Ascending subgraph decompositions of tournaments of order $6n + 5$

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Abstract

In 1987, Alavi, Boals, Chartrand, Erdös, and Oellermann conjectured that all graphs have an ascending subgraph decomposition (ASD). In previous papers, we showed that all tournaments of order congruent to 1, 2, or 3 mod 6 have an ASD. In this paper, we will consider the case where the tournament has order congruent to 5 mod 6.

1 Introduction

In [1], Alavi, Boals, Chartrand, Erdös, and Oellermann defined a type of graph decomposition called an ascending subgraph decomposition (ASD).

Definition 1. *A graph G with $\binom{k+1}{2} + r$ edges where $0 \leq r \leq k$ has an ascending subgraph decomposition if there exists a partition of the edge set of G such that the graphs $G_1, G_2, \ldots, G_k$ induced by the sets of edges in the partition satisfy the properties that G_i is isomorphic to a subgraph of G_{i+1} for all $1 \leq i \leq k-1$ and $|E(G_i)| = i$ for all $i = 1, 2, \ldots k$.*

For digraphs, we can define an ASD similarly.

Definition 2. *A digraph D with $\binom{k+1}{2} + r$ arcs where $0 \leq r \leq k$ has an ascending subgraph decomposition if there exists a partition of the arc set of D such that the digraphs $D_1, D_2, \ldots, D_k$ induced by the sets of arcs in the partition satisfy the properties that D_i is isomorphic to a subgraph of D_{i+1} for all $1 \leq i \leq k-1$ and $|E(D_i)| = i$ for all $i = 1, 2, \ldots k$.*

Since in this paper we consider tournaments, there will be exactly $\binom{k+1}{2}$ arcs in our digraph. Specifically, we will show that all tournaments of order $6n + 5$ for $n \geq 2$ have an ASD.

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We will also need the following definition of a 2-factorization of a graph.

Definition 3. *A graph G on N vertices has a 2-factorization if the edge set of G can partitioned into subsets of N edges where all N vertices in the subgraph induced by each set of edges in the partition have degree 2.*

For oriented graphs, we use a similar definition as above, but we refer to arcs instead of edges and the sum of the indegree and outdegree of each vertex in the induced subgraphs is 2.

See [2] for all terms and notation not specifically defined in this paper.

2 Strategy and Definitions

In [6, 7], we showed that all tournaments of orders $6n+3$, $6n+2$, or $6n+1$ have an ASD. We used Kirkman Triple Systems guaranteed by the following theorem by Ray-Chaudhuri and Wilson [4] to obtain 2-factorization of the tournament of order congruent to 3 modulo 6.

Theorem 1. (*Ray-Chaudhuri, Wilson*) *A complete graph on N vertices, K_N, has a 2-factorization into factors containing only triangles if and only if $N \equiv 3 (\text{mod } 6)$.*

We used the 2-factorization to construct an ASD with the terms in the sequence first consisting of matchings, then directed paths of length 2 and finally triangles.

In this paper, we will use the following result in [5] by Sui and Du.

Theorem 2. (*Sui, Du*) *A complete graph on $6n+5$ vertices where $n \geq 2$ has a 2-factorization where each 2-factor contains exactly one 5-cycle and the rest triangles.*

Since any orientation of a triangle or 5-cycle will contain a directed path of length 2, we will construct the ASD using a similar method as in [6, 7]. We shall keep track of the number of each type of triangle in each 2-factor by assigning each 2-factor an ordered pair (x, y) where x is the number of transitive triangles and y is the number of cyclic triangles in the subgraph.

The following definition, first defined in [6], gives us a tool to construct the terms containing triangles so that they meet the subgraph containment condition of an ASD.

Definition 4. *Let S be a finite multiset $\{(x_i, y_i)\}_{i=1}^{m}$ where x_i and y_i are nonnegative integers for all $i = 1, 2, \ldots, m$. We say that S has an ascending sequence of height h and cap c if there exists a sequence $S' = \{(a_j, b_j)\}_{j=1}^{h+c-1}$ satisfying the following:*

1. $a_j + b_j = j$ *for all* $j = 1, 2, \cdots, h$

2. $a_j + b_j = h$ *for all* $j = h, h+1, \cdots, h+c-1$

3. $a_j \leq a_{j+1}$ *and* $b_j \leq b_{j+1}$ *for all* $j = 1, 2, \cdots h+c-2$

4. $a_j \leq x_j$ *and* $b_j \leq y_j$ *for some ordering of* S.

The value h from the definition will be the number of triangles in the last term of the ASD. The cap c will be the number of terms with exactly h triangles. The values a_i and b_i from the definition will be the number of transitive triangles and cyclic triangles in last terms of the ASD. For the result in this paper, to form an ASD for a tournament of order $6n+5$, we will need an ascending sequence with height of $2n$ and cap of 2.

We need to know the minimum number of transitive triangles in our decomposition so that from the resulting multiset of ordered pairs, we can find an ascending sequence of the desired height and cap. The following theorem of Moon from [3] allows us to find a lower bound for the number of transitive triangles.

Theorem 3. (*Moon*) *Any tournament on* $2k+1$ *vertices contains at least* $(2k+1)\binom{k}{2}$ *transitive triangles. Any tournament on* $2k$ *vertices contains at least* $k(k-1)^2$ *transitive triangles.*

We will use the following result that was proven in [6] using Theorem 3.

Lemma 4. *Let* T *be a tournament of order* $n \geq 2$ *with* $V(T) = [n]$. *Let* $\mathcal{F}$ *be a decomposition of* T. *Then there is a permutation* $\sigma \in S_n$ *where* $\sigma[\mathcal{F}]$ *has at least* $\frac{3(n-3)}{4(n-2)}$ *proportion of the triangles being transitive if* n *is odd and has at least* $\frac{3(n-2)}{4(n-1)}$ *portion of the triangles being transitive if* n *is even.*

3 ASD for tournaments of order $6n+5$

We now begin a series of technical lemmas that show that we have the needed ascending sequence of height $2n$ and cap 2.

Lemma 5. *Let* n *be a positive integer. If the multiset of ordered pairs* $S = \{(x_1, y_1), (x_2, y_2), \ldots, (x_{2n+1}, y_{2n+1})\}$ *with* $x_i + y_i = n$ *for all* $i = 1, 2, \ldots, 2n+1$, *then* S *has an ascending sequence of height* n *and cap* 2.

Proof. We will prove the lemma by induction on n. Suppose first that $n = 1$ and so the multiset S consists of 3 ordered pairs of which there are only two possibilities $(1,0)$ or $(0,1)$. By the pigeonhole principle, at least two of the ordered pairs is the same. These two ordered pairs are the ascending sequence of height 1 and cap 2.

Now suppose that $n \geq 2$ and that the lemma holds for all values less than n. By the pigeonhole principle, at least $n+1$ of the ordered pairs have $x_i \geq \lceil \frac{n}{2} \rceil$ or at least $n+1$ of the ordered pairs have $y_i \geq \lceil \frac{n}{2} \rceil$. Without loss of generality, suppose that at least $n+1$ of the ordered pairs have $x_i \geq \lceil \frac{n}{2} \rceil$ and the ordered pairs are $(x_1, y_1), (x_2, y_2), \ldots, (x_{n+1}, y_{n+1})$.

Consider the multiset $S' = \{x_i - \lceil \frac{n}{2} \rceil, y_i)\}_{i=1}^{n+1}$ which has $x_i + y_i = \lfloor \frac{n}{2} \rfloor$. By the induction hypothesis multiset S' (or if n is odd, a subset of it) has an ascending sequence of height $\lfloor \frac{n}{2} \rfloor$ with cap 2. Suppose without loss of generality, that the ascending sequence is $(a_1, b_1), \ldots, (a_{\lfloor \frac{n}{2} \rfloor+1}, b_{\lfloor \frac{n}{2} \rfloor+1})$ where $a_i \leq x_i - \lceil \frac{n}{2} \rceil$ and $b_i \leq y_i$. Since each of the values $x_{\lfloor \frac{n}{2} \rfloor+1}, x_{\lfloor \frac{n}{2} \rfloor+1}, \ldots, x_{n+1}$ is at least $\lceil \frac{n}{2} \rceil$, we can obtain from the multiset S the ascending sequence $(1,0), (2,0), \ldots, (\lceil \frac{n}{2} \rceil, 0), (a_1 + \lceil \frac{n}{2} \rceil, b_1), \ldots, (a_{\lfloor \frac{n}{2} \rfloor+1} + \lceil \frac{n}{2} \rceil, b_{\lfloor \frac{n}{2} \rfloor+1})$ which has height n and cap 2.

□

Lemma 6. *Let S be a multiset of ordered pairs $\{(x_i, y_i)\}_{i=1}^{3n+2}$ where x_i and y_i are nonnegative integers with the following properties:*

1. *$x_i + y_i = 2n$ for all $i = 1, 2, \ldots, 3n+2$.*

2. $$\sum_{i=1}^{3n+2} x_i \geq \frac{(3n+1)(3n+2)n}{2n+1} =: f(n).$$

Then S contains an ascending sequence of height $2n$ and cap 2.

Proof. First order the multiset so that that $x_i \geq x_{i+1}$ for all i. We will prove, by induction on n, that we can construct the ascending sequence required using the first $2n+1$ elements in the ordered multiset S.

If $n = 1$, by Property 2 in the statement of the lemma, we have that $\sum_{i=1}^{5} x_i \geq \frac{20}{3}$. Thus, $\sum_{i=1}^{5} x_i \geq 7$. Then, we must have $x_1 = x_2 = 2$ and $x_3 \geq 1$. Thus, we can take $(1,0), (2,0), (2,0)$ to be our ascending sequence of height 2 and cap 2.

Now let $n \geq 2$ and suppose that we can find an ascending sequence of height $2(n-1)$ and cap 2 using the first $2(n-1)+1$ elements of any ordered multiset S with $x_i \geq x_{i+1}$ for all values of i that satisfy the conditions of the lemma. Now consider an ordered multiset S with $3n+2$ elements that satisfies the conditions of the lemma.

Claim: The term $x_{2n+1} \geq 2$.

Proof. Suppose instead that $x_{2n+1} \leq 1$. Since

$$x_i \leq \begin{cases} 2n & \text{if } i \leq 2n \\ 1 & \text{if } i \geq 2n+1 \end{cases},$$

we have that

$$\sum_{i=1}^{3n+2} x_i \leq 2n(2n) + (n+2) = 4n^2 + n + 2.$$

So, by Property 2 in Lemma 6,

$$4n^2 + n + 2 \geq \sum_{i=1}^{3n+2} x_i \geq \frac{(3n+1)(3n+2)n}{2n+1} = \frac{9n^2}{2} + \frac{9n}{4} - \frac{1}{8} + \frac{1}{8(2n+1)}.$$

Note that for all $n \geq 2$, $\frac{9n^2}{2} + \frac{9n}{4} - \frac{1}{8} + \frac{1}{8(2n+1)} > 4n^2 + n + 2$. That is a contradiction. Hence, $x_{2n+1} \geq 2$. □

We consider two cases:

Case I: Suppose the term $x_{3n-1} \leq n-1$.

Consider a new ordered multiset $S' = \{(x_i', y_i')\}_{i=1}^{3n-1}$ formed from S as follows: if $x_i \geq 2$, then $x_i' = x_i - 2$ and $y_i' = y_i$. If $x_i = 1$, then $x_i' = 0$ and $y_i' = y_i - 1$. If $x_i = 0$, then $x_i' = 0$ and $y_i' = y_i - 2$. Clearly, S' satisfies Property 1 from Lemma 6. Property 2 is proven in the following claim.

Claim: The ordered multiset S' satisfies:

$$\sum_{i=1}^{3n-1} x_i' \geq \frac{(3n-2)(3n-1)n}{2n-1} = f(n-1).$$

Proof. By definition of x_i' and the fact that $x_{3n+2} \leq x_{3n+1} \leq x_{3n} \leq x_{3n-1} \leq n$, we have

$$\sum_{i=1}^{3n-1} x_i' \geq \sum_{i=1}^{3n-1} (x_i - 2) \geq f(n) - 2(3n-1) - 3(n-1).$$

Note that $f(n) - 2(3n-1) - 3(n-1) - f(n-1) = \frac{11n^2-3}{4n^2-1} > 0$ for $n \geq 1$.

So, we have $\sum_{i=1}^{3n-1} x_i' \geq f(n-1)$. □

Thus, S' satisfies Property 2 of Lemma 6. By induction, we can find an ascending sequence of height $2n-2$ and cap 2 using the first $2n-1$ elements of S'. Suppose this sequence is $(a_{k_1}, b_{k_1}), (a_{k_2}, b_{k_2}), \ldots, (a_{k_{2n-1}}, b_{k_{2n-1}})$ where $k_i \in [2n-1]$, $a_{k_i} \leq x'_{k_i} = x_{k_i} - 2$, and $b_{k_i} \leq y'_{k_i} = y_{k_i}$ for all i. Since $x_1 \geq x_2 \geq \cdots \geq x_{2n} \geq x_{2n+1} \geq 2$, we have

$$(1,0), (2,0), (a_{k_1}+2, b_{k_1}), (a_{k_2}+2, b_{k_2}), \ldots, (a_{k_{2n-1}}+2, b_{k_{2n-1}})$$

as our ascending sequence of height $2n$ and cap 2 in S.

Case II: Suppose the term $x_{3n-1} \geq n$.

Note that for $n \geq 2$, $3n-1 \geq 2n+1$. Then we have $x_1 \geq x_2 \geq \cdots \geq x_{2n+1} \geq n$.

Let S'' be the ordered multiset $\{x_i - n, y_i\}_{i=1}^{2n}$. By Lemma 5, S'' has an ascending sequence of height n and cap 2. Without loss of generality, suppose the sequence is $(a_1, b_1), \ldots, (a_{n+1}, b_{n+1})$. But then,

$$(1,0),(2,0)\ldots,(n,0),(a_1+n,b_1),\ldots,(a_{n+1}+n,b_{n+1})$$

is our ascending sequence of height $2n$ and cap 2 in S.

This completes the proof. □

We are now ready to show that we can find an ASD when the tournament has order $6n+5$ for $n \geq 2$. Note that the case where a tournament has 5 vertices ($n=0$) can also be done using a technique similar to what is used in the following proof except that the 2-factorization will have no triangles.

Theorem 7. *For $n \geq 2$, any tournament of order $6n+5$ has an ASD.*

Proof. Let $n \geq 2$ and T be a tournament with $6n+5$ vertices. By applying Theorem 2, we obtain a 2-factorization of T where each 2-factor contains exactly one 5-cycle and the rest triangles. can

By Lemma 4, we may suppose that the 2-factorization has at least $\frac{3n+1}{4n+2}$ portion of the triangles being transitive. Let x_i and y_i be the number of transitive and cyclic triangles in F_i for $i = 1, 2, \ldots, 3n+2$ respectively. Note that the multiset $S = \{(x_i, y_i)\}_{i=1}^{3n+2}$ satisfies Part 1 of Lemma 6. Since the decomposition contains $2n(3n+2)$ triangles of which at least $\frac{3n+1}{4n+2}$ portion of them being transitive, S satisfies Part 2 of Lemma 6 as well.

By applying Lemma 6, we can obtain from S an ascending sequence of height $2n$ and cap 2. Without loss of generality, suppose that the ascending sequence is $(a_{2n+1}, b_{2n+1}), \ldots, (a_1, b_1)$. Note the order of the terms are so that the smaller subscript terms are the larger ordered pairs.

We will now construct the ASD.

First, we will use F_1 to construct both D_1 and D_{6n+4}. The term D_1 is a single arc. Take this arc from the oriented 5-cycle in F_1 in such a way that the remaining oriented path of length 4 contains a directed path of length 2 as a subgraph. What remains from F_1 will be the term D_{6n+4}.

For $2 \leq i \leq 2n+2$, let terms D_{6n+5-i} and D_i for $2 \leq i \leq 2n+2$ be formed from F_i. The term D_{6n+5-i} consists of a_i transitive triangles and b_i cyclic triangles (for $2n+2-i$ triangles total), $i-1$ directed paths of length 2 and an isolated arc. Note that the isolated arc and one of directed paths of length 2 come from the oriented 5-cycle. What remains is the term D_i which consists of a matching of size i.

For convenience, we rename the 2-factors $F_{2n+3}, F_{2n+4}, \cdots, F_{3n+2}$ to be $F_1', F_2', \cdots, F_n'$ respectively.

For $1 \leq j \leq n$, let the terms D_{4n+3-j} and D_{2n+2+j} be formed from F_j'. The term D_{4n+3-j} will consist of $2n+1-j$ directed paths of length 2 and a matching of size $j+1$. Note that one of the arcs in the matching and one of directed paths of length 2 come from the oriented 5-cycle. What remains is the term D_{2n+2+j} which consists of j directed paths of length 2 and a matching of size $2n+2-j$.

As constructed, the terms $D_1, D_2, \ldots, D_{6n+4}$ form an ASD of the tournament T. □

References

[1] Y. Alavi, A.J. Boals, G. Chartrand, P. Erdös, O. Oellermann: The ascending subgraph decomposition problem. Congr. Numer. **58**, 7-14 (1987)

[2] G. Chartrand, L. Lesniak: Graphs & Digraphs, 4th ed. Chapman & Hall/CRC, Boca Raton (2005)

[3] J. Moon: Topics on tournaments. Holt, Rinehart and Winston, New York (1968)

[4] D.K. Ray-Chaudhuri, R.M. Wilson: Solution to Kirkman's schoolgirl problem. Combinatorics, Proc. Sympos. Pure Math., Univ. California, Los Angeles, Calif., 1968. **19**, 187-203 (1971)

[5] Q. Sui, B. Du: The Oberwolfach problem for a unique 5-cycle and all others of length 3. Util. Math **65**, 243-254 (2004)

[6] B. Wagner: Ascending subgraph decompositions of tournaments of orders $6n+2$ and $6n+1$. Graphs Combin. **32**, 813-822 (2016)

[7] B. Wagner: Ascending subgraph decompositions of tournaments of order $6n+3$. Graphs Combin. **29**,1951-1959 (2013)