C_6-DECOMPOSITION OF THE TENSOR PRODUCT OF COMPLETE GRAPHS

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ABSTRACT. Let G be a simple and finite graph. A graph is said to be decomposed into subgraphs H_1 and H_2 which is denoted by $G = H_1 \oplus H_2$, if G is the edge disjoint union of H_1 and H_2. If $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where H_1, H_2, ..., H_k are all isomorphic to H, then G is said to be H-decomposable. Futhermore, if H is a cycle of length m then we say that G is C_m-decomposable and this can be written as $C_m|G$. Where $G \times H$ denotes the tensor product of graphs G and H, in this paper, we prove that the necessary conditions for the existence of C_6-decomposition of $K_m \times K_n$ are sufficient. Using these conditions it can be shown that every even regular complete multipartite graph G is C_6-decomposable if the number of edges of G is divisible by 6.

1. Introduction

Let C_m, K_m and $K_m - I$ denote cycle of length m, complete graph on m vertices and complete graph on m vertices minus a 1-factor respectively. By an m-cycle we mean a cycle of length m. All graphs considered in this paper are simple and finite. A graph is said to be *decomposed* into subgraphs H_1 and H_2 which is denoted by $G = H_1 \oplus H_2$, if G is the edge disjoint union of H_1 and H_2. If $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where H_1, H_2, ..., H_k are all isomorphic to H, then G is said to be H-decomposable. Futhermore, if H is a cycle of length m then we say that G is C_m-decomposable and this can be written as $C_m|G$. A k-factor of G is a k-regular spanning subgraph. A k-factorization of a graph G is a partition of the edge set of G into k-factors. A C_k-factor of a graph is a 2-factor in which each component is a cycle of length k. A *resolvable* k-cycle decomposition (for short k-RCD) of G denoted by $C_k||G$, is a 2-factorization of G in which each 2-factor is a C_k-factor.

For two graphs G and H their tensor product $G \times H$ has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever

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$g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. From this, note that the tensor product of graphs is distributive over edge disjoint union of graphs, that is if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H)$. Now, for $h \in V(H)$, $V(G) \times h = \{(v,h)|v \in V(G)\}$ is called a column of vertices of $G \times H$ corresponding to h. Further, for $y \in V(G)$, $y \times V(H) = \{(y,v)|v \in V(H)\}$ is called a layer of vertices of $G \times H$ corresponding to y. The problem of finding C_k-decomposition of K_{2n+1} or $K_{2n} - I$ where I is a 1-factor of K_{2n}, is completely settled by Alspach, Gavlas and Sagna in two different papers (see [1, 13]). A generalization to the above complete graph decomposition problem is to find a C_k-decomposition of $K_m * \overline{K}_n$, which is the complete m-partite graph in which each partite set has n vertices. The study of cycle decompositions of $K_m * \overline{K}_n$ was initiated by Hoffman et al. [5]. In the case when p is a prime, the necessary and sufficient conditions for the existence of C_p-decomposition of $K_m * \overline{K}_n$, $p \geq 5$ is obtained by Manikandan and Paulraja in [7, 8, 10]. Billington [2] has studied the decomposition of complete tripartite graphs into cycles of lenght 3 and 4. Furthermore, Cavenagh and Billington [4] have studied 4-cycle, 6-cycle and 8-cycle decomposition of complete multipartite graphs. Billington et al. [3] have solved the problem of decomposing $(K_m * \overline{K}_n)$ into 5-cycles. Similarly, when $p \geq 3$ is a prime, the necessary and sufficient conditions for the existence of C_{2p}-decomposition of $K_m * \overline{K}_n$ is obtained by Smith (see [14]). For a prime $p \geq 3$, it was proved in [15] that C_{3p}-decomposition of $K_m * \overline{K}_n$ exists if the obvious necessary conditions are satisfied. As the graph $K_m \times K_n \cong K_m * \overline{K}_n - E(nK_m)$ is a proper regular spanning subgraph of $K_m * \overline{K}_n$. It is natural to think about the cycle decomposition of $K_m \times K_n$. The results in [7, 8, 10] also give necessary and sufficient conditions for the existence of a p-cycle decomposition, (where $p \geq 5$ is a prime number) of the graph $K_m \times K_n$. In [9] it was shown that the tensor product of two regular complete multipartite graph is Hamilton cycle decomposable. Muthusamy and Paulraja in [11] proved the existence of C_{kn}-factorization of the graph $C_k \times K_{mn}$, where $mn \not\equiv 2 \pmod 4$ and k is odd. While Paulraja and Kumar [12] showed that the necessary conditions for the existence of a resolvable k-cycle decomposition of tensor product of complete graphs are sufficient when k is even.

In this paper, we prove that the obvious necessary conditions for $K_m \times K_n$, $2 \leq m, n$, to have a C_6-decomposition are also sufficient. Among other results, here we prove the following main results.

It is not surprising that the conditions in Theorem 1.1 are "symmetric" with respect to m and n since $K_m \times K_n \cong K_n \times K_m$.

Theorem 1.1. *For $2 \leq m, n$, $C_6|K_m \times K_n$ if and only if $m \equiv 1$ or $3 \pmod 6$ or $n \equiv 1$ or $3 \pmod 6$.*

Theorem 1.2. *Let m be an even integer and $m \geq 6$, then $C_6|K_m - I \times K_n$ if and only if $m \equiv 0$ or $2 \pmod 6$*

2. C_6 Decomposition of $C_3 \times K_n$

Theorem 2.1. *For all n, $C_6|C_3 \times K_n$.*

Proof. Following from the definition of tensor product of graphs, let $U^1 = \{u_1, v_1, w_1\}$, $U^2 = \{u_2, v_2, w_2\}, \ldots, U^n = \{u_n, v_n, w_n\}$ form the partite set of vertices in $C_3 \times K_n$. Also, U^i and U^j has an edge in $C_3 \times K_n$ for $1 \leq i, j \leq n$ and $i \neq j$ if the subgraph induce $K_{3,3} - I$, where I is a 1-factor of $K_{3,3}$. Now, each subgraph $U^i \cup U^j$ is isomorphic to $K_{3,3} - I$. But $K_{3,3} - I$ is a cycle of length six. Hence the proof. □

Example 2.2. *The graph $C_3 \times K_7$ can be decomposed into cycles of length 6.*

Solution. *Let the partite sets (layers) of the tripartite graph $C_3 \times K_7$ be $U = \{u_1, u_2, \ldots, u_7\}$, $V = \{v_1, v_2, \ldots, v_7\}$ and $W = \{w_1, w_2, \ldots, w_7\}$. We assume that the vertices of U, V and W having same subscripts are the corresponding vertices of the partite sets. A 6-cycle decomposition of $C_3 \times K_7$ is given below:*

$\{u_1, v_2, w_1, u_2, v_1, w_2\}, \{u_1, v_3, w_1, u_3, v_1, w_3\}, \{u_2, v_3, w_2, u_3, v_2, w_3\},$
$\{u_1, v_4, w_1, u_4, v_1, w_4\}, \{u_2, v_4, w_2, u_4, v_2, w_4\}, \{u_3, v_4, w_3, u_4, v_3, w_4\},$
$\{u_1, v_5, w_1, u_5, v_1, w_5\}, \{u_2, v_5, w_2, u_5, v_2, w_5\}, \{u_3, v_5, w_3, u_5, v_3, w_5\},$
$\{u_4, v_5, w_4, u_5, v_4, w_5\}, \{u_1, v_6, w_1, u_6, v_1, w_6\}, \{u_2, v_6, w_2, u_6, v_2, w_6\},$
$\{u_3, v_6, w_3, u_6, v_3, w_6\}, \{u_4, v_6, w_4, u_6, v_4, w_6\}, \{u_5, v_6, w_5, u_6, v_5, w_6\},$
$\{u_1, v_7, w_1, u_7, v_1, w_7\}, \{u_2, v_7, w_2, u_7, v_2, w_7\}, \{u_3, v_7, w_3, u_7, v_3, w_7\},$
$\{u_4, v_7, w_4, u_7, v_4, w_7\}, \{u_5, v_7, w_5, u_7, v_5, w_7\}, \{u_6, v_7, w_6, u_7, v_6, w_7\}.$

Theorem 2.3. [6] *Let m be an odd integer and $m \geq 3$. If $m \equiv 1$ or $3 \pmod 6$ then $C_3|K_m$.*

Theorem 2.4. [13] *Let n be an even integer and m be an odd integer with $3 \leq m \leq n$. The graph $K_n - I$ can be decomposed into cycles of length m whenever m divides the number of edges in $K_n - I$.*

3. C_6 Decomposition of $C_6 \times K_n$

Theorem 3.1. [13] *Let n be an odd integer and m be an even integer with $3 \leq m \leq n$. The graph K_n can be decomposed into cycles of length m whenever m divides the number of edges in K_n.*

Lemma 3.2. *$C_6|C_6 \times K_2$*

Proof. Let the partite set of the bipartite graph $C_6 \times K_2$ be $\{u_1, u_2, \ldots, u_6\}$, $\{v_1, v_2, \ldots, v_6\}$. We assume that the vertices having the same subscripts are the corresponding vertices of the partite sets. Now $C_6 \times K_2$ can be decomposed into 6-cycles which are $\{u_1, v_2, u_3, v_4, u_5, v_6\}$ and $\{v_1, u_2, v_3, u_4, v_5, u_6\}$. □

Theorem 3.3. *For all n, $C_6|C_6 \times K_n$.*

Proof. Let the partite set of the 6-partite graph $C_6 \times K_n$ be $U = \{u_1, u_2, ..., u_n\}$, $V = \{v_1, v_2, ..., v_n\}$, $W = \{w_1, w_2, ..., w_n\}$, $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\}$ and $Z = \{z_1, z_2, ..., z_n\}$: we assume that the vertices of U, V, W, X, Y and Z having the same subscripts are the corresponding vertices of the partite sets. Let $U^1 = \{u_1, v_1, w_1, x_1, y_1, z_1\}$, $U^2 = \{u_2, v_2, w_2, x_2, y_2, z_2\}$, ..., $U^n = \{u_n, v_n, w_n, x_n, y_n, z_n\}$ be the sets of these vertices having the same subscripts. By the definition of the tensor product, each U^i, $1 \leq i \leq n$ is an independent set and the subgraph induced by each $U^i \cup U^j$, $1 \leq i, j \leq n$ and $i \neq j$ is isomorphic to $C_6 \times K_2$. Now by Lemma 3.2 the graph $C_6 \times K_2$ admits a 6-cycle decomposition. This completes the proof. □

4. C_6 Decomposition of $K_m \times K_n$

Proof of Theorem 1.1. Assume that $C_6|K_m \times K_n$ for some m and n with $2 \leq m, n$. Then every vertex of $K_m \times K_n$ has even degree and 6 divides in the number of edges of $K_m \times K_n$. These two conditions translate to $(m-1)(n-1)$ being even and $6|m(m-1)n(n-1)$ respectively. Hence, by the first fact m or n has to be odd, i.e., has to be congruent to 1 *or* 3 *or* 5 (mod 6). The second fact can now be used to show that they cannot both be congruent to 5 (mod 6). It now follows that $m \equiv 1$ *or* 3 (mod 6) or $n \equiv 1$ *or* 3 (mod 6). Conversely, let $m \equiv 1$ *or* 3 (mod 6). By Theorem 2.3, $C_3|K_m$ and hence $K_m \times K_n = ((C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n))$. Since $C_6|C_3 \times K_n$ by Theorem 2.1.
Finally, if $n \equiv 1$ *or* 3 (mod 6), the above argument can be repeated with the roles of m and n interchanged to show again that $C_6|K_m \times K_n$. This completes the proof.
Proof of Theorem 1.2. Assume that $C_6|K_m - I \times K_n, m \geq 6$. Certainly, $6|mn(m-2)(n-1)$. But we know that if $6|m(m-2)$ then $6|mn(m-2)(n-1)$. But m is even therefore $m \equiv 0$ *or* 2 (mod 6).
Conversely, let $m \equiv 0$ *or* 2 (mod 6). Notice that for each m, $\frac{m(m-2)}{2}$ is a multiple of 3. Thus by Theorem 2.4 $C_3|K_m - I$ and hence $K_m - I \times K_n = ((C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n))$. From Theorem 2.1 $C_6|C_3 \times K_n$. The proof is complete.

5. CONCLUSION

In view of the results obtained in this paper we draw our conclusion by the following corollary.

Corollary 5.1. *For any simple graph G. If*

(1) $C_3|G$ *then* $C_6|G \times K_n$, *whenever* $n \geq 2$.
(2) $C_6|G$ *then* $C_6|G \times K_n$, *whenever* $n \geq 2$.

Proof. We only need to show that $C_3|G$. Applying Theorem 2.1 gives the result. □

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References

[1] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$, J. Combin. Theory Ser. B 81 (2001) 7799.

[2] E.J. Billington, Decomposing complete tripartite graphs into cycles of length 3 and 4, Discrete Math. 197/198 (1999) 123-135.

[3] E.J. Billington, D.G. Hoffman and B.M. Maenhaut, Group divisible pentagon systems, Util. Math. 55 (1999) 211-219.

[4] N.J. Cavenagh and E.J. Billington, Decompositions of complete multipartite graphs into cycles of even length, Graphs Combin. 16 (2000) 49-65.

[5] D. G. Hoffman, C. C. Linder and C. A. Rodger, On the construction of odd cycle systems, J. Graph Theory 13 (1989) 417-426.

[6] C.C. Lindner and C.A. Rodger, Design Theory (CRC Press New York, 1997).

[7] R.S. Manikandan and P. Paulraja, C_p-decompositions of some regular graphs, Discrete Math. 306 (2006) 429-451.

[8] R.S. Manikandan and P. Paulraja, C_5-decompositions of the tensor product of complete graphs, Australas. J. Combin. 37 (2007) 285-293.

[9] R.S. Manikandan and P. Paulraja, Hamilton cycle decompositions of the tensor product of complete multipartite graphs, Discrete Math. 308 (2008)3586-3606.

[10] R.S. Manikandan and P. Paulraja, C_7-decompositions of the tensor product of complete graphs, Discussiones Mathematicae Graph Theory 37 (2017) 523-535.

[11] A. Muthusamy and P. Paulraja, Factorizations of product graphs into cycles of uniform length, Graphs Combin. 11 (1995) 69-90.

[12] P. Paulraja and S. S. Kumar, Resolvable even cycle decompositions of the tensor product of complete graphs. Discrete Math. 311 (2011) 1841-1850.

[13] M. Sajna, Cycle decompositions III: complete graphs and fixed length cycles, J. Combin. Designs 10 (2002) 27-78.

[14] B. R. Smith, Decomposing complete equipartite graphs into cycles of lenght $2p$, J. Combin. Des. 16 (2006) 244-252.

[15] B. R. Smith, Complete equipartite $3p$-cycle systems, Australas J. Combin. 45 (2009) 125-138.

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