Domination Number and Hamiltonicity of Graphs

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Submitted on Dec. 26, 2017; accepted on May 19, 2018

Abstract

Let G be a k - connected ($k \geq 2$) graph of order n. If $\gamma(G^c) \geq n - k$, then G is Hamiltonian or $K_k \vee K^c_{k+1}$, where $\gamma(G^c)$ is the domination number of the complement of the graph G.

2010 *Mathematics Subject Classification* : 05C45, 05C69

Keywords : *Hamiltonicity, domination number*

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. Let G be a graph. We use G^c to denote the complement of G. We also use $\gamma(G)$, $\omega(G)$, and $\alpha(G)$ to denote the domination number, the clique number, and the independent (or stability) number of G, respectively. We use $G \vee H$ to denote the the join of two disjoint graphs G and H. If C is a cycle of G, we use $\overrightarrow{C}$ to denote the cycle C with a given direction. For two vertices x, y in C, we use $\overrightarrow{C}[x, y]$ to denote the consecutive vertices on C from x to y in the direction specified by $\overrightarrow{C}$. We use x^+ and x^- to denote respectively the successor and predecessor of a vertex x on C along the direction of C. We also use x^{++} to denote $(x^+)^+$. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle.

In this note, we will present a sufficient condition for the Hamiltonicity of graphs. The main result is as follows.

Theorem 1. Let G be a k - connected ($k \geq 2$) graph of order n. If $\gamma(G^c) \geq n - k$, then G is Hamiltonian or $K_k \vee K^c_{k+1}$.

2. The Lemmas

We will use the following results as our lemmas. The first one is from Theorem 1 in [3].

Lemma 1. Let G be a graph of order n. Then $\gamma(G) + \chi(G) \leq n + 1$.

Notice that in Theorem 1 in [3] it is claimed that $\gamma(G) + \chi(G) = n + 1$ if and only if G is a complete graph of order n. It seems that this claim is not completely correct since if G is the disjoint union of a complete graph and a collection of isolated vertices then we still have $\gamma(G) + \chi(G) = n + 1$.

The second one is the main result in [1].

Lemma 2. Let G be a k - connected ($k \geq 2$) graph with independent number $\alpha = k + 1$. Let C be the longest cycle in G. Then $G[V(G) - V(C)]$ is complete.

3. Proofs

Proof of Theorem 1. Let G be a k - connected ($k \geq 2$) graph satisfying the conditions in Theorem 1. Assume that G is not Hamiltonian. Since $k \geq 2$, G contains a cycle. Choose a longest cycle C in G and give a direction on C. Since G is not Hamiltonian, there exists a vertex $x_0 \in V(G) - V(C)$. By Menger's theorem, we can find s ($s \geq k$) pairwise disjoint (except for x_0) paths P_1, P_2, ..., P_s between x_0 and $V(C)$. Let u_i be the end vertex of P_i on C, where $1 \leq i \leq s$. We assume that the appearance of u_1, u_2, ... u_s agrees with the given direction on C. We use u_i^+ to denote the successor of u_i along the direction of C, where $1 \leq i \leq s$. Then a standard proof in Hamiltonian graph theory yields that $T := \{x_0, u_1^+, u_2^+, ..., u_s^+\}$ is independent (otherwise G would have cycles which are longer than C). Since $s \geq k$, we have an independent set $S := \{x_0, u_1^+, u_2^+, ..., u_k^+\}$ of size $k + 1$ in G and a clique S of size $k + 1$ in G^c. From Lemma 1, we have that

$$\begin{aligned} n + 1 = n - k + k + 1 &\leq \gamma(G^c) + \alpha(G) \\ &= \gamma(G^c) + \omega(G^c) \leq \gamma(G^c) + \chi(G^c) \leq n + 1. \end{aligned}$$

Then $\gamma(G^c) = n - k$ and $\alpha(G) = \omega(G^c) = \chi(G^c) = k + 1$. Next we will present two claims and their proofs.

Claim 1. $G^c[V(G) - S]$ is an empty graph. Namely, $G[V(G) - S]$ is a complete graph.

Proof of Claim 1. Suppose, to the contrary, that $G^c[V(G) - S]$ is not an empty graph. Then there exist vertices x, $y \in V(G) - S$ such that $xy \in E(G^c)$. Notice that $G^c[S]$ is complete. Then $(V(G) - S - \{x\}) \cup \{z\}$ is a domination set in G^c, where z is a vertex in S. Thus $n - k = \gamma(G^c) \leq |V(G) - S| - 1 + 1 = n - k - 1$, a contradiction. ◇

Claim 2. There are no edges between S and $V(G) - S$ in G^c. Namely, for any vertex $x \in S$ and any vertex $y \in V(G) - S$, $xy \in E(G)$.

Proof of Claim 2. Suppose, to the contrary, that there exist vertices $x \in S$ and $y \in V(G) - S$ such that $xy \in E(G^c)$. Notice that $G^c[S]$ is complete. Then $(V(G) - S - \{y\}) \cup \{x\}$ is a domination set in G^c. Thus $n - k = \gamma(G^c) \leq |V(G) - S| - 1 + 1 = n - k - 1$, a contradiction. ◇

Set $T_i := \overrightarrow{C}[u_i^{++}, u_{i+1}]$, where $1 \leq i \leq k$ and the index $k+1$ is regarded as 1. Obviously, $|T_i| \geq 1$ for each i with $1 \leq i \leq k$. Set $T := \{i : |T_i| \geq 2\}$. Next we, according to the different sizes of $|T|$, divide the remainder of the proofs into three cases.

Case 0 $|T| = 0$.

Since $|T| = 0$, we have $C = u_1 u_1^+ u_2 u_2^+ ... u_k u_k^+ u_1$. Next we will prove that $V(G) - V(C) = \{x_0\}$. Suppose, to the contrary, that $V(G) - V(C) \neq \{x_0\}$. Then there exists a vertex, say z, in $V(G) - V(C) - \{x_0\}$. Since $\alpha(G) = k+1$, we have, by Lemma 2, that $G[V(G) - V(C)]$ is complete. Thus $x_0 z \in E(G)$ Since $z \in V(G) - S$ and $u_2 \in V(G) - S$, we, by Claim 1, have that $zu_2 \in E(G)$. Therefore G has a cycle $u_1 x_0 z u_2 u_2^+ ... u_k u_k^+ u_1$ which is longer than C, a contradiction. Now we, by Claim 2, have that G is $K_k \vee K_{k+1}^c$.

Case 1 $|T| = 1$.

Without loss of generality, we assume that $|T_1| \geq 2$, $|T_r| = 1$ for each r with $2 \leq r \leq k$. Since $u_1^+ \in S$, $u_2^+ \in S$, $u_2 \in V(G) - S$, and $u_2^- \in V(G) - S$, we, by Claim 2, have that $u_1^+ u_2 \in E(G)$ and $u_2^- u_2^+ \in E(G)$. Then G has a cycle $x_0 P_2 u_2 \overrightarrow{C}[u_1^+, u_2^-] \overrightarrow{C}[u_2^+, u_1] P_1 x_0$ which is longer than C, a contradiction.

Case 2 $|T| \geq 2$.

Notice that $T_i \subseteq V(G) - S$ for each i with $1 \leq i \leq k$. We, by Claim 1, have that $G[T_1 \cup T_2 \cup \cdots \cup T_k]$ is complete. Since $|T| \geq 2$, there exist two different indexes i and j such that $u_i^- \in T_i$, $u_j^- \in T_j$ and therefore $u_i^- u_j^- \in E(G)$, where $1 \leq i, j \leq k$. Then we can easily find a cycle in G which is longer than C, a contradiction.

So the proof of Theorem 1 is completed.

References

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