \cup (t_1)(t_2)K_{7,7}$. Now, $K_{4,5}$ has a maximal packing given by the following $\{[1_1, 2_2, 2_1, 1_2; (3_2, 3_1, 4_2)], [4_2, 3_1, 3_2, 4_1; (1_1, 5_2, 2_1)]\}$ with leave $L = \{\{1_1, 5_2\}, \{2_1, 5_2\}, \{3_1, 1_2\}, \{4_1, 1_2\}, \{4_1, 2_2\}, \{4_1, 5_2\}\}$, $|E(L)| = 6$. Therefore, there is packing of $K_{m,n}$ with leave L; where $|E(L)| = 6 = |E(K_{m,n})| \pmod 7$.

Case 18. Suppose $m \equiv 4 \pmod 7$ and $n \equiv 6 \pmod 7$ then $K_{m,n} = K_{4,6} \cup (t_2)K_{4,7} \cup (t_1)K_{6,7} \cup (t_1)(t_2)K_{7,7}$. Now, $K_{4,6}$ has a maximal packing given by $\{[1_1, 2_2, 1_1, 1_2; (3_2, 4_1, 4_2)], [3_2, 3_1, 4_2, 4_1; (6_1, 1_2, 5_1)],$
$[2_2, 5_1, 1_2, 6_1; (3_1, 3_2, 4_1)]\}$ with leave $L = \{\{1_1, 4_2\}, \{2_1, 3_2\}, \{6_1, 4_2\}\}$. Thus, there is a maximal packing of $K_{m,n}$ with leave L; where $|E(L)| = 3 = |E(K_{m,n})| \pmod 7$.

Case 19. Suppose $m \equiv 5 \pmod 7$ and $n \equiv 5 \pmod 7$ then $K_{m,n} = K_{5,5} \cup (t_2)K_{5,7} \cup (t_1)K_{5,7} \cup (t_1)(t_2)K_{7,7}$. Now, $K_{5,5}$ has a maximal packing given by the following $\{[1_2, 1_1, 2_2, 2_1; (5_1, 4_2, 4_1)], [3_1, 4_2, 4_1, 3_2; (1_2, 5_1, 5_2)],$
$[2_1, 3_2, 5_1, 5_2; (4_2, 1_1, 2_2)]\}$ with leave $L = \{\{3_1, 2_2\}, \{3_1, 5_2\}, \{4_1, 1_2\},$
$\{1_1, 5_2\}\}$; $|E(L)| = 4$. Therefore, there is packing of $K_{m,n}$ with leave L where $|E(L)| = 4 = |E(K_{m,n})| \pmod 7$.

Case 20. Suppose $m \equiv 5 \pmod 7$ and $n \equiv 6 \pmod 7$ then $K_{m,n} = K_{5,6} \cup (t_2)K_{5,7} \cup (t_1)K_{7,6} \cup (t_1)(t_2)K_{7,7}$. Now, $K_{5,6}$ has a maximal packing given by the following $\{[1_1, 1_2, 2_1, 2_2; (3_2, 4_1, 6_2)], [4_2, 3_1, 3_2, 4_1; (1_1, 5_2, 2_1)],$
$[5_2, 5_1, 6_2, 4_1; (2_1, 3_2, 1_1)], [5_1, 2_2, 3_1, 1_2; (4_2, 4_1, 6_2)]\}$ with leave
$L = \{\{1_1, 5_2\}, \{2_1, 4_2\}\}$. Thus, there is a maximal packing of $K_{m,n}$ with leave L where $|E(L)| = 2 = |E(K_{m,n})| \pmod 7$.

Case 21. Suppose $m \equiv 6 \pmod 7$ and $n \equiv 6 \pmod 7$ then $K_{m,n} = K_{6,6} \cup (t_2)K_{6,7} \cup (t_1)K_{6,7} \cup (t_1)(t_2) \cup K_{7,7}$. $K_{6,6}$ has a maximal packing given by the following $\{[1_1, 1_2, 2_1, 2_2; (3_2, 3_1, 5_2)], [3_2, 3_1, 4_2, 2_1; (6_1, 2_2, 1_1)],$
$[5_2, 4_1, 6_2, 3_1; (1_1, 4_2, 2_1)], [5_1, 6_2, 6_1, 5_2; (1_2, 6_1, 4_2)],$
$[4_1, 2_2, 5_1, 3_2; (1_2, 6_1, 4_2)]\}$ with leave $L = \{\{6_1, 1_2\}\}$, $|E(L)| = 1$. Thus, there is packing of $K_{m,n}$ with leave L where $|E(L)| = 1 = |E(K_{m,n})| \pmod 7$. □

Theorem 5.2. *A minimal H-covering of $K_{m,n}$, where $m, n \notin \{1, 2, 3\}$ and $m + n \geq 7$ has padding P, where $|E(P)| = -|E(K_{mn})| \pmod 7$.*

Proof. If $m, n \in \{1, 2, 3\}$ there is no subgraph of $K_{m,n}$ exist. An H-Covering of $K_{m,n}$ with padding P where $|E(P)| = -|E(K_{m,n})| \pmod 7$ would be minimal. For $m \geq 3$ and $n \geq 4$ there is covering of $K_{m+7i,n+7j}$ with padding P for all $i, j \in N$. There is an H-decomposition of $K_{m,7j}$, $K_{7i,7j}$ and $K_{7i,n}$. However, $K_{m,n}$ has a minimal H-covering with padding P by adding a copy of H that contains the edges in the leave in Theorem 5.1. □

6 Acknowledgements

The authors wish to thank the referees for their careful reading of the paper.

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