A generalization of Dirac's Theorem for claw-free graphs

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Abstract

For a graph H, let $\delta_t(H) = \min\{|\bigcup_{i=1}^{t} N_H(v_i)| \mid \{v_1, \cdots, v_t\}$ are t vertices in $H\}$. We show that for a given number ϵ and given integers $p \ge t > 0$ and $k \in \{2, 3\}$, the family of k-connected Hamiltonian claw-free graphs H of sufficiently large order n with $\delta(H) \ge 3$ and $\delta_t(H) \ge t(n+\epsilon)/p$ has a finite obstruction set in which each member is a k-edge-connected K_3-free graph of order at most $\max\{p/t + 2t, 3p/t + 2t - 7\}$ and without spanning closed trails. We found the best possible values of p and ϵ for some $t \ge 2$ when the obstruction set is empty or has the Petersen graph only. In particular, we prove the following for such graphs H:

(a) For $k = 2$ and a given t ($1 \le t \le 4$), if $\delta_t(H) \ge \frac{n+1}{3}$ and $\delta(H) \ge 3$, then H is Hamiltonian.

(b) For $k = 3$ and $t = 2$, (i) if $\delta_2(H) \ge \frac{(n+12)}{9}$, then H is Hamiltonian; (ii) if $\delta_2(H) \ge \frac{(n+9)}{10}$, then either H is Hamiltonian, or H can be characterized by the Petersen graph.

(c) For $k = 3$ and $t = 3$, (i) if $\delta_3(H) \ge \frac{(n+9)}{8}$, then H is Hamiltonian; (ii) if $\delta_3(H) \ge \frac{(n+6)}{9}$, then either H is Hamiltonian, or H can be characterized by the Petersen graph.

These bounds on $\delta_t(H)$ are sharp. Since the number of graphs of orders at most $\max\{p/t + 2t, 3p/t + 2t - 7\}$ is finite for given p and t, improvements to (a), (b) or (c) by increasing the value of p are possible with the help of a computer.

Keywords: Claw-free graph, Hamiltonicity, generalized t-degree condition

1 Introduction

In Hamiltonian graph theory, a classical result is Dirac's Theorem.

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Theorem 1.1 *(Dirac [8]) A graph H of order $n \geq 3$ with $\delta(H) \geq n/2$ is Hamiltonian.*

This result inspired much research on degree conditions for Hamiltonian properties in graphs. Many generalization of Dirac's Theorem have been obtained (see [12, 14]). Faudree et al. [10] define the *generalized t-degree* of a graph H by

$$\delta_t(H) = \min\left\{\left|\bigcup_{i=1}^{t} N_H(v_i)\right| \mid \{v_1, \cdots, v_t\} \text{ is a set of } t \text{ vertices in } H\right\}.$$

In [11], they used δ_2 to give a sufficient condition for the hamiltonicity of claw-free graphs.

Theorem 1.2 *(Faudree et al. [11]). Let H be a 2-connected claw-free graph of order n and $\delta_2(H) \geq (n+1)/3$. Then for n sufficiently large, H is Hamiltonian.*

For given constants $p \geq 4$, t and ϵ, many graphs H with $\delta_t(H) \geq t(n+\epsilon)/p$ are Hamiltonian, however these traditional results cannot distinguish between non-Hamiltonian graphs and Hamiltonian graphs that share these conditions.

In this paper, we generalized Dirac's Theorem in two ways. First, we show that similar to the planar graphs have the obstruction set $\{K_5, K_{3,3}\}$, for given constant p, t, ϵ and $k \in \{2,3\}$, k-connected claw-free Hamiltonian graphs H of order n with $\delta_t(H) \geq t(n+\epsilon)/p$ have a finite obstruction set in which each graph has order at most $\max\{p/t + 2t, 3p/t + 2t - 7\}$. Second, we obtain new $\delta_t(H)$ conditions for claw-free graphs to be Hamiltonian by determining the best possible values of p and ϵ when the obstruction set is empty or has only one graph for some $t \geq 2$.

1.1 Notation

We shall use the notation of Bondy and Murty [1], except when otherwise stated. Graphs considered in this paper are finite and loopless, but multiple edges are allowed. As in [1], $\kappa'(G)$ and $d_G(v)$ denote the edge-connectivity of G and the degree of a vertex v in G, respectively. For a vertex $v \in V(G)$, let $E_G(v)$ be the set of edges incident with v in G. We define $\sigma_2(G) = \min\{d_G(x) + d_G(y) \mid \text{for every } xy \in E(G)\}$ and $D_i(G) = \{v \in V(G) \mid d_G(v) = i\}$. An edge cut X of a graph G is *essential* if each component of $G - X$ has some edges. A graph G is *essentially k-edge-connected* if G is connected and does not have an essential edge cut of size less than k. An edge $e = uv$ is called a *pendant edge* if $\min\{d_G(u), d_G(v)\} = 1$. A set with t vertices is called a *t-vertex set*. A graph H is *claw-free* if H does not contain an induced subgraph isomorphic to $K_{1,3}$. A connected graph Ψ is a *closed trail* if the degree of each vertex in Ψ is even. A closed trail Ψ is called a *spanning closed trail (SCT)* in G if $V(G) = V(\Psi)$, and is called a *dominating closed trail* (DCT) if $E(G - V(\Psi)) = \emptyset$. A graph is Hamiltonian if it has a spanning cycle. Throughout this paper, we use P for the Petersen graph and use P_{14} for the graph obtained from P by replacing a vertex v in P by a $K_{2,3}$ such that the three edges incident with v in P are incident with the three degree 2 vertices in $K_{2,3}$, respectively.

For a graph G, the line graph $L(G)$ has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

1.2 Ryjáček closure concept

For a claw-free graph H, a vertex $v \in V(H)$ is *locally connected* if its neighborhood $N_H(v)$ induces a connected graph. The closure of a claw-free graph H introduced by Ryjáček [16] is the graph obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of H as long as this is possible and is denoted by $cl(H)$. A claw-free graph H is said to be *closed* if $H = cl(H)$.

Theorem 1.3 *(Ryjáček [16]). Let H be a claw-free graph and $cl(H)$ its closure. Then*

(a) $cl(H)$ is well defined, and $\kappa(cl(H)) \geq \kappa(H)$;
(b) there is a K_3-free graph G such that $cl(H) = L(G)$;
(c) both graphs H and $cl(H)$ have the same circumference.

The following theorem shows a relationship between Hamiltonian cycles and DCTs.

Theorem 1.4 *(Harary and Nash-Willams [13]). The line graph $H = L(G)$ of a graph G with at least three edges is Hamiltonian if and only if G has a DCT.*

It is known that a connected line graph $H \neq K_3$ has a unique graph G with $H = L(G)$. For a claw-free graph H, the closure $cl(H)$ of H can be obtained in polynomial time [16] and the preimage graph of a line graph can be obtained in linear time [15]. We can compute G efficiently for $cl(H) = L(G)$ and call G the *preimage graph of H*. By Theorems 1.4 and 1.3, finding a Hamiltonian cycle in a claw-free graph H is equivalent to finding a DCT in the preimage graph G of H.

1.3 Catlin's reduction method

For $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge $e \in X$ and deleting the resulting loops. G/X may not be simple. If Γ is a connected subgraph of G, then Γ is contracted to a vertex in G/Γ and we write G/Γ for $G/E(\Gamma)$.

Let $O(G)$ be the set of vertices of odd degree in G. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph Γ_R of G with $O(\Gamma_R) = R$. When $R = \emptyset$, Γ_R is an SCT in G. As always, K_1 is regarded as a collapsible graph.

In [2], Catlin showed that every graph G has a unique collection of maximal collapsible subgraphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_c$. The *reduction* of G is $G' = G/(\cup_{i=1}^{c}\Gamma_i)$, the graph obtained from G by contracting each Γ_i into a single vertex v_i $(1 \le i \le c)$. For a vertex $v \in V(G')$, there is a unique maximal collapsible subgraph $\Gamma_0(v)$ such that v is the contraction image of $\Gamma_0(v)$ and $\Gamma_0(v)$ is the *preimage* of v. A vertex $v \in V(G')$ is a *contracted* vertex if $\Gamma_0(v) \ne K_1$. A graph G is *reduced* if $G' = G$.

Theorem 1.5 *(Catlin, et al. [2, 3]). Let G be a connected graph and let G′ be the reduction of G.*

(a) $G \in \mathcal{CL}$ *if and only if* $G' = K_1$, *and G has an SCT if and only if G′ has an SCT.*

(b) G has a DCT if and only if G′ has a DCT containing all the contracted vertices of G′.

(c) If G is a reduced graph, then G is simple and K_3*-free with* $\delta(G) \le 3$. *For any subgraph H of G, H is reduced and either* $H \in \{K_1, K_2, K_{2,t}(t \ge 2)\}$ *or* $|E(H)| \le 2|V(H)| - 5$.

Let $k \ge 1$ be an integer. Let H be a k-connected claw-free graph with $\delta(H) \ge 3$. By Theorem 1.3, there is a K_3-free graph G such that $cl(H) = L(G)$. Then $V(H) = V(cl(H))$ and $\delta(cl(H)) \ge \delta(H) \ge 3$. For an edge $e = xy$ in G, let v_e be the vertex in H defined by e in G. Then $d_{cl(H)}(v_e) + 2 = d_G(x) + d_G(y)$. Thus, G is an essentially k-edge-connected K_3-free simple graph with $\sigma_2(G) \ge 5$ and $D_1(G) \cup D_2(G)$ is an independent set. Let E_1 be the set of pendant edges in G. For each $x \in D_2(G)$, there are two edges e_x^1 and e_x^2 incident with x. Let $X_2(G) = \{e_x^1 \,|\, x \in D_2(G)\}$. Define

$$G_0 = G/(E_1 \cup X_2(G)) = (G - D_1(G))/X_2(G).$$

In other words, G_0 is obtained from G by deleting the vertices in $D_1(G)$ and replacing each path of length 2 whose internal vertex is a vertex in $D_2(G)$ by an edge. Note that G_0 may not be simple.

Let $W = D_1(G) \cup D_2(G)$. In [18], G_0 is denoted by $I_W(G)$. In [17], Shao defined G_0 for essentially 3-edge-connected graphs G. Following [17], we call G_0 the *core* of G.

Let G_0' be the reduction of G_0. For a vertex $v \in V(G_0')$, let $\Gamma_0(v)$ be the maximum collapsible preimage of v in G_0 and let $\Gamma(v)$ be the preimage of v in G which is the graph induced by edges in $E(\Gamma_0(v))$ and some edges in $E_1 \cup X_2(G)$. For a vertex v in G_0', v is a *contracted* vertex if $|E(\Gamma(v))| \ge 1$ and v is a *nontrivial* vertex if $|E(\Gamma(v))| \ge 1$ or v is adjacent to a vertex in $D_1(G) \cup D_2(G)$.

Convenience Assumption: In the definition of G_0, each edge in $X_2(G)$ is selected arbitrarily from two edges incident with a vertex $x \in D_2(G)$. To avoid unnecessary cases in our proofs, we assume that the edges in $X_2(G)$ are chosen such that the number of nontrivial preimages $\Gamma(v)$ for each $v \in V(G_0')$ is as large as possible.

For instance, if uv is an edge in G_0' that is obtained from G by replacing the path uxv in G by uv, $\Gamma(u)$ has edges other than ux and $\Gamma(v)$ may be equal to K_1 if xv is not counted, then we assume that $e_x^1 = xv$ and so both $\Gamma(u)$ and $\Gamma(v)$ contain at least one edge.

Using Theorem 1.5, Veldman [18] and Shao [17] proved the following:

Theorem 1.6 *([18, 17]) Let G be a connected and essentially k-edge-connected graph ($k \geq 2$) with $\sigma_2(G) \geq 5$ where $L(G)$ is not complete. Let G_0 be the core of graph G. Let G_0' be the reduction of G_0. Then each of the following holds:*
(a) G_0 is well defined, nontrivial and $\kappa'(G_0') \geq \kappa'(G_0) \geq \min\{3, k\}$.
(b) (Lemma 5 [18]) G has a DCT if and only if G_0' has a DCT containing all the nontrivial vertices.

2 Main Results

Let $Q_0(r, k)$ be the family of k-edge-connected K_3-free graphs of order at most r and without an SCT. It is known that $Q_0(5, 2) = \{K_{2,3}\}$ and $Q_0(13, 3) = \{P\}$ (see Theorem 3.1). For a given integer $p > 0$ and a real number ϵ, define

$$N(p,\epsilon)=\max\{36p^2-34p-\epsilon p-\epsilon,\ 10p(2p-1)-\epsilon p-\epsilon,\ (3p+1)(-\epsilon-4p)\}. \quad (1)$$

The following two parameters are closely related to $\delta_t(H)$. For a graph H and $t \geq 1$, we define

- $\sigma_t(H) = \min\{\Sigma_{i=1}^{t} d_H(v_i) \mid \{v_1, v_2, \cdots, v_t\}$ is an independent set in $H\}$ (if $t > \alpha(H)$, $\sigma_t(H) = \infty$);
- $U_t(H) = \min\{|\bigcup_{i=1}^{t} N_H(v_i)| \mid \{v_1, v_2, \cdots, v_t\}$ is an independent set in $H\}$.

Let $\Omega(H) = \{\sigma_t(H), U_t(H)\}$. Degree conditions involved parameters in $\Omega(H)$ for the hamiltonicity of claw-free graphs have been the subjects of many papers (see [7, 9, 12, 14]). Recently, we obtained a result which unifies several prior results.

Theorem 2.1 *([4]) Let H be a k-connected claw-free graph of order n ($k \geq 2$) and $\delta(H) \geq 3$. For given integers $p \geq t > 0$ and a given number ϵ, if $d_t(H) \geq \frac{t(n+\epsilon)}{p}$ where $d_t(H) \in \Omega(H)$ and $n > N(p, \epsilon)$, then either H is Hamiltonian or $cl(H) = L(G)$ where G is an essentially k-edge-connected K_3-free graph without a DCT and G_0' satisfies one of the following:*
(a) if $k = 2$, $G_0' \in Q_0(c, 2)$ where $c \leq \max\{4p-5, 2p+1\}$;
(b) if $k = 3$, $G_0' \in Q_0(c, 3)$ where $c \leq \max\{3p-5, 2p+1\}$.

Since $\sigma_t(H) \geq U_t(H) \geq \delta_t(H)$, we have the following corollary.

Corollary 2.2 *Let H be a k-connected claw-free graph of order n ($k \geq 2$) and $\delta(H) \geq 3$. For given integers $p \geq t > 0$ and a given number ϵ, if $\delta_t(H) \geq \frac{t(n+\epsilon)}{p}$ and $n > N(p, \epsilon)$, then either H is Hamiltonian or $cl(H) = L(G)$ where G is an essentially k-edge-connected K_3-free graph without a DCT and G_0' satisfies one of the following:*

(a) if $k = 2$, $G_0' \in \mathcal{Q}_0(c, 2)$ where $c \leq \max\{4p-5, 2p+1\}$;

(b) if $k = 3$, $G_0' \in \mathcal{Q}_0(c, 3)$ where $c \leq \max\{3p-5, 2p+1\}$.

Since the condition $\delta_t(H) \geq \frac{t(n+\epsilon)}{p}$ defines the structure of graphs differently than conditions involving parameters in $\Omega(H)$, we have a much better upper bounds on $|V(G_0')|$ in Theorem 2.3.

Note that it is not necessary to use Corollary 2.2 and (1) to prove Theorem 2.3 and other results in this paper. One may obtain a different expression on $N(p, \epsilon)$ other than the one defined by (1) from [4] to prove Theorem 2.3. However, Corollary 2.2 provides a good starting point for our proofs and allows us to avoid some tedious arguments. So in this paper when we say "n is large enough" or ""n is sufficiently large", we mean "$n > N(p, \epsilon)$".

Theorem 2.3 *Let H be a k-connected claw-free graph of order n ($k \in \{2, 3\}$) and $\delta(H) \geq 3$. Let $cl(H) = L(G)$. For given integers $p \geq t > 0$ and a number ϵ, if $\delta_t(H) \geq \frac{t(n+\epsilon)}{p}$ and $n > N(p, \epsilon)$, then either H is Hamiltonian or $G_0' \in \mathcal{Q}_0(c, k)$ where $c \leq \max\{p/t + 2t, 3p/t + 2t - 7\}$ and G_0' does not have a DCT containing all the nontrivial vertices.*

For 2-connected claw-free graphs, we have

Theorem 2.4 *Let H be a 2-connected claw-free graph of order n with $\delta(H) \geq 3$ and n is sufficiently large. For given p and t with $2 \leq t \leq 4$ and $p/t \leq 3$, if $\delta_t(H) \geq \frac{t(n+1)}{p}$ (i.e., $\delta_t(H) \geq \frac{n+1}{3}$), then H is Hamiltonian.*

For 3-connected claw-free graphs and $t = 2$, we have

Theorem 2.5 *Let H be a 3-connected claw-free graph of order n and n is sufficiently large. Let G be the preimage of H, i.e., $cl(H) = L(G)$. Let G_0' be the reduction of the core of G. Then each of the following holds:*

(a) if $\delta_2(H) \geq \frac{n+12}{9}$, then H is Hamiltonian;

(b) if $\delta_2(H) \geq \frac{n+9}{10}$, then either H is Hamiltonian or $G_0' = P$ and one of the following holds:

(i) for each $v \in V(P)$, the preimage $\Gamma(v)$ is a K_{1,s_v} with $s_v \geq \frac{n+9}{10} - 3$ and $15 + \sum_{v \in V(P)} s_v = n$;

(ii) *one preimage* $\Gamma(u)$ *is a* K_2 *and for each* $v \in V(P) - \{u\}$ *the preimage* $\Gamma(v)$ *is a* K_{1,s_v} *with* $s_v \geq \frac{n+9}{10} - 3$ *and* $16 + \sum_{v \in V(P)-\{u\}} s_v = n$;

(iii) *one preimage* $\Gamma(w)$ *is not a tree with* $s_w = |E(\Gamma(w))| \geq 2\left(\frac{n+9}{10}\right) - 8$, *one preimage* $\Gamma(u)$ *is a* K_2 *and for each* $v \in V(P) - \{u, w\}$ *the preimage* $\Gamma(v)$ *is a* K_{1,s_v} *with* $s_v \geq \frac{n+9}{10} - 3$ *and* $16 + s_w + \sum_{v \in V(P)-\{u,w\}} s_v = n$.

For 3-connected claw-free graphs and $t = 3$, we have the following:

Theorem 2.6 *Let* H *be a 3-connected claw-free graph of order* n *and* n *is sufficiently large. Let* G *be the preimage of* H, *i.e.*, $cl(H) = L(G)$. *Let* G_0' *be the reduction of the core of* G. *Then one of the following holds:*

(a) *If* $\delta_3(H) \geq \frac{n+9}{8}$, *then* H *is Hamiltonian;*

(b) *If* $\delta_3(H) \geq \frac{n+6}{9}$, *then either* H *is Hamiltonian or* $G_0' = P$ *and one of the following holds:*

(i) *there is a vertex* $v_1 \in V(P)$ *such that the preimage* $\Gamma(v_1) = K_{1,s_1}$ *with* $1 \leq s_1 \leq 2$ *and for each* $v \in V(P) - \{v_1\}$, *the preimage* $\Gamma(v)$ *is a* K_{1,s_v} *with* $s_v \geq \frac{n+6}{9} - 3$ *and* $15 + s_1 + \sum_{v \in V(P)-\{v_1\}} s_v = n$;

(ii) *there are two vertices (say* v_1 *and* v_2*) in* $V(P)$ *such that each preimage* $\Gamma(v_i)$ $(i = 1, 2)$ *is a* K_2 *and for each* $v \in V(P) - \{v_1, v_2\}$ *the preimage* $\Gamma(v)$ *is a* K_{1,s_v} *with* $s_v \geq \frac{n+6}{9} - 3$ *and* $17 + \sum_{v \in V(P)-\{v_1,v_2\}} s_v = n$;

(iii) *there is a vertex* w *in* $V(P)$ *such that the preimage* $\Gamma(w)$ *is not a tree with* $s_w = |E(\Gamma(w))| \geq 2\left(\frac{n+6}{9}\right) - 13$, *there are two vertices (say* v_1 *and* v_2*) in* $V(P)$ *such that each preimage* $\Gamma(v_i)$ $(i = 1, 2)$ *is a* K_2 *and for each* $v \in V(P) - \{u, w\}$ *the preimage* $\Gamma(v)$ *is a* K_{1,s_v} *with* $s_v \geq \frac{n+6}{9} - 3$ *and* $17 + s_w + \sum_{v \in V(P)-\{v_1,v_2,w\}} s_v = n$.

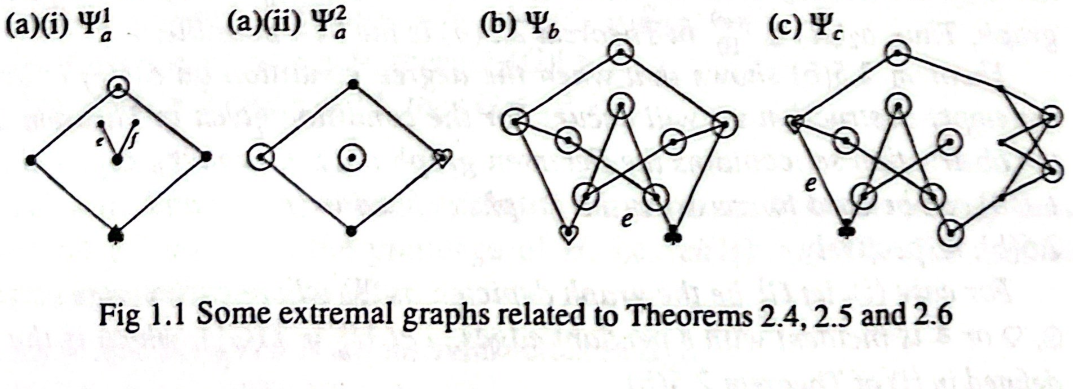

Fig 1.1 Some extremal graphs related to Theorems 2.4, 2.5 and 2.6

Remark 1 *(a) Theorem 2.4 is an improvement of Theorem 1.2. The degree conditions and* $t \leq 4$ *in Theorem 2.4 are the best possible. By the definition of* $\delta_t(H)$, $\delta_4(H) \geq \delta_3(H) \geq \delta_2(H)$. *Thus* $\delta_2(H) \geq \frac{n+1}{3}$ *implies* $\delta_4(H) \geq \frac{n+1}{3}$ *but the reverse is not true. Theorem 2.4 shows that for a 2-connected claw-free graph* H *if* $\delta_4(H) \geq \frac{n+1}{3}$ *(even* $\delta_2(H) < \frac{n+1}{3}$*),* H *is Hamiltonian.*

(i) Let $G_a^1 = \Psi_a^1$ as depicted in Fig 1.1 (a)(i), where the vertex marked by ⊙ is incident with 2r pendant edges and the one marked by ♠ is incident with r pendant edges. Let $H_a^1 = L(G_a^1)$. Then H_a^1 is a Hamiltonian graph with $n = |V(H_a^1)| = |E(G_a^1)| = 3r + 8$, $\delta_4(H_a^1) = \delta_3(H_a^1) \geq r + 3 = \frac{n+1}{3}$ but $\delta_2(H_a^1) = 4$. Theorem 2.4 can determine that H_a^1 is Hamiltonian but Theorem 1.2 cannot.

(ii) Let $G_a^2 = \Psi_a^2$ as depicted in Fig 1.1 (a)(ii), where each vertex marked by ⊙ or ♡ is incident with r pendant edges. Let $H_a^2 = L(G_a^2)$. Since G_a^2 does not have a DCT, H_a^2 is not a Hamiltonian graph with $n = |V(H_a^2)| = |E(G_a^2)| = 3r + 6$. For $t \geq 2$ $\delta_t(H_a^2) \geq r + 2 = \frac{n}{3}$. Thus, $\delta_t(H) \geq \frac{n+1}{3}$ in Theorem 2.4 is the best possible.

Next, we show that $t \leq 4$ cannot be extended to $t \geq 5$ for the degree condition.

For $t = 5$, let $G_a = \Psi_a^2$ as depicted in Fig 1.1(ii), in which each of the two vertices marked by ⊙ is incident with $r \geq 5$ pendant edges and the vertex marked by ♡ is incident with 2 pendant edges. Let $H_a = L(G_a)$. Then $n = |V(H_a)| = 2r+8$, $\delta(H_a) = 3$, $\delta_2(H_a) = 4$, $\delta_3(H_a) = 6$ and $\delta_4(H_a) = 8$ and $\delta_5(H_a) = r+2 = \frac{n-4}{2} > \frac{n+1}{3}$ (when $n \geq 15$). However, H_a is not Hamiltonian. Thus, $\delta_t(H) \geq \frac{n+1}{3}$ for $2 \leq t \leq 4$ in Theorem 2.4 cannot be extended to $t \geq 5$.

(b) Let $G_b = \Psi_b$ as depicted in Fig 1.1(b), where each vertex marked by ⊙ or ♣ is incident with r pendant edges, and the vertex marked by ♡ is incident with one pendant edge. Let $H_b = L(G_b)$. Then H_b is a 3-connected claw-free graph of order $n = |V(H_b)| = |E(G_b)| = 9r + 16$ with $\delta_2(H_b) = r + 3 = \frac{n+11}{9}$. Since G_b does not have a DCT, H_b is not Hamiltonian. This shows that $\delta_2(H) \geq \frac{n+12}{9}$ in Theorem 2.5(a) is the best possible.

(c) Let $G_c = \Psi_c$ as depicted in Fig 1.1(c), where each vertex marked by ⊙ or ♣ is incident with r pendant edges and the vertex marked by ♡ is incident with one pendant edge. Let $H_c = L(G_c)$. Then H_c is a 3-connected claw-free graph of order $n = |V(H_c)| = |E(G_c)| = 10r + 22$ with $\delta_2(H_c) = r + 3 = \frac{n+8}{10}$. Since G_c does not have a DCT, H_c is not Hamiltonian. However, $(G_c)'_0 = P_{14}$, not the Petersen graph. Thus, $\delta_2(H) \geq \frac{n+9}{10}$ in Theorem 2.5(b) is the best possible.

Theorem 2.5(b) shows that when the degree condition on $\delta_2(H)$ is lower, a nonempty obstruction set will occur. For the condition given in Theorem 2.5(b), the obstruction set contains the Petersen graph only. Graph Ψ_b depicted in Fig. 1.1(b) can be used to construct the graphs defined in (i), (ii) and (iii) of Theorem 2.5(b), respectively.

For case (i), let G_b^1 be the graph depicted as Ψ_b where each vertex marked by ⊙, ♡ or ♣ is incident with r pendant edges. Let $H_b^1 = L(G_b^1)$, which is the graph defined in (i) of Theorem 2.5(b).

For case (ii), let G_b^2 be the graph depicted as Ψ_b where each vertex marked by ⊙ or ♡ is incident with r pendant edges and the vertex marked by ♣ is incident with one pendant edge. Let $H_b^2 = L(G_b^2)$, which is the graph defined in (ii) of Theorem 2.5(b).

For case (iii), let G_b^3 be the graph depicted as Ψ_b where each vertex marked *by ⊙ is incident with r pendant edges, the vertex marked by ♡ is a $K_{2,r+2}$ subgraph and the vertex marked by ♣ is incident with one pendant edge. Let $H_b^3 = L(G_b^3)$, which is the graph defined in (iii) of Theorem 2.5(b).*

(d) Let G_d be the graph obtained from Ψ_b (depicted in Fig. 1.1(b)) by subdividing the edge e, where each vertex marked by ⊙ is incident with r pendant edges. Then G_d is a 2-edge-connected and essentially 3-edge-connected graph with $|E(G_d)| = 8r + 16$. Let $H_d = L(G_d)$. Then H_d is a 3-connected non-Hamiltonian claw-free graph of order $n = |V(H_d)| = |E(G_d)| = 8r + 16$ with $\delta_3(H_d) = \frac{n+8}{8}$. Thus, $\delta_3(H) = \frac{n+9}{8}$ in Theorem 2.6(a) is the best possible.

(e) Let G_e be a graph obtained from Ψ_c (depicted in Fig. 1.1(c)) by subdividing the edge e, where each vertex marked by ⊙ is incident with r and each vertex marked by ♡ or ♣ is adjacent to a vertex of degree two. Let $H_e = L(G_e)$. Then H_e is a 3-connected non-Hamiltonian claw-free graph of order $n = |V(H_e)| = |E(G_e)| = 9r + 22$ with $\delta_3(H_e) = \frac{n+5}{9}$ and $(G_e)'_0 = P_{14}$. Thus, $\delta_3(H) = \frac{n+6}{9}$ in Theorem 2.6(b) is the best possible.

Similar to the discussion of case (c) above, the graph Ψ_b depicted in Fig. 1.1(b) can be used to construct the graphs defined in (i), (ii) and (iii) of Theorem 2.6(b), respectively.

In Section 3, we give a brief discussion on reduced graphs and prove some technical lemmas. The proofs of the main results will be given in Section 4.

3 Properties on reduced graphs and some lemmas

Some facts concerning reduced graphs are summarized in the following theorem.

Theorem 3.1 *Let G be a connected reduced graph of order n and without an SCT.*
(a) if $\kappa'(G) \geq 2$, then $n \geq 5$ and $n = 5$ only if $G = K_{2,3}$;
(b) ([5]) if $\kappa'(G) \geq 2$ and $n \leq 9$, then $|D_2(G)| \geq 3$;
(c) ([5]) if $\kappa'(G) \geq 3$ and $n \leq 14$, then $G \in \{P, P_{14}\}$.

In the rest of the paper, we assume that H is a graph satisfying the assumptions of Theorem 2.3 and G is the preimage of H, i.e., $cl(H) = L(G)$. We use the following notation related to G'_0:

- $S_0 = \{v \in V(G'_0) \mid v$ is a nontrivial vertex in $G'_0\}$;
- $S_t = \{v \in S_0 \mid |E(\Gamma(v))| \geq t\}$;
- $S_1 = \{v \in S_0 \mid 1 \leq |E(\Gamma(v))| \leq t - 1\}$;
- $S^* = S_0 - (S_t \cup S_1)$, the set of vertices $v \in S_0$ with $\Gamma(v) = K_1$ and adjacent to some vertices in $D_2(G)$;
- $V_0 = V(G'_0) - (S_t \cup S_1)$, the set of vertices v in G'_0 with $\Gamma(v) = K_1$ in G which includes S^*;

- $\Phi_0 = G'_0[V_0 \cup S_1]$;
- $E_0 = E(\Phi_0)$ is the set of edges in Φ_0;
- V_E is the set of vertices incident with some edges in E_0;
- $E_R = \bigcup_{v \in S_1} E(\Gamma(v))$ and $\Phi^* = G[E_0 \cup E_R]$ (and $E_0 \cup E_R = E(\Phi^*)$);
- $U_0 = V_0 - V_E$ and so $V(G'_0) = S_t \cup S_1 \cup V_0 = S_t \cup S_1 \cup V_E \cup U_0$. For $v \in V_0$, $E(\Gamma(v)) = \emptyset$. Then

$$E(G) = \bigcup_{v \in S_t} E(\Gamma(v)) \bigcup_{v \in S_1} E(\Gamma(v)) \bigcup E(G'_0). \tag{2}$$

Since $\sigma_2(G) \geq 5$, $D_2(G'_0) \subseteq S_t \cup S_1$. U_0 is an independent set and $N_{G'_0}(x) \subseteq S_t$ for $x \in U_0$.

By the Convenience Assumption, we have the following easy lemma.

Lemma 3.2 *For a vertex $v \in S^*$, v must be adjacent to a vertex $u \in S_1$ such that $|E(\Gamma(u))| = 1$ and the edge (say xu) in $E(\Gamma(u))$ is an edge in $X_2(G)$ and x is in $D_2(G)$ and is adjacent to v in G.*

Proof. By the definition of S^*, v is adjacent to a vertex x in $D_2(G)$. Let vxy be a path of length 2 in G. Since $\Gamma(v) = K_1$, $xy \in X_2(G)$. Thus, xy is one of the edges in a $\Gamma(u)$ where $u \in N_{G'_0}(v)$. By the Convenience Assumption, if $|E(\Gamma(u))| > 1$, then we shall use vx as an edge in $X_2(G)$ instead of xy. Thus, $|E(\Gamma(u))| = 1$. □

For an edge $e \in E(G)$, let $E_G(e)$ be the set of edges incident with exactly one end of edge e (so $e \notin E_G(e)$). For $cl(H) = L(G)$, if v in H corresponds to edge e in G then $d_{cl(H)}(v) = |E_G(e)|$. If $\{v_1, \cdots, v_t\}$ is a t-vertex set in H with the corresponding t-edge set $\{e_1, \cdots, e_t\}$ in G, then

$$\delta_t(H) \leq \left| \bigcup_{i=1}^{t} N_{cl(H)}(v_i) \right| = \left| \bigcup_{i=1}^{t} E_G(e_i) \right|. \tag{3}$$

Lemma 3.3 *With the notation defined above, each of the following holds:*

(a) for each $v \in S_t$, $|E(\Gamma(v))| \geq \delta_t(H) - d_{G'_0}(v)$;

(b) $|S_t| \leq p/t$. If $|S_t| = p/t$ then $|E(G'_0)| \geq \epsilon + \sum_{v \in S_1} d_{G'_0}(v) + \sum_{v \in V_0} d_{G'_0}(v) + \sum_{v \in S_1} |E(\Gamma(v))|$;

(c) $|E_0| + |E_R| = |E(\Phi^*)| \leq t - 1$ and $|S_1 \cup V_E| \leq 2|S_1| + |V_E| \leq 2(t-1)$;

(d) $|U_0| \leq \max\{2, 2|S_t| - 5\} \leq \max\{2, 2p/t - 5\}$;

Proof. (a) For $v \in S_t$, $|E(\Gamma(v))| \geq t$. For $\{e_v^1, \cdots, e_v^t\} \subseteq E(\Gamma(v))$, $\bigcup_{i=1}^{t} E_G(e_v^i) \subseteq E(\Gamma(v)) \cup E_{G'_0}(v)$. Then by (3),

$$\delta_t(H) \leq \left| \bigcup_{i=1}^{t} E_G(e_v^i) \right| \leq |E(\Gamma(v))| + |E_{G'_0}(v)| \leq |E(\Gamma(v))| + d_{G'_0}(v). \tag{4}$$

Thus, (a) is proved.

(b) Let $s = |S_t|$. By (2), (4) and $n = |E(G)|$,

$$\begin{aligned} n = |E(G)| &= \sum_{v\in S_t} |E(\Gamma(v))| + \sum_{v\in S_1} |E(\Gamma(v))| + |E(G_0')| \\ &\geq \sum_{v\in S_t} (\delta_t(H) - d_{G_0'}(v)) + \sum_{v\in S_1} |E(\Gamma(v))| + |E(G_0')|; \\ n &\geq s\delta_t(H) - \sum_{v\in S_t} d_{G_0'}(v) + \sum_{v\in S_1} |E(\Gamma(v))| + |E(G_0')|. \end{aligned} \tag{5}$$

By $2|E(G_0')| = \sum_{v\in V(G_0')} d_{G_0'}(v) = \sum_{v\in S_t} d_{G_0'}(v) + \sum_{v\in S_1} d_{G_0'}(v) + \sum_{v\in V_0} d_{G_0'}(v)$, (5) and $\delta_t(H) \geq \frac{t(n+\epsilon)}{p}$,

$$\begin{aligned} n &\geq s\delta_t(H) - \left(2|E(G_0')| - \sum_{v\in S_1} d_{G_0'}(v) - \sum_{v\in V_0} d_{G_0'}(v)\right) + \sum_{v\in S_1} |E(\Gamma(v))| + |E(G_0')| \\ &\geq s\frac{t(n+\epsilon)}{p} - |E(G_0')| + \sum_{v\in S_1} d_{G_0'}(v) + \sum_{v\in V_0} d_{G_0'}(v) + \sum_{v\in S_1} |E(\Gamma(v))|. \end{aligned} \tag{6}$$

By Corollary 2.2 and $p \geq 3$, $|V(G_0')| \leq 4p - 5$. By Theorem 1.5, $|E(G_0')| \leq 2|V(G_0')| - 4$. Thus, $|E(G_0')| \leq 8p - 14$. By (6),

$$\begin{aligned} s\frac{t(n+\epsilon)}{p} &\leq n + |E(G_0')| \leq n + 8p - 14; \\ st &\leq \frac{p(n + 8p - 14)}{n+\epsilon} = p + \frac{p(8p - 14 - \epsilon)}{n+\epsilon}. \end{aligned}$$

Since st is an integer, $st \leq p$ when $n > p(8p - 14 - \epsilon) - \epsilon$. Thus, $|S_t| = s \leq p/t$.

If $|S_t| = p/t$, by (6) $|E(G_0')| \geq \epsilon + \sum_{v\in S_1} d_{G_0'}(v) + \sum_{v\in V_0} d_{G_0'}(v) + \sum_{v\in S_1} |E(\Gamma(v))|$.

(c) To the contrary, suppose that $|E(\Phi^*)| \geq t$. Let $X_t = \{e_1, \cdots, e_t\}$ be a t-edge set in $E(\Phi^*) = E_R \cup E_0$. Let $W = \{v \in S_1 \mid E(\Gamma(v)) \cap X_t \neq \emptyset\}$. Then $|W| \leq t$ and $\bigcup_{i=1}^{t} E_G(e_i) \subseteq \bigcup_{v\in W} E(\Gamma(v)) \cup E(G_0')$.

Since $|E(\Gamma(v))| \leq t - 1$ for each $v \in W$, by (3) and $|E(G_0')| \leq 8p - 14$,

$$\frac{t(n+\epsilon)}{p} \leq \left|\bigcup_{i=1}^{t} E_G(e_i)\right| \leq \left|\bigcup_{v\in W} E(\Gamma(v))\right| + |E(G_0')| \leq t(t-1) + 8p - 14,$$

a contradiction, since $n > N(p, \epsilon) \geq \frac{p(t(t-1)+8p-14)}{t} - \epsilon$. Thus, $|E_R| + |E_0| = |E(\Phi^*)| \leq t - 1$.

Since $|E(\Gamma(v))| \geq 1$ for $v \in S_1$, $|S_1| \leq |E_R|$. In the worst case, E_0 is a matching and so $|V_E| \leq 2|E_0| = 2(|E(\Phi^*)| - |E_R|) \leq 2(t - 1 - |E_R|)$. Then $2|E_R| + |V_E| \leq 2(t-1)$ and $|S_1 \cup V_E| \leq |S_1| + |V_E| \leq 2|S_1| + |V_E| \leq 2|E_R| + |V_E| \leq 2(t - 1)$. (c) is proved.

(d) If $|U_0| \le 2$, the statement is true trivially. Thus, we assume that $|U_0| \ge 3$. Let $Y = \bigcup_{u \in U_0} N_{G_0'}(u)$. By the definition of U_0, U_0 is an independent set and $Y \subseteq S_t$. Let $\Phi = G_0'[U_0 \cup Y]$. Then $|V(\Phi)| = |U_0| + |Y| \le |U_0| + |S_t|$. Since $D_2(G_0') \subseteq S_t \cup S_1$, $d_{G_0'}(u) \ge 3$ for $u \in U_0$. Then $|E(\Phi)| \ge 3|U_0|$. Since $|U_0| \ge 3$, $\Phi \notin \{K_1, K_2, K_{2,r}\}$. By Theorem 1.5, $|E(\Phi)| \le 2|V(\Phi)| - 5$. Then $3|U_0| \le 2|U_0| + 2|Y| - 5$. Hence, $|U_0| \le 2|Y| - 5 \le 2|S_t| - 5 \le 2p/t - 5$. □

4 Proofs of Theorems 2.3, 2.4, 2.5 and 2.6

Proof of Theorem 2.3. Suppose that H is not Hamiltonian. By Theorem 1.6, G_0' does not have a DCT containing all the nontrivial vertices. By Lemma 3.3, we have $|V(G_0')| = |S_t| + |S_1 \cup V_E| + |U_0| \le p/t + 2(t-1) + \max\{2, 2p/t - 5\} = \max\{p/t + 2t, 3p/t + 2t - 7\}$. □

Proof of Theorem 2.4. Suppose that H is not Hamiltonian. By Theorem 1.6, G_0' does not have an SCT. By Theorem 3.1, $|V(G_0')| \ge 5$. By Lemma 3.3 and $t \le 4$, and $E_R = \cup_{v \in S_1} E(\Gamma(v))$

$$|E_0| + |\cup_{v \in S_1} E(\Gamma(v))| = |E(\Phi^*)| \le t - 1 = 3 \text{ and } |S_1 \cup V_E| \le 2(t-1) = 6. \quad (7)$$

Claim 1. $|S_t| < p/t = 3$.

To the contrary, suppose that $|S_t| = p/t = 3$. Then $|V(G_0')| = |S_t| + |S_1| + |V_0| = 3 + |S_1| + |V_0|$. By Lemma 3.3 with $\epsilon = 1$, $|E(G_0')| \ge 1 + \sum_{v \in S_1} d_{G_0'}(v) + \sum_{v \in V_0} d_{G_0'}(v) + \sum_{v \in S_1} |E(\Gamma(v))|$. For each $v \in S_1$, $d_{G_0'}(v) \ge 2$ and $|E(\Gamma(v))| \ge 1$. For each $v \in V_0$, $d_{G_0'}(v) \ge 3$. Then $|E(G_0')| \ge 1 + 2|S_1| + 3|V_0| + |S_1| = 1 + 3|S_1| + 3|V_0|$.

Since $G_0' \notin \{K_1, K_2\}$, by Theorem 1.5, $|E(G_0')| \le 2|V(G_0')| - 4 = 2 + 2|S_t| + 2|V_0|$. Then

$$\begin{aligned} 2 + 2|S_t| + 2|V_0| &\ge 1 + 3|S_1| + 3|V_0|; \\ 1 &\ge |S_1| + |V_0|. \end{aligned}$$

Hence, $|V(G_0')| = |S_t| + |S_1| + |V_0| \le 4$, contrary to $|V(G_0')| \ge 5$. Claim 1 is proved.

Then $|S_t| \le 2$. If U_0 has a vertex x, then $d_{G_0'}(x) \ge 3$ and $N_{G_0'}(x) \subseteq S_t$, a contradiction. Thus, $U_0 = \emptyset$ and $V(G_0') = S_t \cup S_1 \cup V_E$. By (7), $|V(G_0')| = |S_t| + |S_1 \cup V_E| \le 2 + 6 = 8$.

By Theorem 3.1, $|D_2(G_0')| \ge 3$. Since $D_2(G_0') \subseteq S_t \cup S_1$, there is a vertex v_1 in $D_2(G_0') \cap S_1$. Since $\overline{\sigma_2}(G) \ge 5$, $|E(\Gamma(v_1))| \ge 2$. By (7), $|\cup_{v \in S_1} E(\Gamma(v))| \le 3$. Thus, $|D_2(G_0') \cap S_1| = 1$, $|S_t| = 2$, $|E_0| \le 1$ and $|S_1| \le 2$. Then $|S_t| + |S_1| \le 4$. Since $|V(G_0')| \ge 5$, $|V_0| \ge 1$.

Let v be a vertex in V_0. Then $d_{G_0'}(v) \ge 3$. Since $|S_t| = 2$ and $|E_0| \le 1$, v is adjacent to exactly one vertex u in $S_1 \cup V_0$ as well as the two vertices in S_t.

However, u must be also adjacent to at least one of the two vertices in S_t, G_0' contains a K_3, contrary to that G_0' is K_3-free. The proof is complete. □

To prove Theorems 2.5 and 2.6, we need the following theorem in which P is the Petersen graph.

Theorem 4.1 *([6]). Let G be a 3-edge-connected graph and let $S \subseteq V(G)$ be a vertex subset with $|S| \leq 12$. Then either G has a closed trail C such that $S \subseteq V(C)$, or G can be contracted to P in such a way that the preimage of each vertex of P contains at least one vertex in S.*

Suppose that G_0' is contracted to P. For each $v \in V(P)$, we use $\Gamma_P(v)$ as the preimage of v in G_0'.

Lemma 4.2 *Let G be an essentially 3-edge-connected graph and let G_0' be the reduction of the core of G. Suppose that G_0' can be contracted to P. For a vertex v in P, if $\Gamma_P(v) \neq K_1$, then $\kappa'(\Gamma_P(v)) \geq 2$ and $|D_2(\Gamma_P(v))| \leq 3$.*

Proof. Since G is essentially 3-edge-connected, by Theorem 1.6 $\kappa'(G_0') \geq 3$. Since $d_P(v) = 3$, only three edges join $\Gamma_P(v)$ to $G_0' - V(\Gamma_P(v))$, $\Gamma_P(v)$ must be 2-edge-connected and $|D_2(\Gamma_P(v))| \leq 3$. □

Lemma 4.3 *Let G be a K_3-free graph. Let $\Theta_i \cong K_{1,t}$ $(t \geq 2)$ be a subgraph of G $(i = 1, 2)$ and $V(\Theta_1) \cap V(\Theta_2) = \emptyset$. Let $E(\Theta_i) = \{e_1^i, \cdots, e_t^i\}$. Then $|(\bigcup_{j=1}^t E_G(e_j^1)) \cap (\bigcup_{j=1}^t E_G(e_j^2))| \leq t^2 + 1$.*

Proof. Each edge in $(\bigcup_{j=1}^t E_G(e_j^1)) \cap (\bigcup_{j=1}^t E_G(e_j^2))$ has one end in $V(\Theta_1)$ and the other end in $V(\Theta_2)$. Let $\Theta = G[V(\Theta_1) \cup V(\Theta_2)]$. Then Θ is a K_3-free graph of order $2(t+1)$ and $(\bigcup_{j=1}^t E_G(e_j^1)) \cap (\bigcup_{j=1}^t E_G(e_j^2)) \subseteq E(\Theta) - (E(\Theta_1) \cup E(\Theta_2))$. By Turán's Theorem, $|E(\Theta)| \leq (t+1)^2$. Then

$$\left|\left(\bigcup_{j=1}^t E_G(e_j^1)\right) \cap \left(\bigcup_{j=1}^t E_G(e_j^2)\right)\right| \leq |E(\Theta)| - |E(\Theta_1) \cup E(\Theta_2)| \leq (t+1)^2 - 2t = t^2 + 1.$$

□

In the following, we assume that G is a graph satisfying Theorem 2.3.

Lemma 4.4 *Suppose that G does not have a DCT and $G_0' = P$ and $t \in \{2, 3\}$. For $v \in V(P)$, let $\Gamma(v)$ be the preimage of v in G.*

(a) If $\Gamma(v)$ is not a tree, then $|E(\Gamma(v))| \geq 2\delta_t(H) - t^2 - 4$.

(b) If $|S_t| = p/t$, then $\Gamma(v) = K_{1,s_v}$ with $s_v \geq \delta_t(H) - 3$ for each $v \in S_t$.

(c) If $|S_t| = p/t - 1$, there is at most one vertex u in $G_0' = P$ such that $\Gamma(u)$ is not a tree.

(d) $V_0 - S^* = \emptyset$ and so $V_E - S^* = \emptyset$

Proof. (a) We prove the case $t = 3$ only (it is easier to prove the case $t = 2$). Then $\delta_3(H) \geq \frac{3(n+\epsilon)}{p}$.

Since G is K_3-free, if $\Gamma(v)$ is not a tree then it contains a cycle of at least length 4. Let $C = v_1e_1v_2e_2v_3e_3v_4e_4\cdots v_1$ be a cycle in $\Gamma(v)$.

Since $|\cup_{i=1}^{3} E_G(e_i)| \geq \delta_3(H)$, $\max\{|E_G(e_1)|, |E_G(e_2)|, |E_G(e_3)|\} \geq \frac{\delta_3(H)}{3} \geq \frac{n+\epsilon}{p}$. Without loss of generality, we assume $|E_G(e_1)| = \max\{|E_G(e_1)|, |E_G(e_2)|, |E_G(e_3)|\}$ and $d_G(v_1) \geq d_G(v_2)$. Then since $|N_G(v_1)| = d_G(v_1) \geq (d_G(v_1) + d_G(v_2))/2$

$$|N_G(v_1)| \geq \frac{d_G(v_1) + d_G(v_2)}{2} = \frac{|E_G(e_1)| + 2}{2} \geq \frac{\delta_3(H)}{6} + 1 \geq \frac{n + \epsilon + 2p}{2p}. \tag{8}$$

We need another vertex $u \neq v_1$ that satisfies (8).

If C is a cycle of length at least 5, then there is a vertex $u \in \{v_2, v_3, v_4, v_5, \cdots\}$ such that $|N_G(u)| \geq \frac{n+\epsilon+2p}{2p}$.

Suppose that $C = v_1e_1v_2e_2v_3e_3v_4e_4v_1$, a cycle of length 4.

If one of the vertices in $\{v_2, v_3, v_4\}$ (say v_2) is not incident with any edges in $E(G_0') = E(P)$, then since $d_{G_0}(v_i) \geq 3$ there is an edge (say $e_u = uv_2$) in $E(\Gamma(v)) - E(C)$ incident with v_2 in $\Gamma(v)$.

If every vertex in $\{v_2, v_3, v_4\}$ is incident with an edge in $E(P)$, then since $d_{G_0'}(v) = 3$, v_1 is not incident with any edges in $E(P)$. Since G is essentially 3-edge-connected, $\Gamma(v) - \{e_1, e_4\}$ is connected. Thus, there is a path joining v_1 to $C - \{e_1, e_4\}$ and so there is an edge (say $e_u = uv_2$) in $E(\Gamma(v)) - E(C)$ incident with a vertex in $\{v_2, v_3, v_4\}$.

Thus, in either case, we have $|E_G(e_u) \cup E_G(e_2) \cup E_G(e_3)| \geq \delta_3(H)$. Similarly to the way we obtained (8), we have a vertex in $\{u, v_2, v_3, v_4\}$ (say u) such that

$$|N_G(u)| \geq \frac{n + \epsilon + 2p}{2p}.$$

Thus, when n is large enough (say $n > 10p - \epsilon$), there are three edges incident with v_1 in $\Gamma(v)$ (say $e_i^v = x_iv_1$, $i = 1, 2, 3$) and there are another three edges incident with u in $\Gamma(v)$ (say $e_i^u = a_iu$).

By Lemma 4.3, $|\left(\cup_{i=1}^{3} E_G(e_i^v)\right) \cap \left(\cup_{i=1}^{3} E_G(e_i^u)\right)| \leq t^2 + 1$. Since $(\cup_{i=1}^{3} E_G(e_i^v)) \cup (\cup_{i=1}^{3} E_G(e_i^u)) \subseteq E(\Gamma(v)) \cup E_{G_0'}(v)$ and $|E_{G_0'}(v)| = 3$,

$$\begin{aligned} 2\delta_3(H) - t^2 - 1 &\leq |\cup_{i=1}^{3} E_G(e_i^v)| + |\cup_{i=1}^{3} E_G(e_i^u)| - \left|\left(\cup_{i=1}^{3} E_G(e_i^v)\right) \cap \left(\cup_{i=1}^{3} E_G(e_i^u)\right)\right| \\ &\leq |E(\Gamma(v))| + |E_{G_0'}(v)|; \\ 2\delta_3(H) - t^2 - 4 &\leq |E(\Gamma(v))|. \end{aligned}$$

Thus (a) is proved.

(b) By Lemma 3.3, for each $v \in S_t$, $|E(\Gamma(v))| \geq \delta_t(H) - d_{G_0'}(v) = \delta_t(H) - 3$. To the contrary, suppose that there is a vertex $w \in S_t$ such that $\Gamma(w)$ is not a tree. By (a) above, $|E(\Gamma(w))| \geq 2\delta_t(H) - t^2 - 4$. Since $\delta_t(H) \geq \frac{t(n+\epsilon)}{p}$, $|S_t| = p/t$ and $|E(P)| = 15$.

$$n = |E(G)| = |E(\Gamma(w))| + \sum_{v \in S_t - \{w\}} |E(\Gamma(v))| + \sum_{v \in S_1} |E(\Gamma(v))| + |E(P)|;$$

$$n \geq 2\delta_t(H) - t^2 - 4 + (|S_t| - 1)(\delta_t(H) - 3) + 15 = (|S_t| + 1)\delta_t(H) - t^2 - 3|S_t| + 14;$$

$$n \geq (n + \epsilon) + \frac{t(n + \epsilon)}{p} - t^2 - 3p/t + 14,$$

a contradiction, since $n > N(p, \epsilon) > \frac{p}{t}(t^2 + 3\frac{p}{t} - 14 - \epsilon) - \epsilon$ where $t \in \{2, 3\}$.

Thus, for each $v \in S_t$, $\Gamma(v)$ is a tree. Since G is essentially 3-edge-connected, $\Gamma(v) = K_{1,s_v}$ with $s_v = |E(\Gamma(v))| \geq \delta_t(H) - 3$. (b) is proved.

(c) The proof is very similar to case (b) above. Hence, we skip the details here.

(d) Since G does not have a DCT, G_0' does not have a closed trail containing all the nontrivial vertices. Since for any given 9 vertices, $G_0' = P$ has a cycle containing them, all the vertices in $V(G_0')$ must be nontrivial. Thus, $V_0 - S^* = V(G_0') - (S_t \cup S_1 \cup S^*)$ must be empty. □

For $t \leq 3$ and S^* defined in Section 3 above, we have

Lemma 4.5 *Let $p \geq 8$. If $t = 2$, then $S^* = \emptyset$. If $t = 3$, then $|S^*| \leq 1$. If $|S^*| = 1$, then $|S_1| = 1$.*

Proof. By Lemma 3.2, for a vertex $v \in S^*$, there is a vertex $u \in N_{G_0'}(v) \cap S_1$ such that $|E(\Gamma(u))| = 1$ and $E(\Gamma(u))$ only contains an edge $xu \in X_2(G)$ such that $x \in D_2(G)$ and is adjacent to v in G. Then $d_G(v) = d_{G_0'}(v)$ and $d_G(u) = d_{G_0'}(u)$ (we regard u as a vertex in G as well as in G_0'). Let $e_1 = vx$ and $e_2 = xu$. Then $|E_G(e_1) \cup E_G(e_2)| = d_G(v) + d_G(u) = d_{G_0'}(v) + d_{G_0'}(u)$. Since G_0' is K_3-free and 2-edge-connected, for any $z \in V(G_0')$, by Theorem 2.3 with $t \in \{2, 3\}$ and $p \geq 8$,

$$d_{G_0'}(z) \leq |V(G_0')| - 2 \leq \max\{p/t + 2t, 3p/t + 2t - 7\} - 2 \leq \frac{3p}{2} - 5. \quad (9)$$

If $t = 2$, then by (9) and (3) with $t = 2$, $2(\frac{3p}{2} - 5) \geq d_{G_0'}(v) + d_{G_0'}(u) = |E_G(e_1) \cup E_G(e_2)| \geq \delta_2(H) \geq \frac{2(n+\epsilon)}{p}$, a contradiction when $n > N(p, \epsilon) \geq p(\frac{3p}{2} - 5) - \epsilon$. Thus, $S^* = \emptyset$ if $t = 2$.

For $t = 3$ case, suppose that $S^* \neq \emptyset$. By Lemma 3.2, $S_1 \neq \emptyset$. We only need to show that $|S_1| = 1$. To the contrary, suppose that $|S_1| \geq 2$. Let $\{u, u_1\} \subseteq S^*$. Let $e_3 = u_1x_1$ be an edge in $E(\Gamma(u_1))$. Then by Lemma 3.3 with $t = 3$, $|E_G(e_3)| \leq |E(\Gamma(u_1)| + d_{G_0'}(u_1) \leq 2 + d_{G_0'}(u_1)$. Then by (9) and (3) with $t = 3$, $3(\frac{3p}{2} - 5) + 2 \geq d_{G_0'}(v) + d_{G_0'}(u) + d_{G_0'}(u_1) = |E_G(e_1) \cup E_G(e_2) \cup E_G(e_3)| \geq \delta_3(H) \geq \frac{3(n+\epsilon)}{p}$, a contradiction when $n > N(p, \epsilon) \geq p(\frac{3p}{2} - 5) + \frac{2}{3} - \epsilon$. □

In the following, we assume $t \in \{2, 3\}$. By Lemma 3.3, $|E_0| = |E(\Phi_0)| \le 2$.

Let Z be the set of vertices by selecting one end of each edge in E_0 with both ends in $V_E - (S_1 \cup S^*)$; in the case that $\Phi_0 = K_{1,2}$ and $V(\Phi_0) \subseteq V_E - (S_1 \cup S^*)$, only the center vertex is selected. Then $|Z| \le |V_E - (S_1 \cup S^*)|/2$. Define

$$V_a = S_t \cup S_1 \cup S^* \cup Z.$$

Since U_0 is an independent set and $N_{G'_0}(x) \subseteq S_t$ for each $x \in U_0$, V_a is a vertex-covering of G'_0 containing all the nontrivial vertices and

$$|S_t| + |S_1| + |S^*| \le |V_a| \le |S_t| + |S_1| + |S^*| + \frac{|V_E - (S_1 \cup S^*)|}{2}. \tag{10}$$

Proof of Theorem 2.5. Suppose that H is not Hamiltonian. Then G'_0 does not have a DCT containing all the nontrivial vertices and $\kappa'(G'_0) \ge 3$. Since $t = 2$, by Lemma 4.5, $S^* = \emptyset$. By Lemma 3.3, $|E_0| + |E_R| \le |E(\Phi^*)| \le 1$. If $|S_1| = 1$, then $E_0 = \emptyset$ and $V_E = \emptyset$. If $|S_1| = 0$, since $|E_0| \le 1$, $|V_E| \le 2$. Then $|S_1| + |S^*| + \frac{|V_E - (S_1 \cup S^*)|}{2} \le 1$. Since $t = 2$ and $p \in \{18, 20\}$, by Lemma 3.3 and (10),

$$|V_a| \le |S_t| + 1 \le p/t + 1 \le 11. \tag{11}$$

If G'_0 has a closed trail C such that $V_a \subseteq V(C)$, then by Theorem 1.6, G has a DCT, a contradiction. Thus, G'_0 does not have such closed trails C. By Theorem 4.1 and by (11), we have

$$10 \le |V_a| \le 11 \text{ and} \tag{12}$$

$$G'_0 \text{ can be contracted to } P \text{ and } V(\Gamma_P(v)) \cap V_a \ne \emptyset \text{ for each } v \text{ in } P, \tag{13}$$

where P is the Petersen graph.

Let $V_b = V(G'_0) - V_a$, which contains all the vertices in U_0 and the one vertex in $V_E - V_a$ (if $V_E \ne \emptyset$). For each $u \in V_b$, $N_{G'_0}(u) \subseteq V_a$.

(a) Since $p = 18$, $\epsilon = 12$ and $t = 2$, by Lemma 3.3, (11) and (12), $|S_t| = 9 = p/t$ and so $|V_a| = 10$. Each $\Gamma_P(v)$ contains only one vertex in V_a.

Claim 1. $V_b = \emptyset$ and so $V_E = \emptyset$.

To the contrary, let u be a vertex in V_b. Let $\Gamma_P(v)$ be the preimage of a vertex v in P containing u. Since $\Gamma_P(v)$ contains a vertex in V_a and vertex u, $\Gamma_P(v) \ne K_1$. By Lemma 4.2, $\kappa'(\Gamma_P(v)) \ge 2$. $\Gamma_P(v)$ contains at least two vertices in $N_{G'_0}(u) \subseteq V_a$, contrary to that $\Gamma_P(v)$ contains only one vertex in V_a. Claim 1 is proved.

Thus, $V(G'_0) = V_a = S_t \cup S_1$ with $|V(G'_0)| = |V_a| = 10$. By Theorem 3.1, $G'_0 = P$. By Lemma 3.3, and by $d_{G'_0}(v) = 3$ and $\delta_2(H) \ge \frac{n+12}{9}$, $|E(\Gamma(v))| \ge \delta_2(H) - d_{G'_0}(v) \ge \frac{n+12}{9} - 3$ for each $v \in S_t$. Let v_s be the vertex in S_1. Then

$|E(\Gamma(v_s))| = 1$. Therefore, with $|S_t| = 9$ and $|E(G_0')| = |E(P)| = 15$,

$$n = |E(G)| = \sum_{v \in S_t} |E(\Gamma(v))| + |E(\Gamma(v_s))| + |E(G_0')| \geq 9\left(\frac{n+12}{9} - 3\right) + 1 + 15 = n + 1,$$

a contradiction. Theorem 2.5(a) is proved.

(b) Since $p = 20$, $\epsilon = 9$ and $t = 2$, by Lemma 3.3 and (12), $|S_t| \leq 10 = p/t$ and $10 \leq |V_a| = 11$.

Let $A = \{v \in V(P) \mid \Gamma_P(v) \neq K_1\}$.

Case 1. $|A| = 0$. Then $G_0' = P$.

By Lemma 4.4, $V_0 - S^* = \emptyset$. Since $|S_1| \leq 1$ and $|S^*| = 0$, $|V(G_0')| = |V_a| = |S_t| + |S_1| = 10$ and $|S_t| \geq 9$. Either $|S_t| = 10$ or $|S_t| = 9$ and $|S_1| = 1$.

Subcase 1. $|S_t| = 10 = p/t$.

By Lemma 4.4, $\Gamma(v) = K_{1,s_v}$ for each $v \in V(P)$. Then $|E(G)| = \sum_{v \in V(P)} |E(\Gamma(v))| + |E(P)| = \sum_{v \in V(P)} s_v + 15$. This is the graph defined in Theorem 2.5(b)(i).

Subcase 2. $|S_t| = 9$ and $|S_1| = 1$.

Let u be the vertex in S_1. Since $|E(\Gamma(u))| = 1$, $\Gamma(u) = K_2$. By Lemma 4.4, there is at most one vertex $w \in V(P)$ such that $\Gamma(w)$ is not a tree.

If $\Gamma(v) = K_{1,s_v}$ for all the vertices v in S_t, then $n = |E(G)| = \sum_{v \in S_t} |E(\Gamma(v))| + |E(\Gamma(u))| + |E(P)| = \sum_{v \in V(P)-\{u\}} s_v + 16$. This is the graph defined in Theorem 2.5(b)(ii).

If there is a vertex w in S_t such that $\Gamma(w)$ is not a tree, then by Lemma 4.4 $s_w = |E(\Gamma(w))| \geq 2\left(\frac{n+9}{10}\right) - 8$ and for vertices v in $S_t - \{w\}$ $\Gamma(v) = K_{1,s_v}$. Thus, $n = |E(G)| = |E(\Gamma(w))| + \sum_{v \in S_t - \{w\}} E(\Gamma(v))| + |E(\Gamma(u))| + |E(P)| = s_w + \sum_{v \in V(P)-\{u,w\}} s_v + 16$. This is the graph defined in Theorem 2.5(b)(iii).

Case 2. $|A| \geq 1$.

Let v_1 be a vertex in A. Let $\Gamma_P(v_1)$ be the preimage of v_1 in G_0'. Then $|V(\Gamma_P(v_1))| \geq 2$. By Lemma 4.2, $\kappa'(\Gamma_P(v_1)) \geq 2$.

Since G_0' is K_3-free and $\kappa'(G_0') \geq 3$, $|V(\Gamma_P(v_1))| \geq 5$. Since $|V_a| \leq 11$ and for each $v \in V(P)$ $\Gamma_P(v)$ contains at least one vertex in V_a, $\Gamma_P(v_1)$ contains at most two vertices in V_a. There is a vertex u_1 in $V(\Gamma_P(v_1)) - V_a$. Since $d_{G_0'}(u_1) \geq 3$ and $\kappa'(\Gamma_P(v_1)) \geq 2$, $\Gamma_P(v_1)$ contains at least two vertices in $N_{G_0'}(u_1) \subseteq V_a$. Thus, $\Gamma_P(v_1)$ contains exactly two vertices in V_a.

Thus, $|A| = 1$, $|V_a| = 11$, $|S_t| = 10$ and $|S_1| = 1$, and so $|E(\Phi^*)| = 1$. This also shows that if $u \in V(\Gamma_P(v_1)) - V_a$, u is only adjacent to the two vertices in $V(\Gamma_P(v_1)) \cap V_a$, i.e., $d_{\Gamma_P(v_1)}(u) = 2$. Since only three edges join $\Gamma_P(v_1)$ to $G_0' - \Gamma_P(v_1)$, there are at most three vertices of degree two in $V(\Gamma_P(v_1)) - V_a$. Thus, $|V(\Gamma_P(v_1)) - V_a| \leq 3$ and so $|V(\Gamma_P(v_1))| = |V(\Gamma_P(v_1)) - V_a| + |V(\Gamma_P(v_1)) \cap V_a| \leq 5$. Thus, $|V(\Gamma_P(v_1))| = 5$ and $|D_2(\Gamma_P(v_1))| = 3$. By Theorem 3.1, $\Gamma_P(v_1) = K_{2,3}$.

Hence, $G'_0 = P_{14}$. Then for each $v \in S_t$, $|E(\Gamma(v))| \geq \frac{n+9}{10} - 3$. Since $|S_t| = 10$ and $|E(\Phi^*)| = 1$,

$$n = |E(G)| \geq \sum_{v \in S_t} |E(\Gamma(v))| + |E(\Phi^*)| + |E(P_{14})| \geq 10\left(\frac{n+9}{10} - 3\right) + 22 = n + 1,$$

a contradiction. This shows that Case 2 is impossible. The proof is completed. □

Proof of Theorem 2.6. Since $t = 3$ and $p = 24$ or $p = 27$, $p/t = 8$ or 9.

By Lemma 3.3, $|E_0| + |E_R| \leq |E(\Phi^*)| \leq 2$. By Lemma 4.5, if $|S_1| = 2$, then $E_0 = \emptyset$, $|V_E| = 0$ and $|S^*| = 0$; if $|S_1| = 1$ then $|E_0| \leq 1$ and so $|S^*| + \frac{|V_E - (S_1 \cup S^*)|}{2} \leq 2$; if $|S_1| = 0$ then $|E_0| \leq 2$, $|S^*| = 0$ and so $|S^*| + \frac{|V_E - (S_1 \cup S^*)|}{2} \leq 2$. Thus, $|S_1| + |S^*| + \frac{|V_E - (S_1 \cup S^*)|}{2} \leq 2$. By (10),

$$|V_a| \leq |S_t| + 2 \leq p/t + 2 \leq 11. \tag{14}$$

Let $V_b = V(G'_0) - V_a$. Thus, for each vertex $u \in V_b$, $N_{G'_0}(u) \subseteq V_a$.

Similar to the proof of Theorem 2.5, G'_0 does not have a DCT containing V_a and we have

$$10 \leq |V_a| \leq |S_t| + 2 \leq 11 \text{ and} \tag{15}$$

$$G'_0 \text{ can be contracted to } P \text{ and } V(\Gamma_P(v)) \cap V_a \neq \emptyset \text{ for each } v \text{ in } P. \tag{16}$$

(a) In this case, $t = 3$, $p = 24$ and $\epsilon = 9$. By Lemma 3.3 and (15), $|S_t| = 8$ and $|V_a| = 10$. Thus, each $\Gamma_P(v)$ contains exactly one vertex of V_a. By Lemma 3.3, $|E(\Gamma(v))| \geq \frac{n+9}{8} - 3$ for each $v \in S_t$.

Following the same arguments in the proof of Claim 1 in Theorem 2.5, we have $V_b = \emptyset$. Then $|V(G'_0)| = |V_a| = 10$. By Theorem 3.1, $G'_0 = P$.

If $S_1 \neq \emptyset$, then $|E_R| = \sum_{v \in S_1} |E(\Gamma(v))| \geq 1$. Hence,

$$n = |E(G)| = \sum_{v \in S_t} |E(\Gamma(v))| + \sum_{v \in S_1} |E(\Gamma(v))| + |E(G'_0)| \geq 8\left(\frac{n+9}{8} - 3\right) + 1 + 15 = n + 1,$$

a contradiction.

Thus, $S_1 = \emptyset$. By Lemma 4.5, $S^* = \emptyset$. By Lemma 4.4(d), $V_0 = V_0 - S^* = \emptyset$. However, since $|S_t| = 8$ and $|V(G'_0)| = |V_a| = 10$, $V_0 = V(G'_0) - S_t \neq \emptyset$, a contradiction. Theorem 2.6(a) is proved.

(b) In this case we have $p = 27$, $\epsilon = 5$ and $t = 3$. By Lemma 3.3 and (15), $|S_t| \leq 9 = p/t$ and $10 \leq |V_a| \leq 11$. For each $v \in S_t$, by Lemma 3.3, $|E(\Gamma(v))| \geq \frac{n+6}{9} - 3$.

Let $A = \{v \in V(P) \mid \Gamma_P(v) \neq K_1\}$.

Case 1. $|A| = 0$. Then $G'_0 = P$ and $|V(G'_0)| = |V_a|$. By Lemma 4.4, $V_E - S^* = \emptyset$. By (10), $|V_a| = |S_t| + |S_1| + |S^*| = 10$. Since $|S_1| + |S^*| \le 2$, $9 \ge |S_t| \ge 8$. We have three subcases.

Subcase 1. $|S_t| = 9$ and $|S_1| = 1$.

Let v_1 be the vertex in S_1. Then $\Gamma(v_1) = K_{1,s_1}$ where $1 \le s_1 \le 2$. By Lemma 4.4, for each $v \in S_t$, $\Gamma(v) = K_{1,s_v}$ with $s_v = |E(\Gamma(v))| \ge \frac{n+6}{9} - 3$. Then $n = |E(G)| = \sum_{v\in S_t} E(\Gamma(v))| + |E(\Gamma(v_1))| + |E(P)| = \sum_{v\in S_t} s_v + s_1 + 15$. This is the graph defined in Theorem 2.6(b)(i).

Subcase 2. $|S_t| = 8$ and $|S_1| = 2$.

Let $S_1 = \{v_1, v_2\}$. Since $|E(\Phi^*)| \le t - 1 = 2$, $|E(\Gamma(v_i))| = 1$ $(i = 1, 2)$.

By Lemma 4.4(c), there is at most one vertex $w \in S_t \subseteq V(P)$ such that $\Gamma(w)$ is not a tree.

If for all the vertices v in S_t $\Gamma(v)$ is a tree, then $\Gamma(v) = K_{1,s_v}$ and so $n = |E(G)| = \sum_{v\in S_t} E(\Gamma(v))| + \sum_{i=1}^{2} |E(\Gamma(v_i)| + |E(P)| = \sum_{v\in V(P)-\{u\}} s_v + 17$. This is the graph defined in Theorem 2.6(b)(ii).

If there is a vertex w in S_t such that $\Gamma(w)$ is not a tree, then $s_w = |E(\Gamma(w))| \ge 2\left(\frac{n+6}{9}\right) - 13$ and for all vertices v in $S_t - \{w\}$, $\Gamma(v) = K_{1,s_v}$ and so $n = |E(G)| = |E(\Gamma(w))| + \sum_{v\in S_t-\{w\}} E(\Gamma(v))| + \sum_{i=1}^{2} |E(\Gamma(v_i)| + |E(P)| = s_w + \sum_{v\in V(P)-\{u,w\}} s_v + 17$. This is the graph defined in Theorem 2.6(b)(iii).

Subcase 3. $|S_t| = 8$, $|S_1| = 1$ and $|S^*| = 1$.

This is similar to Subcase 1 above. In this case we have $\Gamma(v_1) = K_{1,s_1} = K_2$ where $s_1 = 1$ and the edge in $E(\Gamma(v_1))$ is an edge in $X_2(G)$. This is a graph defined in Theorem 2.6(b)(i).

Case 2. $|A| \ge 1$.

Let v_1 be a vertex in A. By the same argument in the proof of Case 2 in Theorem 2.5, we have $|V_a| = 11$, $|S_t| = 9$ and $\Gamma_P(v_1) = K_{2,3}$ and so $G'_0 = P_{14}$.

Since for any 9 vertices which includes v_1 in P, P has a dominating cycle that can be extended as a dominating cycle in $G'_0 = P_{14}$. Since G'_0 does not have a DCT containing all the nontrival vertices, all the vertices in $V(P) - \{v_1\}$ must be nontrivial. By Lemma 4.5, $|S_1| \ge 1$ and so $|E(\Phi^*)| \ge 1$. Thus,

$$n = |E(G)| \ge \sum_{v\in S_t} |E(\Gamma(v))| + |E(\Phi^*)| + |E(P_{14})| \ge 9\left(\frac{n+6}{9} - 3\right) + 22 = n + 1,$$

a contradiction. Case 2 is impossible. The proof is complete. □

Remark 2 *Using the similar arguments in proofs of Theorems 2.4, 2.5 and 2.6, one can obtain new $\delta_t(H)$ conditions with other values of p or t for the hamiltonicity of k-connected claw-free graphs ($k \in \{2, 3\}$). For given t, when p is increasing,* *the number of graphs in the obstruction set will increase as well. For k = 3, in addition to the Petersen graph, P_{14} may be included in the obstruction set for larger values of p. For k = 2, the smallest graph in the obstruction set is $K_{2,3}$. Since for any given p and t the obstruction set is finite, the members of the obstruction set can be determined with the help of a computer. Thus, the problem of finding new $\delta_t(H)$ for the hamiltonicity of k-connected claw-free graphs is solvable by using computers.*

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