Coloring and Domination of Vertices in Triangle-free Graphs

Ronald Dutton
Department of Computer Science
University of Central Florida, Orlando FL 32816

Abstract

Any dominating set of vertices in a triangle-free graph can be used to specify a graph coloring with at most one color class more than the number of vertices in the dominating set. This bound is sharp for many graphs. Properties of graphs for which this bound is achieved are presented.

Keywords: Triangle-free graphs, Chromatic number, Domination number, Extremal graphs

1 Introduction

A k-coloring c is a partition of the vertices of a graph G into k independent sets $\{c_1, c_2, \ldots, c_k\}$, denoted as the color classes of c. The minimum number of color classes required for a proper coloring is the *chromatic number* $\chi(G)$. For additional terminology and results on graph coloring see *Chromatic Graph Theory* [1] by G. Chartrand and P. Zhang.

In a graph G, $N_G(v)$ is the set of vertices adjacent to vertex v and is known as the open neighborhood of v. The closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. When the underlying graph is understood, the subscript is usually omitted. A graph is *triangle-free* when $N(v)$ is an independent set for every vertex v. Graphs considered throughout this document are triangle-free.

A vertex dominates itself and its neighbors. That is, vertex v dominates $N[v]$. This is extended to a set of vertices D where $N[D]$ is the union of the closed neighborhoods of the vertices in D. Then, D dominates $N[D]$. A dominating set of G is a set D where $N[D] = V(G)$. The minimum number of vertices in a dominating set of G is $\gamma(G)$ and such a set is referred to as a γ-set. The minimum number of vertices in an independent dominating set is a γ_i-set.

It is straightforward that $\gamma(G) \leq \gamma_i(G)$ and, if $\gamma(G) < \gamma_i(G)$, every γ-set contains at least one pair of adjacent vertices. For additional terminology and results concerning domination in graphs see *Fundamentals of Domination in Graphs* [3] by T. Haynes, S. Hedetniemi and P. Slater.

A relationship between the chromatic number of a triangle-free graph and its domination number is established by an algorithm in Section 2 which produces a coloring based upon a given dominating set and shows $\chi(G) \leq \gamma(G) + 1$ for all triangle-free graphs. While the colorings of the algorithm are not necessarily optimum for many triangle-free graphs, it will give a proper coloring for every triangle-free graph and will be exact in many cases.

Section 3 shows, for every $k \geq 2$, there are triangle-free graphs G with $\chi(G) = k$ and $\chi(G) = \gamma(G) + 1$, referred to here as *extremal* graphs. In Section 4, additional properties of extremal graphs are presented.

2 Coloring Algorithm

Assume G is a triangle-free graph with a dominating set $D = \{x_1, x_2, \ldots, x_s\}$. A proper coloring c of G is algorithmically constructible.

Algorithm

1. $c_1 \leftarrow N(x_1)$

2. for $2 \leq i \leq s$,
$$c_i \leftarrow \{\{x_{i-1}\} \cup N(x_i)\} - \{c_1 \cup c_2 \cup \ldots \cup c_{i-1}\}$$

3. if x_s has no neighbor in D, then
$$c_{s+1} \leftarrow \{x_s\}$$

Note that c_{s+1} is created only when $N(x_s) \cap D = \emptyset$, for otherwise x_s will have already been placed in c_j for some $j \leq s - 1$, and the color class c_{s+1} is not created. Also notice that there is no specification on the size of the dominating set or of the order in which the vertices are examined.

Lemma 1 *The Algorithm provides a proper s- or $(s+1)$-coloring of G.*

Proof: It must be shown that each c_i is independent and every vertex of G is in exactly one of the sets.

Since G is triangle-free, for every vertex v, $N(v)$ is an independent set. Thus from Step 1 of the algorithm, c_1 is an independent set of vertices. In Step 2 every vertex in $N(x_i)$ is either in c_i or in $\{c_1 \cup c_2 \cup \ldots \cup c_{i-1}\}$, but not in both. When $i \geq 2$, x_{i-1} has no neighbors in $N(x_i) - \{c_1 \cup c_2 \cup \ldots \cup c_{i-1}\}$. Thus, if x_{i-1} is not previously colored, it will be colored i. Then, c_i is independent and every vertex in $N(x_i)$ is assigned to exactly one color class in $\{c_1, c_2, \ldots, c_i\}$.

Therefore, Steps 1 and 2 assign colors to every vertex in G except x_s when x_s has no neighbor in D. When that occurs, Step 3 places x_s in c_{s+1}. Thus, c is either a proper s-coloring or $(s+1)$-coloring of G. □

As far as the correctness of the algorithm is concerned, the choice of the dominating set D and the ordering of the vertices in D are irrelevant, although different dominating sets and different orderings of the vertices may alter the colorings produced.

Theorem 2 *For a triangle-free graph G, if there is a non-independent γ-set then*

$$\chi(G) \leq \gamma(G) \leq \gamma_i(G).$$

Otherwise, every γ-set is independent and

$$\chi(G) \leq \gamma(G) + 1 = \gamma_i(G) + 1.$$

Proof: Assume $D = \{x_1, x_2, \ldots, x_s\}$ is a γ-set of G that is not independent. Since the order of the vertices in D is irrelevant, assume x_s is a vertex with a neighbor in D, say x_1. Then, from Step 1 of the algorithm, x_s is in c_1, Therefore, in Step 3, c_{s+1} will not be created. From Lemma 1, c is an s-coloring of G. By assumption $s = \gamma(G)$. Therefore,

$$\chi(G) \leq s = \gamma(G) \leq \gamma_i(G).$$

When every γ-set is independent, $\gamma(G) = \gamma_i(G)$. Regardless of the minimum dominating set and the ordering of the vertices in the set, x_s will not be colored by Steps 1 and 2 of the algorithm. Therefore, Step 3 assigns x_s to c_{s+1} and, from Lemma 1, c is an $(s+1)$-coloring of G. Thus,

$$\chi(G) \leq s + 1 = \gamma(G) + 1 = \gamma_i(G) + 1.$$

□

There are graphs with a significant difference between their chromatic and domination numbers. For instance, let G be the graph obtained by subdividing each edge of $K_{1,n}$ exactly once. Then

$$\chi(G) = 2 \text{ and } \gamma(G) = \gamma_i(G) = n.$$

Another case is when G is obtained by adding an edge between the roots of two $K_{1,n}$'s. In this case

$$\chi(G) = \gamma(G) = 2 \text{ and } \gamma_i(G) = n + 1.$$

Simple examples of triangle-free extremal graphs include

1. $K_{1,n}$ for $n \geq 1$, where $\chi(K_{1,n}) = 2$ and $\gamma(K_{1,n}) = \gamma_i(K_{1,n}) = 1$, and
2. a 5-cycle C_5, where $\chi(C_5) = 3$ and $\gamma(C_5) = \gamma_i(C_5) = 2$.

A more complex version is shown in Figure 1, where G is the well known Grötzsch graph where $\chi(G) = 4$ and $\gamma(G) = \gamma_i(G) = 3$. The numbers on the vertices in Figure 1 are the colors assigned by the Algorithm based upon the indicated γ-set x_1, x_2, and x_3.

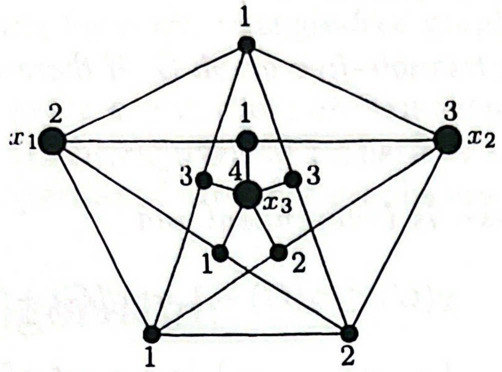

Figure 1: Grötzsch graph

3 Extremal Graphs

In 1955, Jan Mycielski [5] described a graph $\mu(G)$, constructed from a base graph G as follows: $V(\mu(G)) = V \cup W \cup \{z\}$ where $V = \{v_1, v_2, \ldots, v_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$. The subgraph induced by V is isomorphic to G, and W is a set of n independent vertices. For every combination of vertices w_i and v_j, w_i is adjacent to v_j if and only if v_i is adjacent to v_j. Finally, $N(z) = W$.

Mycielski showed $\chi(\mu(G)) = \chi(G) + 1$, $\mu(G)$ is triangle-free when G is triangle-free and, for any positive integer k, there is a triangle-free graph with chromatic number k. Fisher, McKenna and Boyer [2] showed $\gamma(\mu(G)) = \gamma(G) + 1$. Furthermore, Mojdeh and Rad [4] showed a corresponding result for independent domination $\gamma_i(\mu(G)) = \gamma_i(G) + 1$.

To iterate Mycielski graphs from a base graph G, let $\mu^0(G) = G$ and for $i \geq 1$, $\mu^i(G) = \mu(\mu^{i-1}(G))$. Lemma 3 then follows directly from the results cited in the last paragraph.

Lemma 3 *For any graph G and integer $i \geq 0$,*

1. $\chi(\mu^i(G)) = \chi(G) + i$,

2. $\gamma(\mu^i(G)) = \gamma(G) + i$, *and*

3. $\gamma_i(\mu^i(G)) = \gamma_i(G) + i$.

Theorem 4 shows the existence of extremal graphs having chromatic number k for any integer $k \geq 2$.

Theorem 4 *For $k \geq 2$, there is a triangle-free graph G such that*

$$\chi(G) = k \text{ and } \chi(G) = \gamma(G) + 1.$$

Proof: Let $G = \mu^{k-2}(K_{1,n})$ for any $n \geq 1$ and $k \geq 2$. From Lemma 3,

$$\chi(\mu^{k-2}(K_{1,n})) = \chi(K_{1,n}) + (k-2) = 2 + (k-2) = k, \text{ and}$$
$$\gamma(\mu^{k-2}(K_{1,n})) = \gamma(K_{1,n}) + (k-2) = 1 + (k-2) = k-1.$$

Therefore $\chi(G) = \gamma(G) + 1 = k$. □

A supergraph of a triangle-free graph G is a triangle-free graph H such that G is an edge-deleted subgraph of H. The fact that the extremal graphs constructed in Theorem 4 have diameter 2 follows from Theorem 7 of [2], $diam(\mu(G)) = min\{max\{2, diam(G)\}, 4\}$.

The graph in Figure 2, without the dotted edge, is a diameter 3 extremal graph with $\chi(G) = 3$ and $\gamma(G) = \gamma_i(G) = 2$. When the dotted edge is added, $G + xy$ is a supergraph of G, and is also an extremal graph. Theorem 5 shows that every supergraph of an extremal graph is also an extremal graph.

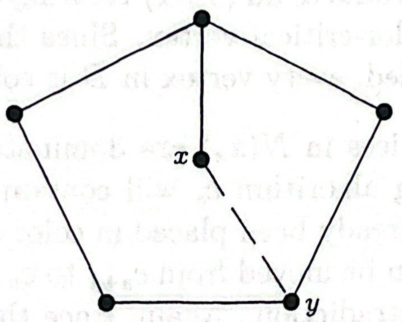

Figure 2: A diameter 3 extremal graph

Theorem 5 *If G is a triangle-free graph where $\chi(G) = \gamma(G) + 1$, then for every supergraph H, $\chi(H) = \chi(G)$ and $\chi(H) = \gamma(H) + 1$.*

Proof: Assume $\chi(G) = \gamma(G) + 1$ and let H be a supergraph of G. Since G is a subgraph of H, $\chi(G) \leq \chi(H)$ and every dominating set of G is also a dominating set of H. Therefore, $\gamma(H) \leq \gamma(G)$ and, by Theorem 2, $\gamma(H) + 1 \geq \chi(H)$. It follows that $\chi(H) \geq \chi(G) = \gamma(G) + 1 \geq \gamma(H) + 1 \geq \chi(H)$. Thus,

$$\chi(H) = \chi(G) \text{ and } \chi(H) = \gamma(H) + 1.$$

□

4 Properties of Extremal Graphs

For a dominating set D, if there is a vertex w in $V(G) - D$ for which $N(w) \cap D = \{v\}$, then w is said to be a *private neighbor* of v. A vertex v is *color-critical* if $\chi(G - v) = \chi(G) - 1$. Equivalently, there is a $\chi(G)$-coloring with a color class containing only vertex v.

Theorem 6 *If G is a triangle-free graph and $\chi(G) = \gamma(G) + 1$ then, for any γ-set D,*

1. *D is independent,*
2. *every vertex in D is color-critical,*
3. *every vertex in D has a private neighbor, and*
4. *every pair of vertices in D have a common neighbor.*

Proof: Let $s = \gamma(G)$ and $D = \{x_1, x_2, \ldots, x_s\}$.

1. If G has a γ-set that is not independent then, from Theorem 2, $\chi(G) \leq \gamma(G)$ and contradicts $\chi(G) = \gamma(G) + 1$.
2. Since $\chi(G) = \gamma(G) + 1$, from Part 1, every γ-set is independent and the Algorithm produces an $(s+1)$-coloring c of G where $c_{s+1} = \{x_s\}$. Thus, x_s is a color-critical vertex. Since the ordering of the vertices in D is unspecified, every vertex in D is color-critical.
3. Suppose all vertices in $N(x_s)$ are dominated by $D - \{x_s\}$. In Step 2 of the coloring algorithm c_s will contain no neighbors of x_s since they will have already been placed in color classes $c_1 \cup c_2 \cup \ldots \cup c_{s-1}$. Therefore, x_s can be moved from c_{s+1} to c_s. Since c_{s+1} is now empty, $\chi(G) \leq s$, a contradiction. Again, since the ordering of the vertices in D is unspecified, every vertex in D has a private neighbor.
4. If x_s has no common neighbor with x_1, in Step 1 of the algorithm x_s can be added to c_1 rather than c_{s+1} and implies $\chi(G) \leq s$, a contradiction since it is assumed that $\chi(G) = s+1$. Since the ordering of vertices in D is not an issue, this applies to every pair of vertices in D. □

References

[1] G. Chartrand and P. Zhang, *Chromatic Graph Theory*, CRC Press, Boca Raton, FL (2009).

[2] D. C. Fisher, P. A. McKenna, and E. D. Boyer, Hamiltonicity, diameter, domination, packing, and biclique partitions of Mycielski's graphs, Discrete Appl. Math. 84 (1998) 93-105.

[3] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).

[4] D. A. Mojdeh and N. J. Rad, On domination and its forcing in Mycielski's graphs, Scientia Iranica, 15 (2008) 218-222.

[5] J. Mycielski, Sur le coloriage des graphes, Colloq. Math. 3 (1955) 161-162.