Italian domination in digraphs

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Abstract

An *Italian dominating function* on a digraph D with vertex set $V(D)$ is defined as a function $f : V(D) \to \{0, 1, 2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under f or one in-neighbor w with $f(w) = 2$. The weight of an Italian dominating function is the sum $\sum_{v \in V(D)} f(v)$, and the minimum weight of an Italian dominating function f is the *Italian domination number*, denoted by $\gamma_I(D)$. We initiate the study of the Italian domination number for digraphs, and we present different sharp bounds on $\gamma_I(D)$. In addition, we determine the Italian domination number of some classes of digraphs. As applications of the bounds and properties on the Italian domination number in digraphs, we give some new and some known results of the Italian domination number in graphs.

Keywords: Digraph, Italian dominating function, Italian domination number

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1 Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [6]. Specifically, let D be a finite digraph with neither loops nor multiple arcs (but pairs of opposite arcs are allowed) with vertex set $V(D) = V$ and arc set $A(D) = A$. The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the *order* and the *size* of the digraph D, respectively. For two different vertices $u, v \in V(D)$, we use uv to denote the arc with tail u and head v, and we also call v an *out-neighbor* of u and u an *in-neighbor* of v. For $v \in V(D)$, the *out-neighborhood* and *in-neighborhood* of v, denoted by $N_D^+(v) = N^+(v)$ and $N_D^-(v) = N^-(v)$, are the sets of out-neighbors and in-neighbors of v, respectively. The *closed out-neighborhood* and *closed in-neighborhood* of a vertex $v \in V(D)$ are the sets $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$, respectively. In general, for a set $X \subseteq V(D)$, we define $N_D^+(X) = N^+(X) = \bigcup_{v \in X} N^+(v)$ and $N_D^-(X) = N^-(X) = \bigcup_{v \in X} N^-(v)$. The *out-degree* and *in-degree* of a vertex v are defined by $d_D^+(v) = d^+(v) = |N^+(v)|$ and $d_D^-(v) = d^-(v) = |N^-(v)|$. The *maximum out-degree*, *maximum in-degree*, *minimum out-degree* and *minimum in-degree* of a digraph D are denoted by $\Delta^+(D) = \Delta^+$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\delta^-(D) = \delta^-$, respectively. A digraph D is *r-out-regular* when $\Delta^+(D) = \delta^+(D) = r$ and *r-in-regular* when $\Delta^-(D) = \delta^-(D) = r$. If D is r-out-regular and r-in-regular, then D is called *r-regular*. The *underlying graph* of a digraph D is that graph obtained by replacing each arc uv or symmetric pairs uv, vu of arcs by the edge uv. If X is a nonempty subset of the vertex set $V(D)$ of a digraph D, then $D[X]$ is the subdigraph of D *induced* by X. A digraph D is *bipartite* if its underlying graph is bipartite. Let K_n^* be the complete digraph of order n, C_n the oriented cycle of order n and $K_{p,q}^*$ the complete bipartite digraph with partite sets X and Y, where $|X| = p$ and $|Y| = q$.

A set $S \subseteq V(D)$ of a digraph D is a *dominating set* of D if $N^+[S] = V(D)$. The *domination number* $\gamma(D)$ of a digraph D is the minimum cardinality of a dominating set of D. The domination number of a digraph was introduced by Fu [5]. A set $S \subseteq V(D)$ of a digraph D is a *2-dominating set* of D if every vertex of $V(D) \setminus S$ has at least two in-neighbors in S. The *2-domination number* $\gamma_2(D)$ of a digraph D is the minimum cardinality of a 2-dominating set of D.

In [1], a *2-rainbow dominating function* of a digraph D is defined as a function f from $V(D)$ to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V(D)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N^-(v)} f(u) = \{1, 2\}$ is fulfilled. The *weight* of a 2-rainbow dominating function f is the value $\sum_{v \in V(D)} |f(v)|$. The *2-raimbow domination number* $\gamma_{r2}(D)$ is the minimum weight of 2-rainbow dominating function of D.

A *Roman dominating function* on a digraph D is defined in [8] as a function $f : V(D) \to \{0, 1, 2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has an in-neighbor w with $f(w) = 2$. The weight of a Roman dominating function is the sum $\sum_{v \in V(D)} f(v)$, and the minimum weight of a Roman dominating function f is the *Roman domination number*, denoted by $\gamma_R(D)$.

In this paper we continue the study of Roman dominating functions in graphs and digraphs (see, for example, [2, 3, 4, 7, 9]). Inspired by an idea of the work [2], we define the Italian domination number of a digraph as follows. An *Italian dominating function* on a digraph D is defined as a function $f : V(D) \to \{0,1,2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under f or one in-neighbor w with $f(w) = 2$. The weight of an Italian dominating function is the value $\omega(f) = \sum_{v \in V(D)} f(v)$, and the minimum weight of an Italian dominating function f is the *Italian domination number*, denoted by $\gamma_I(D)$. An Italian dominating function of D with weight $\gamma_I(D)$ is called a $\gamma_I(D)$-*function* of D. An Italian dominating function f of a digraph D can be represented by the ordered partition (V_0, V_1, V_2) of $V(D)$, where $V_i = \{v \in V(D) \mid f(v) = i\}$ for $i \in \{0,1,2\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. In [9], we find the inequality chain

$$\gamma(D) \leq \gamma_R(D) \leq 2\gamma(D).$$

Clearly, every Roman dominating function is an Italian dominating function of D and thus $\gamma_I(D) \leq \gamma_R(D)$. Furthermore, since the set $V_1 \cup V_2$ in an Italian dominating function is a dominating set of D, we observe that $\gamma(D) \leq \gamma_I(D)$. Altogether, we obtain

$$\gamma(D) \leq \gamma_I(D) \leq \gamma_R(D) \leq 2\gamma(D). \tag{1}$$

Our purpose in this paper is to initiate the study of the Italian domination number for digraphs. We present different sharp bounds on $\gamma_I(D)$. In addition, we determine the Italian domination number of some classes of digraphs. As applications of these results, we give some new and known properties of the Italian domination number (also known as the Roman $\{2\}$-domination number [2]) of graphs.

We make use of the following results in this paper.

Proposition A. [8] If D is a digraph of order n, then $\gamma_R(D) \leq n - \Delta^+(D) + 1$.

Proposition B. [9] Let D be a digraph. Then $\gamma_R(D) = 2\gamma(D)$ if and only if D has a Roman dominating function $f = (V_0, V_1, V_2)$ of weight $\gamma_R(D)$ with $V_1 = \emptyset$.

2 Bounds and properties on the Italian domination number

Theorem 1. Let D be a digraph of order n. Then $\gamma_I(D) \leq n$ and $\gamma_I(D) = n$, if and only if $\Delta^+(D), \Delta^-(D) \leq 1$.

Proof. Define the function $g : V(D) \to \{0,1,2\}$ by $g(x) = 1$ for $x \in V(D)$. Obviously, g is an Italian dominating function on D of weight n and thus $\gamma_I(D) \le n$.

Assume next that $\gamma_I(D) = n$. If $\Delta^+(D) \ge 2$, then $\gamma_I(D) \le \gamma_R(D)$ and Proposition A lead to the contradiction

$$n = \gamma_I(D) \le \gamma_R(D) \le n - \Delta^+(D) + 1 \le n - 1.$$

If $\Delta^-(D) \ge 2$, then let w be a vertex with $d^-(w) = \Delta^-(D)$. Define the function $f : V(D) \to \{0,1,2\}$ by $f(w) = 0$ and $f(x) = 1$ for $x \in V(D)\setminus\{w\}$. Then f is an Italian dominating function on D of weight $n-1$ and thus $\gamma_I(D) \le n-1$. This contradiction shows that $\Delta^+(D), \Delta^-(D) \le 1$.

Now assume that $\Delta^+(D), \Delta^-(D) \le 1$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_I(D)$-function. Suppose to the contrary that $\gamma_I(D) < n$. Then

$$|V_0| + |V_1| + |V_2| = n > \gamma_I(D) = |V_1| + 2|V_2|$$

implies $|V_0| \ge |V_2| + 1$. If there exists a vertex $w \in V_0$ which has no in-neighbor in V_2, then w has at least two in-neighbors in V_1, and we obtain the contradiction $\Delta^-(D) \ge d^-(w) \ge 2$. So we may assume that each vertex of V_0 has at least one in-neighbor in V_2. Since $\Delta^+(D) \le 1$, we obtain the contradiction

$$|V_2| \ge \sum_{x \in V_2} d^+(x) \ge |V_0| \ge |V_2| + 1,$$

and the proof is complete. □

Corollary 2. If D is a directed cycle or a directed path of order n, then $\gamma_I(D) = n$.

Proposition 3. Let D be a digraph of order $n \ge 2$. Then $\gamma_I(D) = 2$ if and only if $\Delta^+(D) = n-1$ or there exist two different vertices u and v such that $N^+(u) \cap N^+(v) = V(D) \setminus \{u, v\}$.

Proof. Since $n \ge 2$, we observe that $\gamma_I(D) \ge 2$. If $\Delta^+(D) = n-1$, then let w be a vertex with $d^+(w) = \Delta^+(D)$. Define the function $f : V(D) \to \{0,1,2\}$ by $f(w) = 2$ and $f(x) = 0$ for $x \in V(D) \setminus \{w\}$. Then f is an Italian dominating function on D of weight 2 and thus $\gamma_I(D) \le 2$ and so $\gamma_I(D) = 2$. If there exist two different vertices u and v such that $N^+(u) \cap N^+(v) = V(D) \setminus \{u, v\}$, then define the function $g : V(D) \to \{0,1,2\}$ by $f(u) = g(v) = 1$ and $f(x) = 0$ for $x \in V(D) \setminus \{u, v\}$. Then g is an Italian dominating function on D of weight 2 and thus $\gamma_I(D) = 2$.

Conversely, assume that $\gamma_I(D) = 2$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_I(D)$-function. Then $|V_2| = 1$ and $|V_1| = 0$ or $|V_2| = 0$ and $|V_1| = 2$. Assume first that $|V_2| = 1$ and $|V_1| = 0$, and let $V_2 = \{w\}$. It follows that w is an in-neighbor of all vertices $x \in V(D) \setminus \{w\}$ and thus $\Delta^+(D) = d^+(w) = n-1$. Second assume that $|V_2| = 0$ and $|V_1| = 2$, and let $V_1 = \{u, v\}$. Then u and v are in-neighbors of all vertices $x \in V(D) \setminus \{u, v\}$ and thus $N^+(u) \cap N^+(v) = V(D) \setminus \{u, v\}$. □

Proposition 4. Let D be a digraph of order $n \geq 3$ such that $\Delta^+(D) \leq n-2$ and there doesn't exist two different vertices a and b such that $N^+(a) \cap N^+(b) = V(D) \setminus \{a, b\}$. Then $\gamma_I(D) = 3$ if and only if $\Delta^+(D) = n-2$ or there exist three pairwise different vertices u, v and w such that each vertex $x \in V(D) \setminus \{u, v, w\}$ has at least two in-neighbors in the set $\{u, v, w\}$.

Proof. According to Proposition 3, we observe that $\gamma_I(D) \geq 3$.

If $\Delta^+(D) = n-2$, then let w be a vertex with $d^+(w) = \Delta^+(D)$, and let $\{z\} = V(D) \setminus N^+[w]$. Define the function $f : V(D) \to \{0, 1, 2\}$ by $f(w) = 2$, $f(z) = 1$ and $f(x) = 0$ for $x \in V(D) \setminus \{w, z\}$. Then f is an Italian dominating function on D of weight 3 and thus $\gamma_I(D) = 3$.

If there exist three pairwise different vertices u, v and w such that each vertex $x \in V(D) \setminus \{u, v, w\}$ has at least two in-neighbors in the set $\{u, v, w\}$, then define the function $g : V(D) \to \{0, 1, 2\}$ by $f(u) = g(v) = g(w) = 1$ and $f(x) = 0$ for $x \in V(D) \setminus \{u, v, w\}$. Then g is an Italian dominating function on D of weight 3 and thus $\gamma_I(D) = 3$.

Conversely, assume that $\gamma_I(D) = 3$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_I(D)$-function. Then $|V_2| = 1$ and $|V_1| = 1$ or $|V_2| = 0$ and $|V_1| = 3$. Assume first that $|V_2| = 1$ and $|V_1| = 1$, and let $V_2 = \{w\}$ and $V_1 = \{u\}$. It follows that w is an in-neighbor of all vertices $x \in V(D) \setminus \{u, w\}$ and thus $\Delta^+(D) = d^+(w) = n-2$. Second assume that $|V_2| = 0$ and $|V_1| = 3$, and let $V_1 = \{u, v, w\}$. Then each vertex $x \in V(D) \setminus \{u, v, w\}$ has at least two in-neighbors in the set $\{u, v, w\}$. □

Proposition 5. For any complete bipartite digraph we have:

1. $\gamma_I(K^*_{1,n}) = \gamma_I(K^*_{2,n}) = 2$,

2. $\gamma_I(K^*_{3,n}) = 3$ for $n \geq 3$,

3. $\gamma_I(K^*_{m,n}) = 4$ for $n \geq 4$.

Proof. Propositions 3 and 4 imply 1. and 2. immediately.

3. By Propositions 3 and 4, we deduce that $\gamma_I(K^*_{m,n}) \geq 4$. If X, Y is a bipartition of $K^*_{m,n}$, then let $u \in X$ and $v \in Y$. Define the function $f : V(K^*_{m,n}) \to \{0, 1, 2\}$ by $f(u) = f(v) = 2$ and $f(x) = 0$ for $x \in V(K^*_{m,n}) \setminus \{u, v\}$. Then f is an Italian dominating function on $K^*_{m,n}$ of weight 4 and thus $\gamma_I(K^*_{m,n}) = 4$. □

Propositions 3 and 4 also imply the next result.

Example 6. Let $G = K^*_{n_1,n_2,\ldots,n_r}$ be the complete r-partite digraph with $r \geq 3$ and $n_1 \leq n_2 \leq \ldots \leq n_r$. If $n_1 \leq 2$, then $\gamma_I(D) = 2$, and if $n_1 \geq 3$, then $\gamma_I(D) = 3$.

Theorem 7. If D is a digraph of order n, then

$$\gamma_I(D) \geq \left\lceil \frac{2n}{\Delta^+(D)+2} \right\rceil.$$

Proof. Let $\Delta^+ = \Delta^+(D)$. If $\Delta^+ = 0$, then $\gamma_I(D) = n$, and the desired lower bound is valid.

Let now $\Delta^+ \geq 1$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_I(D)$-function. Let $V_0' = \{x \in V_0 \mid N^-(x) \cap V_2 \neq \emptyset\}$ and $V_0'' = V_0 \setminus V_0'$. Since every vertex of V_2 can have at most Δ^+ out-neighbors in V_0, we observe $|V_0'| \leq \Delta^+|V_2|$. Using the fact that each vertex of V_0'' has at least two in-neighbors in V_1 and every vertex of V_1 has at most Δ^+ out-neighbors in V_0'', we deduce that $2|V_0''| \leq \Delta^+|V_1|$. Therefore we obtain

$$\begin{aligned}\gamma_I(D)(\Delta^+ + 2) &= (|V_1| + 2|V_2|)(\Delta^+ + 2)\\ &= \Delta^+|V_1| + 2\Delta^+|V_2| + 2|V_1| + 4|V_2|\\ &\geq 2|V_0''| + 2|V_0'| + 2|V_1| + 4|V_2|\\ &= 2n + 2|V_2| \geq 2n,\end{aligned}$$

and thus $\gamma_I(D) \geq \lceil (2n)/(\Delta^+ + 2) \rceil$. □

The proof of Theorem 7 leads to $\gamma_I(D) \geq \lceil (2n+2)/(\Delta^+ + 2) \rceil$ if there exists a $\gamma_I(D)$-function $f = (V_0, V_1, V_2)$ with $V_2 \neq \emptyset$. Proposition 5 1. and 2. and Example 6 show that Theorem 7 is sharp.

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u, v the arc uv belongs to $\overline{D}$ if and only if uv does not belong to D. As an application of Proposition A we will prove the following Nordhaus-Gaddum type result.

Theorem 8. If D is a digraph of order n, then

$$\gamma_I(D) + \gamma_I(\overline{D}) \leq n + 3.$$

If $\gamma_I(D) + \gamma_I(\overline{D}) = n + 3$, then D is out-regular.

Proof. Since $\gamma_I(D) \leq \gamma_R(D)$ and $\Delta^+(\overline{D}) = n - 1 - \delta^+(D)$, Proposition A implies that

$$\begin{aligned}\gamma_I(D) + \gamma_I(\overline{D}) &\leq (n - \Delta^+(D) + 1) + (n - \Delta^+(\overline{D}) + 1)\\ &= (n - \Delta^+(D) + 1) + (n - (n - 1 - \delta^+(D)) + 1\\ &= n - \Delta^+(D) + 1 + \delta^+(D) + 2 \leq n + 3,\end{aligned}$$

and this is the desired bound. If D is not out-regular, then $\Delta^+(D) - \delta^+(D) \geq 1$, and thus the inequality chain above leads to the better bound $\gamma_I(D) + \gamma_I(\overline{D}) \leq n+2$. □

If D is the directed cycle of order 3 or the regular tournament of order 5 with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and arc set

$$\{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4, v_3v_5, v_4v_5, v_4v_1, v_5v_1, v_5v_2\},$$

then we have equality in the inequality of Theorem 8.

If D is not out-regular, then Theorem 8 yields to $\gamma_I(D)+\gamma_I(\overline{D}) \leq n+2$. If P_n is a directed path of order $n \geq 2$, then Corollary 2 implies that $\gamma_I(P_n) = n$. Since $\Delta^+(\overline{P_n}) = n-1$, Proposition 3 implies $\gamma_I(\overline{P_n}) = 2$ and thus $\gamma_I(P_n) + \gamma_I(\overline{P_n}) = n+2$. Therefore the inequality above is sharp.

3 Domination, Italian domination and 2-rainbow domination numbers

Proposition 9. Let D be a digraph. Then $\gamma_I(D) = 2\gamma(D)$ if and only if D has a $\gamma_I(D)$-function $f = (V_1, V_2, V_3)$ with $V_1 = \emptyset$.

Proof. First assume that $\gamma_I(D) = 2\gamma(D)$. Then the inequality chain (1) implies that $\gamma_R(D) = 2\gamma(D)$, and therefore it follows from Proposition B that there exists a Roman dominating function $f = (V_0, V_1, V_2)$ of weight $\gamma_R(D)$ with $V_1 = \emptyset$. This is also a $\gamma_I(D)$-function with $V_1 = \emptyset$.

Conversely, let $f = (V_0, V_1, V_2)$ be a $\gamma_I(D)$-function with $V_1 = \emptyset$. Then $\gamma_I(D) = 2|V_2|$, and V_2 is a dominating set of D. It follows that $2\gamma(D) \leq 2|V_2| = \gamma_I(D)$ and so (1) implies $\gamma_I(D) = 2\gamma(D)$. □

Proposition 10. Let D be a digraph with the property that $\gamma_I(D) = \gamma(D)$. If $f = (V_0, V_1, V_2)$ is a $\gamma_I(D)$-function, then $|V_2| = 0$ and the subdigraph $D[V_1]$ is empty.

Proof. Since $\gamma_I(D) = \gamma(D)$, we observe that $\gamma(D) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_I(D) = \gamma(D)$. This leads to $|V_2| = 0$. If $|V_0| = 0$, then $\gamma(D) = |V_1| = n$ and thus the subdigraph $D[V_1]$ is empty. Let now $|V_0| \geq 1$. Then each vertex $v \in V_0$ has at least two in-neighbors in V_1. Suppose that there exists and arc uv in $D[V_1]$.

If neither u nor v has an out-neighbor in V_0, then we see that $g = (V_0 \cup \{v\}, V_1 \setminus \{v\}, \emptyset)$ is dominating set of D, a contradiction. Using the fact that each vertex $v \in V_0$ has at least two in-neighbors in V_1, we observe that the same is valid when u or v has an out-neighbor in V_0, and the proof is complete. □

Example 11. Let H be a digraph consisting of an arbitrary digraph Q with vertex set $V(Q) = \{v_1, v_2, \ldots, v_k\}$ with $k \geq 2$ and a further vertex set $V_1 = \{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$ such that $x_iv_i, y_iv_i \in A(H)$ for $1 \leq i \leq k$. It is easy to see that $\gamma(H) = \gamma_I(H)$, and therefore equality in the first inequality of (1).

Observation 12. If D is a digraph, then $\gamma_I(D) \leq \gamma_{r2}(D)$.

Proof. For every 2-rainbow dominating function f on D, we define the function g on D by $g(u) = 2$ if $f(u) = \{1, 2\}$, $g(u) = 1$ if $f(u) = \{1\}$ or $f(u) = \{2\}$ and $g(u) = 0$ if $f(u) = \emptyset$. Then g is an Italian dominating function on D and thus $\gamma_I(D) \leq \gamma_{r2}(D)$. □

Digraphs D of order n with $\Delta^+(D) = n-1$ or directed paths or directed cycles demonstrate that Observation 12 is sharp. Using Observation 12 and Theorem 7, we obtain the following known result.

Corollary 13. ([1]) If D is a digraph of order n, then

$$\gamma_{r2}(D) \geq \left\lceil \frac{2n}{\Delta^+(D)+2} \right\rceil.$$

Theorem 14. If D is a digraph of order $n \geq 2$, then $\gamma_{r2}(D) \leq 2\gamma_I(D) - 2$.

Proof. Since $n \geq 2$, we observe that $\gamma_{r2}(D), \gamma_I(D) \geq 2$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_I(D)$-function. If $|V_1| \leq 1$, then define g by $g(x) = \{1, 2\}$ if $f(x) = 2$, $g(x) = \{1\}$ if $f(x) = 1$ and $g(x) = \emptyset$ if $f(x) = 0$. Then g is a 2-rainbow dominating function on D and thus $\gamma_{r2}(D) \leq \gamma_I(D) \leq 2\gamma_I(D) - 2$.

Let now $V_1 = \{v_1, v_2, \ldots, v_t\}$ with $t \geq 2$. Define g by $g(x) = \{1, 2\}$ if $f(x) = 2$ or $x \in \{v_1, v_2, \ldots, v_{t-2}\}$, $g(v_{t-1}) = \{1\}$, $g(v_t) = \{2\}$ and $g(x) = \emptyset$ if $f(x) = 0$. Then g is a 2-rainbow dominating function on D and therefore

$$\begin{aligned}\gamma_{r2}(D) &\leq 2|V_2| + 2(|V_1| - 2) + 2 = 2|V_2| + 2|V_1| - 2\\ &\leq 2(|V_1| + 2|V_2|) - 2 = 2\gamma_I(D) - 2.\end{aligned}$$

□

The next example will demonstrate that Theorem 14 is sharp.

Example 15. Let H be a digraph consisting of a vertex set A with $|A| = r \geq 3$ and a vertex set V_0 of $\binom{r}{2}$ further vertices. Let each vertex of V_0 have exactly two in-neighbors in A such that the in-neighborhoods of every two different vertices of V_0 are distinct.

The function f with $f(x) = 1$ for $x \in A$ and $f(x) = 0$ for $x \in V_0$ is a $\gamma_I(H)$-function and thus $\gamma_I(H) = r$.

Let $A = \{v_1, v_2, \ldots, v_r\}$. The function g with $g(x) = \{1,2\}$ for $x \in \{v_1, v_2, \ldots, v_{r-2}\}$, $g(v_{r-1}) = \{1\}$, $g(v_r) = \{2\}$ and $g(x) = \emptyset$ for $x \in V_0$ is a 2-rainbow dominating function on H of weight $\gamma_{r2}(H) = 2r - 2$. This shows that $\gamma_{r2}(H) = 2r - 2 = 2\gamma_I(H) - 2$ and thus Theorem 14 is sharp.

Analogously to Proposition 5 in [2], one can prove the next observation.

Observation 16. If D is a digraph, then $\gamma_I(D) \leq \gamma_2(D)$.

4 Graphs

In the last section we will present some new and some known results on the Italian domination number of graphs. If G is a graph with vertex set $V(G)$, then one can define analogously the *Italian domination number* $\gamma_I(G)$, the *2-rainbow domination number* $\gamma_{r2}(G)$, and the *Roman domination number* $\gamma_R(G)$ (see, for example, [2]). Let $N_G(v)$ be the *neighbohood* of a vertex v, and let $\Delta(G)$ be the *maximun degree* of the graph G.

The *associated digraph* of $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}(v) = N^+_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following observation is valid.

Observation 17. If $D(G)$ is the associated digraph of a graph G, then $\gamma_I(D(G)) = \gamma_I(G)$ and $\gamma_{r2}(D(G)) = \gamma_{r2}(G)$.

There are a lot of interesting applications of Observation 17, as for example the following results. Since $\Delta^+(D(G)) = \Delta^-(D(G)) = \Delta(G)$, Observation 17 and Theorem 1 or Proposition 3 or Theorem 7 or Theorem 14 imply the next corollaries.

Corollary 18. Let G be a graph of order n. Then $\gamma_I(G) = n$ if and only if $\Delta(G) \leq 1$.

Corollary 19. Let G be a graph of order $n \geq 2$. Then $\gamma_I(G) = 2$ if and only if $\Delta(G) = n - 1$ or there exist two different vertices u and v such that $N(u) \cap N(v) = V(G) \setminus \{u, v\}$.

Corollary 20. [2] If G is a graph of order n, then

$$\gamma_I(G) \geq \left\lceil \frac{2n}{\Delta(G) + 2} \right\rceil.$$

Corollary 21. If G is a graph of order $n \geq 2$, then $\gamma_{r2}(G) \leq 2\gamma_I(G) - 2$.

In [2], the authors proved that the associated decision problem for Italian domination is NP-complete for bipartite graphs. This result and Observation 17 show that this problem is also NP-complete for bipartite digraphs.

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