A KRASNOSEL'SKII-TYPE RESULT FOR ANY NON SIMPLY CONNECTED ORTHOGONAL POLYGON

MARILYN BREEN

ABSTRACT. Let S be an orthogonal polygon and let $A_1, \ldots, A_n$ represent pairwise disjoint sets, each the connected interior of an orthogonal polygon, $A_i \subseteq S, 1 \leq i \leq n$. Define $T = S \backslash (A_1 \cup \ldots \cup A_n)$. We have the following Krasnosel'skii-type result: Set T is staircase starshaped if and only if S is staircase starshaped and every 4n points of T see via staircase paths in T a common point of Ker S. Moreover, the proof offers a procedure to select a particular collection of 4n points of T such that the subset of Ker S seen by these 4n points is exactly Ker T. When $n = 1$, the number 4 is best possible.

1. Introduction

We begin with some definitions and comments that also appear in [2]. A set B in $\mathbb{R}^d$ is called a *box* if and only if B is a convex polytope (possibly degenerate) whose edges are parallel to the coordinate axes. A nonempty set S in $\mathbb{R}^d$ is an *orthogonal polytope* if and only if S is a connected union of finitely many boxes. An orthogonal polytope in $\mathbb{R}^2$ is an *orthogonal polygon.* Let λ be a simple polygonal path in $\mathbb{R}^d$ whose edges are parallel to the coordinate axes. That is, let λ be a simple rectilinear path in $\mathbb{R}^d$. For points x and y in S, the path λ is called an *$x - y$ path* if and only if λ lies in S and has endpoints x and y. The $x - y$ path λ is a *staircase path* (or simply a *(staircase)* if and only if, as we travel along λ from x to y, no two edges of λ have opposite directions. That is, for each standard basis vector $e_i, 1 \leq i \leq d$, either each edge of λ parallel to e_i is a positive multiple of e_i or each edge of λ parallel to e_i is a negative multiple of e_i. In the plane, an edge (or subset of an edge) $[v_{i-1}, v_i]$ of path λ will be called *north*, *south*, *east*, or *west* according to the direction of vector $\overrightarrow{v_{i-1}v_i}$. Similarly, we use the terms north, south, east, west, northeast, northwest, southeast, southwest to describe the relative position of points.

Mathematics Subject Classification: Primary 52A30, 52A35.

Keywords: Orthogonal polygons, staircase starshaped sets, Krasnosel'skii-type theorems.

For points x and y in a set S, we say x *sees* y (x is *visible* from y) *via staircase paths* if and only if S contains an $x-y$ staircase path. For $W \subseteq S$, the set of all points seen (via staircase paths in S) by every point of W is the *common visibility set* of W. A set S is *staircase convex (orthogonally convex)* if and only if, for every pair of points x, y in S, x sees y via staircase paths. Similarly, a set S is *staircase starshaped (orthogonally starshaped)* if and only if, for some point p in S, p sees each point of S via staircase paths. The set of all such points p is the *staircase kernel* of S, Ker S.

Many results in convexity that involve the usual notion of visibility via straight line segments have interesting analogues that instead employ the idea of visibility via staircase paths. For instance, the familiar Krasnosel'skii theorem [8] says that, for S a nonempty compact set in the plane, S is starshaped via segments if and only if every three points of S see via segments in S a common point. In the staircase analogue [1], for S a nonempty simply connected orthogonal polygon, S is staircase starshaped if and only if every two points of S see via staircase paths in S a common point. Concerning the kernel itself, when S is starshaped via segments, it is easy to show that the corresponding kernel is a convex set. Similarly, when the simply connected orthogonal polygon S is staircase starshaped, then its staircase kernel will be staircase convex. Moreover, a discussion in [4] describes an easy way to locate the staircase kernel of the simply connected orthogonal polygon S. Without the simple connectedness requirement, every component of the staircase kernel will be staircase convex, as [4, Theorem 2] reveals.

To extend these results, let S be a simply connected orthogonal polygon, and let $A_1, \ldots, A_n$ represent pairwise disjoint sets, each the connected interior of an orthogonal polygon, with $A_i \subseteq S, 1 \leq i \leq n$. Define $T = S \backslash (A_1 \cup \ldots \cup A_n)$. Theorems from [2] explore the relationship between Ker S and Ker T and obtain an upper bound on the number of components of Ker T in terms of n. Considering the special case in which the sets A_i are open boxes, [3] has provided a Krasnosel'skii-type result for T. Specifically, such a set T is staircase starshaped if and only if every $4n$ points of T see via staircase paths in T a common point of Ker S. Furthermore, [3, Example 2] demonstrates that, independent of n, no Krasnosel'skii-type number exists for T.

In this paper we extend the Krasnosel'skii-type bound mentioned above to an arbitrary orthogonal polygon T, eliminating any restriction on the shape of the open sets A_i. In addition, the constructive argument offers a procedure that allows us to locate a particular collection of $4n$ points of T such that the subset of Ker S seen by these $4n$ points is exactly Ker T. Since[4] has shown that Ker S is relatively simple to find, the new results yield a straightforward method to locate the staircase kernel of any orthogonal polygon. When $n = 1$, the number 4 is best possible.

Throughout the paper, for S a set in $\mathbb{R}^2$, $bdry\, S$, $int\, S$, $rel\ int\, S$, and $cl\, S$ will represent the boundary, the interior, the relative interior, and the closure, respectively, for set S. If λ is a simple path containing points a and b, then $\lambda(a, b)$ will denote the subpath of λ from a to b, ordered from a to b.

Readers may refer to Valentine [10], to Lay [9], to Danzer, Grünbaum, Klee [6], and to Eckhoff [7] for discussions concerning visibility via straight line segments and starshaped sets.

2. The Results.

We begin with some preliminary material.

Preliminary notation, definitions, and observations:

Let S be an orthogonal polygon, and let A represent the connected interior of an orthogonal polygon, with $A \subseteq S$. Let D denote the union of all horizontal and vertical lines that meet A. We define D_N to be the subset of D consisting of all points of $D \backslash A$ that lie strictly north of some point of A. Analogously, define sets D_S, D_E, and D_W consisting of all points of $D \backslash A$ that lie strictly south, east, and west, respectively, of some point of A. (Of course, these sets need not be pairwise disjoint.)

For e an edge of $cl\, A$, we say that e is *north facing* if and only if points of A near $rel\ int\, e$ are north of e. That is, for every $x \in rel\ int\, e$, every north vector at x meets A. Parallel definitions hold for *south, east,* and *west facing edges* of $cl\, A$. Clearly all four edge types exist for each A. Certainly no relatively interior point of a north facing edge of $cl\, A$ sees any point of D_N via staircase paths in $S \backslash A$. Analogous statements hold for south, east, west facing edges and D_S, D_E, D_W, respectively. (See edges n, s, w in Figure 1 for examples of a north facing edge, a south facing edge, and a west facing edge, respectively, of the set $cl\, A$.)

Let B represent the smallest box containing A. Then $\mathbb{R}^2 \backslash D$ is a union of four closed (and pairwise disjoint) regions R_1, R_2, R_3, R_4, one at each vertex of B. For convenience of notation, assume R_1 is northwest of B, R_2 southwest, R_3 southeast, R_4 northeast.

Preliminary results.

The following results from [2] will be helpful. The first appears in [2, Lemma 1], the second and third in [2, Theorem 1] and its proof, and the fourth in [2, Theorem 2].

Result 1. With our preliminary notation, Ker $(S\backslash A) \subseteq$ Ker S.

Result 2. With our preliminary notation, let y belong to $S\backslash A$ and let x, z belong to $R_1\cap$ (Ker S). If x sees y via a staircase path in $S\backslash A$, then z sees y via such a path as well.

Result 3. With our preliminary notation, for each $j, 1 \le j \le 4, R_j\cap$ (Ker S) is either disjoint from Ker $(S\backslash A)$ or a subset of Ker $(S\backslash A)$. Furthermore, Ker $(S\backslash A)$ is exactly the union of those sets $R_j\cap$ (Ker S)that lie in Ker $(S\backslash A)$.

Result 4. Let S be an orthogonal polygon, with pairwise disjoint sets $A_1, \ldots, A_n$ each the connected interior of an orthogonal polygon, $A_i \subseteq S, 1 \le i \le n$. Then Ker $(S\backslash(A_i \cup \ldots \cup A_n)) = \cap\{$ Ker $(S\backslash A_i) : 1 \le i \le n\}$.

We are ready for a lemma.

Lemma 1. *Using our preliminary notation, let point x belong to $R_1\cap$ (Ker S). Then x belongs to Ker $(S\backslash A)$ if and only if x sees via staircase paths in $S\backslash A$ every point of every north facing edge of cl A. A similar statement holds for every west facing edge of cl A.*

Proof. If x belongs to Ker $(S\backslash A)$, then certainly x sees via staircase paths in $S\backslash A$ every point of every edge of *cl* A. For the converse, assume that x sees via $S\backslash A$ every point of every north facing edge of *cl* A to prove that $x \epsilon$ Ker $(S\backslash A)$. That is, for $y \epsilon S\backslash A$, show that x sees y via staircase paths in $S\backslash A$. If y is not south, east, or southeast of points in A, then any $x - y$ staircase in S will provide a suitable staircase in $S\backslash A$. Hence we assume that y is south, east, or southeast of points in A. Consider any $x - y$ staircase $\lambda(x, y)$ in S. If $\lambda(x, y)$ does not meet A, then again we have a suitable $x - y$ path in $S\backslash A$. Otherwise, $\lambda(x, y) \cap A$ will be a finite union of relatively open segments. Following the order on λ from x to y, let z denote the last endpoint of the last of these segments. Then $z \epsilon$ *bdry* A and $\lambda(z, y) \subseteq S\backslash A$.

We will show that x sees z via staircase paths in $S\backslash A$. Again following the order on λ from x to y, certainly the edges of $\lambda(x, y)$ are all south or east vectors. Thus z lies either on a north facing edge in *bdry* A or on a west facing edge in *bdry* A. If z lies on a north facing edge in *bdry* A, then by hypothesis x sees z via a staircase in $S\backslash A$, the desired result. Hence we assume that z is on a west facing edge w in *bdry* A and not on a north facing edge. Observe that, if z is an endpoint of edge w, then z lies on a south facing edge s in *bdry* A. Since points of $\lambda(x, z)$ immediately preceding z lie in A, s must lie east of w, so z is the south endpoint of w. Of course, if z is not an endpoint of w, then z is relatively interior to w. We assert that x sees z by a path that travels north and east of A in $S \backslash A$: Otherwise, there would be points of A north of z. But this situation would require points of edge n of A to lie northeast of z, and x could not see such an edge via staircase paths in $S \backslash A$, contradicting our hypothesis. (See Figure 1.)

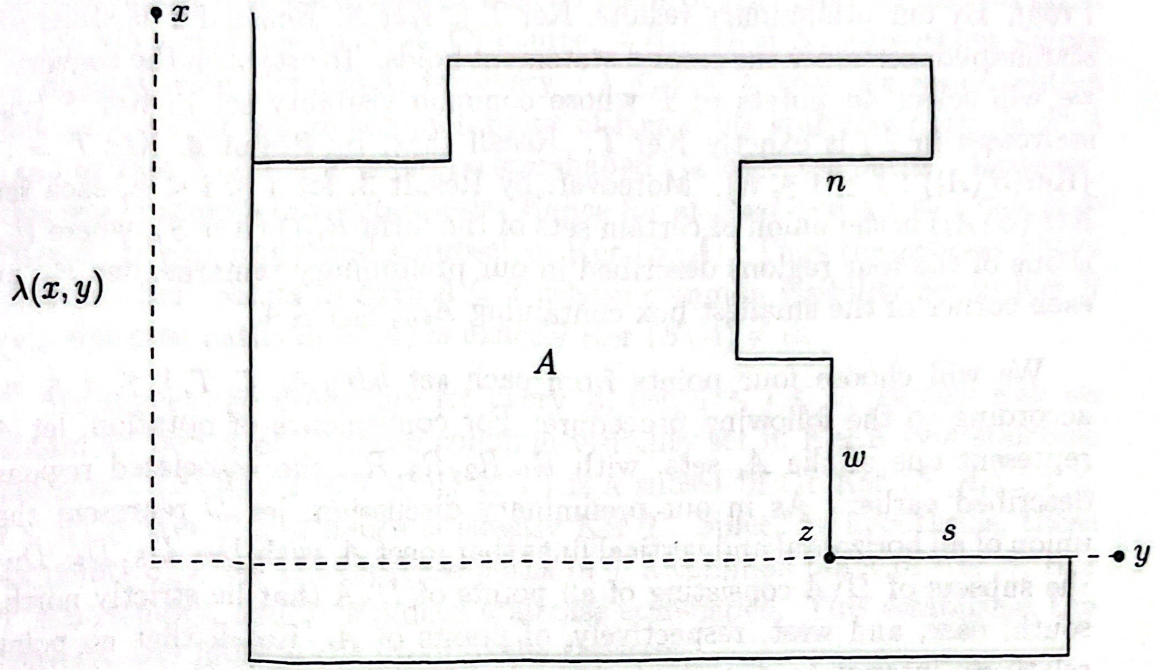

FIGURE 1. Point x cannot see point z via staircase paths in $S \backslash A$

We conclude that x does indeed see z via a staircase $\mu(x, z)$ in $S \backslash A$, again the desired result. Moreover, any two $x - z$ staircases employ vectors in exactly the same directions, namely south and east. Since $\lambda(x, y) = \lambda(x, z) \cup \lambda(z, y)$ is a staircase, so is $\mu(x, z) \cup \lambda(z, y)$, supplying an $x - y$ staircase in $S \backslash A$. Therefore, x sees each point y of $S \backslash A$ via staircase paths in $S \backslash A$, and $x \in \text{Ker}\ (S \backslash A)$.

The argument for west facing edges of *cl* A is similar to the argument above, so we omit the details. This finishes the proof of Lemma 1.

Corollary. An analogue of Lemma 1 holds for each $R_i, 2 \leq i \leq 4$, using south and west facing edges for R_2, south and east for R_3, north and east for R_4.

We have the following Krasnosel'skii-type result.

Theorem 1. *Let S be an orthogonal polygon, with pairwise disjoint sets $A_1, \ldots, A_n$ each the connected interior of an orthogonal polygon, $A_i \subseteq S, 1 \leq i \leq n$. Let $T = S \backslash (A_1 \cup \ldots \cup A_n)$. Set T is staircase starshaped if and only if set S is staircase starshaped and every $4n$ points of T see via staircase paths in T a common point of Ker S. Moreover, some $4n$ points may be selected so that the subset of Ker S seen (via staircase paths in T) by all these points is exactly Ker T.*

Proof. By our preliminary results, Ker $T \subseteq$ Ker S. Hence if T is staircase starshaped, certainly the second statement holds. To establish the converse, we will select $4n$ points of T whose common visibility set in Ker S (via staircases in T) is exactly Ker T. Recall that, by Result 4, Ker $T = \cap$ $\{\text{Ker}(S\backslash A_i) : 1 \leq i \leq n\}$. Moreover, by Result 3, for $1 \leq i \leq n$, each set Ker $(S\backslash A_i)$ is the union of certain sets of the form $R_{ij} \cap$ (Ker S), where R_{ij} is one of the four regions described in our preliminary remarks, one R_{ij} at each corner of the smallest box containing $A_i, 1 \leq j \leq 4$.

We will choose four points from each set *bdry* $A_i \subseteq T, 1 \leq i \leq n$, according to the following procedure: For convenience of notation, let A represent one of the A_i sets, with R_1, R_2, R_3, R_4, the associated regions described earlier. As in our preliminary discussion, let D represent the union of all horizontal and vertical lines that meet A, with D_N, D_S, D_E, D_W the subsets of $D\backslash A$ consisting of all points of $D\backslash A$ that lie strictly north, south, east, and west, respectively, of points of A. Recall that no point relatively interior to a north facing edge of *cl* A sees via staircase paths in $S\backslash A$ any point of D_N. Select a_N relatively interior to such an edge. Similarly, select a_S, a_E, a_W relatively interior to a south, an east, and a west facing edge of *cl* A, respectively. Then a_N, a_S, a_E, a_W see via staircase paths in $S\backslash A$ only points of $R_1 \cup R_2 \cup R_3 \cup R_4$. Hence the only points of Ker S seen by a_N, a_S, a_E, a_W via staircase paths in $S\backslash A$ must belong to $\cup\{R_i \cap \text{(Ker } S) : 1 \leq i \leq 4\}$.

Recall that, if $R_1 \cap$ (Ker S) $\neq \phi$, then $R_1 \cap$ (Ker S) is either disjoint from Ker $(S\backslash A)$ or a subset of Ker $(S\backslash A)$. In case $R_1 \cap$ (Ker S) is nonempty and disjoint from Ker $(S\backslash A)$, choose x_1 in $R_1 \cap$ (Ker S). By Lemma 1, there is some boundary point y_1 on a north facing edge of *cl* A such that x_1 cannot see y_1 via a staircase in $S\backslash A$. Since (by an argument in [5, Lemma 1]) the visibility set of x_1 in $S\backslash A$ is closed, we may choose y_1 relatively interior to a north facing edge of *cl* A. Moreover, by Result 2, no point of $R_1 \cap$ (Ker S) sees y_1 via staircase paths in $S\backslash A$. Hence y_1 sees no point of $R_1 \cap$ (Ker S) as well as no point of D_N via such paths. We exchange a_N for y_1. In case $R_1\cap$ (Ker S) is empty or $R_1\cap$ (Ker S) $\subseteq$ Ker ($S\backslash A$), we leave a_N as our selected point.

Continue the process for R_2, R_3, R_4. Specifically, if $R_2\cap$ (Ker S) is nonempty and disjoint from Ker ($S\backslash A$), we swap a_W for a point y_2 relatively interior to a west facing edge of *cl* A such that y_2 sees no point of $R_2\cap$ (Ker S) and no point of D_W via a staircase in $S\backslash A$. If $R_3\cap$ (Ker S) is nonempty and disjoint from Ker ($S\backslash A$), we swap a_S for y_3 relatively interior to a south facing edge of *cl* A such that y_3 sees no point of $R_3\cap$ (Ker S)and no point of D_S via a staircase in $S\backslash A$. Finally, if $R_4\cap$ (Ker S) is nonempty and disjoint from Ker ($S\backslash A$), we swap a_E for some y_4 relatively interior to an east facing edge such that y_4 sees no point of $R_4\cap$ (Ker S) and no point of D_E via a staircase in $S\backslash A$. Of course, if $R_j\cap$ (Ker S) were either empty or disjoint from Ker ($S\backslash A$) for every $j, 1 \leq j \leq 4$, then we would obtain four points that see no common point of Ker S via staircase paths in $S\backslash A$ (and neither $S\backslash A$ nor T could be starshaped via staircase paths). However, this would violate our hypothesis. Hence for at least one $j, 1 \leq j \leq 4, R_j\cap$ (Ker S) is nonempty and a subset of Ker ($S\backslash A$). Thus the process above produces four points in *bdry* $A \subseteq T$ whose common visibility set in Ker S (via staircase paths in $S\backslash A$) is exactly Ker ($S\backslash A$) $\neq \phi$.

We repeat the procedure for every A_i set, $1 \leq i \leq n$. In this way we obtain $4n$ points of T whose common visibility set in Ker S (via staircase paths in $\cap\{S\backslash A_i : 1 \leq i \leq n\} \equiv T$) is a subset of $\cap\{$ Ker ($S\backslash A_i$) $: 1 \leq i \leq n\} \equiv$ Ker T and hence is exactly Ker T. Since, by hypothesis, these $4n$ points of T see via staircase paths in T a common point of Ker S, Ker T is nonempty, and T is indeed staircase starshaped. This establishes the converse and finishes the proof of the theorem.

□

Note: In the argument above, we could have selected y_1 on a west facing edge of *cl* A and appropriately altered subsequent choices for y_2, y_3, y_4.

As in [2], if orthogonal polygon T (not simply connected) is a union of fully two dimensional boxes, then T may be represented as the set in Theorem 1. Otherwise, T will be the union of such a set (or sets) with line segments contained in $\cup\{A_i : 1 \leq i \leq n\}$. If we replace these segments with sufficiently thin boxes, we produce a new set to which our theorem applies and whose staircase kernel is Ker T.

In conclusion, the following example shows that the Krasnosel'skii-type number $4n$ in Theorem 1 is best when $n = 1$.

Example 1. Let T represent the orthogonal polygon in Figure 2, a rectangular region S with an open set A removed. Every three points of T

see via staircase paths in T a common point of Ker $S = S$. (For example, points x,y, z see, via staircase paths in T, a common point w of R_1.) However, Ker T is empty.

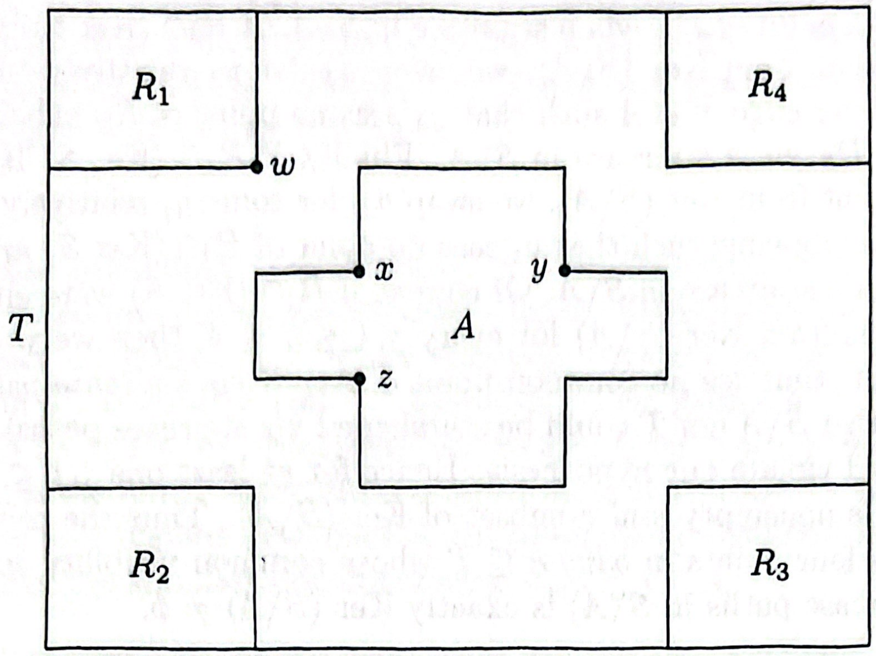

FIGURE 2. The number $4n$ is the best when $n = 1$

Although the constructive argument in Theorem 1 gives a useful technique to locate the staircase kernel of an orthogonal polygon, the Krasnosel'skii-type number $4n$ might not be best in general, leaving an open problem for future work.

REFERENCES

[1] Marilyn Breen, *An improved Krasnosel'skii-type theorem for orthogonal polygons which are starshaped via staircase paths*, Journal of Geometry **51** (1994), 31-35. .

[2] ———, *Components of the kernel in a staircase starshaped polygon*, Journal of Combinatorial Math and Combinatorial Computing (to appear).

[3] ———, *Krasnosel'skii numbers and non simply connected orthogonal polygons*, Ars Combinatoria **57** (2000) 209-216.

[4] ———, *Staircase kernels in orthogonal polygons*, Archiv der Mathematik **59** (1992), 588-594.

[5] ———, *Suitable families of boxes and kernels of staircase starshaped sets in* $\mathbb{R}^d$, Aequationes Mathematicae **87** (2014), 43-52.

[6] Ludwig Danzer, Branko Grünbaum, and Victor Klee, *Helly's theorem and its relatives*, Convexity, Proc. Sympos. Pure Math. **7** (1962), Amer. Math. Soc., Providence, RI, 101-180.

[7] Jürgen Eckhoff, *Helly, Radon, and Carathéodory type theorems*, Handbook of Convex Geometry vol. A, ed. P.M. Gruber and J.M. Wills, North Holland, New York (1993), 389-448.

[8] M. A. Krasnosel'skii, *Sur un critère pour qu'un domaine soit étoilé*, Math. Sb. (61) **19** (1946), 309-310.
[9] Steven R. Lay, *Convex Sets and Their Applications*, John Wiley, New York (1982).
[10] F. A. Valentine, *Convex Sets*, McGraw-Hill, New York (1964).

The University of Oklahoma

Norman, Oklahoma 73019

U.S.A.

Email: mbreen@ou.edu