k-Frugal List Coloring of Planar Graphs Without Small Cycles *

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Abstract: A graph G is k-frugal colorable if there exists a proper vertex coloring of G such that every color appears at most $k-1$ times in the neighborhood of v. The k-frugal chromatic number, denoted by $\chi_k(G)$, is the smallest integer l such that G is k-frugal colorable with l colors. A graph G is L-list colorable if there exists a coloring c of G for a given list assignment $L = \{L(v) : v \in V(G)\}$ such that $c(v) \in L(v)$ for all $v \in V(G)$. If G is k-frugal L-colorable for any list assignment L with $|L(v)| \geq l$ for all $v \in V(G)$, then G is said to be k-frugal l-list-colorable. The smallest integer l such that the graph G is k-frugal l-list-colorable is called the k-frugal list chromatic number, denoted by $ch_k(G)$. It is clear that $ch_k(G) \geq \lceil \frac{\Delta(G)}{k-1} \rceil + 1$ for any graph G with maximum degree $\Delta(G)$. In this paper, we prove that for any integer $k \geq 4$, if G is a planar graph with maximum degree $\Delta(G) \geq 13k - 11$ and girth $g \geq 6$, then $ch_k(G) = \left\lceil \frac{\Delta(G)}{k-1} \right\rceil + 1$; and if G is a planar graph with girth $g \geq 6$, then $ch_k(G) \leq \left\lceil \frac{\Delta(G)}{k-1} \right\rceil + 2$.

Keywords: k-frugal list coloring; Maximum degree; Planar graphs; Discharging

Mathematics Subject Classification: 05C15

1 Introduction

We consider undirected, finite and simple graphs here. Definitions and notations not given here may be found in [2]. For a vertex v, we use $d_G(v)$, $N_G(v)$ (or simply $d(v)$, $N(v)$) to denote the degree of v and the neighborhood of v. For a plane graph G, we use $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$

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and $F(G)$ to denote its vertex set, edge set, maximum degree, minimum degree and face set, respectively. A k^+*-vertex* v (k^-*-vertex*) is a vertex v with degree at least k (at most k). Similarly, we can get the definitions for $d(f)$, k^+-face and k^--face. The boundary of a t-face f is denoted by $\partial(f) = [v_1, \cdots, v_t]$. A *proper* l*-coloring* of a graph G is a mapping c from $V(G)$ to the color set $\{1, 2, ..., l\}$ such that no two adjacent vertices are assigned the same color. We use $c(v)$ to denote the color of the vertex v and $C_i(v)$ to denote the set of colors which appears i times in $N(v)$.

Frugal coloring of graphs is considered by Hind et al. in [8]. In a vertex coloring c of graph G, we say a vertex v is k*-frugal* if every color appears at most $k-1$ times in $N_G(v)$. We say graph G is k*-frugal colorable* if every vertices of G is k-frugal, and the coloring c is said to be a k*-frugal coloring* of graph G. The k*-frugal chromatic number*, denoted by $\chi_k(G)$, of a graph G is the least integer l such that G is k-frugal colorable with l colors.

The frugal coloring can be generalized to a list coloring version. Let L be a function which assigns to each vertex v of G a set $L(v)$ of positive integers, called the *list* of v. A colouring $c : V \to N$ such that $c(v) \in L(v)$ for all $v \in V$ is called a *list coloring of* G *with respect to* L, or an L*- coloring*, and we say that G is L*-colorable.* A graph G is said to be k*-list-colorable* if it has a list coloring whenever all the lists have length k, i.e., G is L-colorable for any k-list L. The k*-frugal list chromatic number*, denoted by $ch_k(G)$, of a graph G is the least integer l such that G is k-frugal L-colorable for any l-list L.

A *linear* k*-coloring* of a graph G is a proper k-coloring of G such that the subgraph induced by the vertices of any two color classes is the union of vertex-disjoint paths. The linear chromatic number, denoted by $lc(G)$, of the graph G is the smallest number k such that G admits a linear k-coloring. The concept of linear coloring was introduced by Yuster [11]. Obviously, a linear coloring is just a 3-frugal coloring. But the converse may not be true since in a 3-frugal coloring, bicolored cycles are permitted.

Esperet et al. [7] generalized the linear coloring of graphs to a list coloring version and got many results on the linear list chromatic number (denoted by $\Lambda^l(G)$) of some special graphs. It is easy to see that for each graph G, $\left\lceil \frac{\Delta(G)}{k-1} \right\rceil + 1$ is a lower bound of $\chi_k(G)$.

In this paper, we mainly talk about the upper bound of the k-frugal list coloring of the planar graph with girth 6. We use $g(G)$, or simply g, to denote the girth of a graph G which is the length of the shortest cycle of graph G. The linear coloring and frugal coloring of planar graphs have been extensively studied in the past. In 2011, Li et al.[9] showed that $lc(G) \leq \lfloor 0.9\Delta(G) + 5 \rfloor$ if G is planar graph with $\Delta(G) \geq 52$. In 2009, Raspaud et al. [10] proved that $lc(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 4$ when G is planar graph with $g \geq 6$. In 2010, Dong et al. [5] renewed the bound by

$lc(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 3$. In 2014, Dong and Lin [6] got the sharp bound that $lc(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$ when G is a planar graph with $g \geq 6$ and $\Delta(G) \geq 39$.

The list coloring version of planar graphs with girth 6 is also studied in the past. In 2011, Cranston et al. [4] showed that the linear list chromatic number $\Lambda^l(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 2$ for every planar graph G with $g \geq 6$ and $\Delta(G) \geq 9$. Cohen et al. [3] showed that $\Lambda^l(G) \leq \lceil \frac{\Delta(G)}{2} \rceil + 4$ for every planar graph G with $g \geq 6$.

What is the $\chi_k(G)$ and $ch_k(G)$ for larger k? In [1], Amini et al. proved that for all $k \geq 1$, every planar graph G with girth 6 has $\chi_k(G) \leq \lceil \frac{\Delta(G)+4}{k-1} \rceil + 6$. In this paper, we will investigate the k-frugal list coloring of planar graph with $g \geq 6$ for larger k. We get a sharp bound on $ch_k(G)$ as follows.

Theorem 1.1. *Let G be a planar graph with maximum degree $\Delta(G) \geq 13k - 11$ and girth $g \geq 6$, then $ch_k(G) = \left\lceil \frac{\Delta(G)}{k-1} \right\rceil + 1$ for any integer $k \geq 4$.*

If there is no restriction on the maximum degree, we get the following bound on $ch_k(G)$.

Theorem 1.2. *Let G be a planar graph with girth $g \geq 6$, then $ch_k(G) \leq \left\lceil \frac{\Delta(G)}{k-1} \right\rceil + 2$ for any integer $k \geq 4$.*

2 Proof of Theorem 1.1

Suppose, by way of contradiction, that graph G_0 is a planar graph with $\Delta(G_0) \geq 13k - 11$, $g(G_0) \geq 6$ and $ch_k(G_0) > \left\lceil \frac{\Delta(G_0)}{k-1} \right\rceil + 1$. Let L be a $(\lceil \frac{\Delta(G_0)}{k-1} \rceil + 1)$-list such that G_0 is not k-frugal L-colorable. In the following proof, we just use Δ to denote the maximum degree $\Delta(G_0)$. By the assumption, we know that the graph set $\mathcal{G} = \{G | G \subseteq G_0, g(G) \geq g(G_0), \Delta(G) \leq \Delta(G_0) = \Delta, ch_k(G) > \left\lceil \frac{\Delta}{k-1} \right\rceil + 1\}$ is nonempty, since $G_0 \in \mathcal{G}$. Let $G \in \mathcal{G}$ be a graph with the fewest edges. Then we have $ch_k(H) \leq \left\lceil \frac{\Delta}{k-1} \right\rceil + 1$ for any subgraph $H \subset G$, which implies that H is k-frugal L-colorable.

Now we will first present some structures of G and then apply a discharging procedure to get a contradiction.

Lemma 2.1. *G is connected, and $\delta(G) \geq 2$.*

Proof. Suppose to the contrary that G has a 1-vertex v. By the choice of G, $G - v$ has a k-frugal L-coloring c. Let $N_G(v) = \{u\}$, then v can receive any color except for $c(u)$ and those colors that appear $k-1$ times in $N_G(u)$ (i.e. $C_{k-1}(u)$), so the number of forbidden colors of v is at most

$$1 + \lfloor \frac{d(u) - 1}{k - 1} \rfloor = \lceil \frac{d(u)}{k - 1} \rceil < \lceil \frac{\Delta}{k - 1} \rceil + 1.$$

We can extend c to the whole graph G, a contraction

□

Lemma 2.2. *For any $(k-1)^-$-vertex v of G with $N_G(v) = \{v_1, v_2, \cdots, v_x\}$, we have* $\sum_{i=1}^{x} \lceil \frac{d(v_i)}{k-1} \rceil \geq \lceil \frac{\Delta}{k-1} \rceil + 1$.

Proof. Suppose to the contrary that $\sum_{i=1}^{x} \lceil \frac{d(v_i)}{k-1} \rceil < \lceil \frac{\Delta}{k-1} \rceil + 1$. By the choice of G , $G - v$ has a k-frugal L-coloring c. Then v can receive any colors except for $c(v_i)$ and those colors that appear $k-1$ times in $N_G(v_i)$, where $i = 1, 2, \cdots, x$. So the number of forbidden colors of v is at most

$$x + \sum_{i=1}^{x} \lfloor \frac{d(v_i) - 1}{k-1} \rfloor = \sum_{i=1}^{x} \lceil \frac{d(v_i)}{k-1} \rceil < \lceil \frac{\Delta}{k-1} \rceil + 1.$$

Thus, we can extend c to the whole graph G, a contraction.

□

Now we give some definitions and notations that we will use in the following section.

Let v be a 2-vertex with $u \in N(v)$, then v is called *a light 2-vertex* if u is also a 2-vertex. Suppose v is a 2-vertex with $N_G(v) = \{v_1, v_2\}$. If v_1 is a $(k-1)^-$-vertex, then $d(v_2) \geq \Delta - (k-2) \geq 13k - 11 - k + 2 = 12k - 9$ by Lemma 2.2. Each $(12k-9)^+$-vertex is called a *Δ^ϵ-vertex*. Let u_1uvv_1 be a path, if u and v are both 2-vertices, then the path is called an *u_1v_1-thread* . For an edge $uv \in E(G)$, it is called *a heavy-edge* if u is a Δ^ϵ-vertex and v is a 4^+-vertex.

For convenient, we divided the faces of G in different types according to its boundary.

For a 6-face f, it is *Type-(I)* if the boundary $\partial(f)$ has 2 threads; it is *Type-(II)* if $\partial(f)$ has exactly 1 thread and at least 1 heavy-edge; it is *Type-(III)* if $\partial(f)$ has exactly 1 thread and is not Type-(II) (that means there exists at least one 3-vertex on the boundary $\partial(f)$); it is *Type-(IV)* if there is no threads on $\partial(f)$.

For a 7-face f, it is *Type-(I)* if the boundary $\partial(f)$ has 2 threads; *Type-(II)* if $\partial(f)$ has exactly 1 thread; and *Type-(III)* if $\partial(f)$ has no threads.

According to the definitions and notations above, we can give some lemmas now.

Lemma 2.3. *Each 3-vertex can be incident with at most two Type(III)-6-faces.*

Proof. Suppose f is a Type-(III)-6-face with boundary $\partial(f) = \{v_1, \cdots, v_6\}$, where $(v_1 \cdots v_4)$-path is a v_1v_4-thread, then v_2 and v_3 are both 2-vertices, v_1

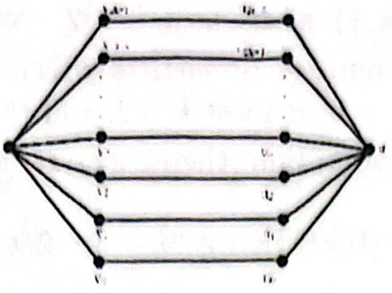

Figure 1: uv-threads

and v_4 are both Δ^{ϵ}-vertices. Without loss of generality, let v_5 be a 3-vertex, then v_6 must be a 3^--vertex since there is no heavy-edge on the boundary of f. Suppose $N(v_5) = \{v_4, v_6, w\}$ and the faces incident with v_4v_5 and v_5v_6 (besides f) are f_1 and f_2, respcetively. If f_1 is a Type-(III)-6-face, then w is a 3^--vertex, which means that f_2 cannot be a Type-(III)-6-face. □

Lemma 2.4. *Let u and v be any two Δ^{ϵ}-vertices of graph G, then the number of uv-threads is at most $2k-3$.*

Proof. Suppose there are at least $2k-2$ uv-threads in G, which are uu_iv_iv-pathes, $i = 0, \cdots, 2k-3$ (see Figure 1.). By the choice of G, $G' := G - \{v_0, u_0\}$ has a k-frugal L-coloring c, then the admit color set of v_0 is $|L^*(v_0)| = |L(v_0) \backslash (C_{k-1}(v) \cup \{c(v)\})| \geq \lceil \frac{\Delta}{k-1} \rceil + 1 - \lfloor \frac{\Delta-1}{k-1} \rfloor - 1 = 1$. Similarly, the admit color set of u_0 is $|L^*(u_0)| \geq 1$.

If $|L^*(v_0)| > 1$ or $|L^*(u_0)| > 1$, we can extend c to the whole graph G easily.

Now let $|L^*(v_0)| = |L^*(u_0)| = 1$ with $L^*(v_0) = \{\alpha\}$ and $L^*(u_0) = \{\beta\}$. If $\alpha \neq \beta$, we can color v_0 with α and u_0 with β, then we get a k-frugal L-coloring of the whole graph G.

Thus, we assume $\alpha = \beta$, $L(v_0) = \{\alpha_1, \alpha_2, ..., \alpha_{\lceil \frac{\Delta}{k-1} \rceil}, \alpha\}$ and $L(u_0) = \{\beta_1, \beta_2, ..., \beta_{\lceil \frac{\Delta}{k-1} \rceil}, \alpha\}$. Morever, since $|L^*(v_0)| = |L^*(u_0)| = 1$, we know that $c(v) \cup C_{k-1}(v) \subseteq L(v_0)$ and $|c(v) \cup C_{k-1}(v)| = \lfloor \frac{\Delta-1}{k-1} \rfloor + 1 = \lceil \frac{\Delta}{k-1} \rceil$. It is similar to u. Without loss of generality, suppose $c(v) = \alpha_1$ and $c(u) = \beta_1$, then we know that $\forall \alpha_i \subseteq C_{k-1}(v)$ and $\forall \beta_i \subseteq C_{k-1}(u)$, where $i = 2, 3, \cdots, \lceil \frac{\Delta}{k-1} \rceil$.

On the other hand, $d_{G'}(v) \leq \Delta - 1$, thus we have

$$|N_{G'}(v)| - |C_{k-1}(v)| \cdot (k-1) \leq \Delta - 1 - (\lceil \frac{\Delta}{k-1} \rceil - 1) \times (k-1) \leq k-2,$$

which means that the number of vertices which satisfy $w \in N_{G'}(v)$ and

$c(w) \notin \{\alpha_1, \alpha_2, ..., \alpha_{\lceil \frac{\Delta}{k-1} \rceil}\}$ is at most $k-2$. We denote the veretx set by W_v. Similarly, the number of vertices which satisfy $\mu \in N_{G'}(u)$ and $c(\mu) \notin \{\beta_1, \beta_2, ..., \beta_{\lceil \frac{\Delta}{k-1} \rceil}\}$ is at most $k-2$, and the set is denoted by W_u.

According to the assumption, there exist at least $2k-3$ uv-thread in G', and

$$2k-3-|W_v \cup W_u| \geq 2k-3-2(k-2)=1.$$

So we can find a pair of vertices v_i and u_i, $(i=1,2,...,2k-3)$, such that $c(v_i) \in \{\alpha_1, \alpha_2, ..., \alpha_{\lceil \frac{\Delta}{k-1} \rceil}\}$ and $c(u_i) \in \{\beta_1, \beta_2, ..., \beta_{\lceil \frac{\Delta}{k-1} \rceil}\}$.

Now, we can erease the color of v_i first, then color v_0 with $c(v_i)$, u_0 with α, and recolor v_i with a color in $L(v_i) \setminus \{c(u_i), c(v)(=\alpha_1), \alpha_2, ..., \alpha_{\lceil \frac{\Delta}{k-1} \rceil}\}$ if there exists one. And then we obtain a k-frugal L-coloring of G. Similarly, we can extend the coloring c to the whole graph G if $|L(u_i) \setminus \{c(v_i), c(u)(=\beta_1), \beta_2, ..., \beta_{\lceil \frac{\Delta}{k-1} \rceil}\}| \geq 1$.

Now suppose that $L(v_i) = \{c(u_i), c(v)(=\alpha_1), \alpha_2, ..., \alpha_{\lceil \frac{\Delta}{k-1} \rceil}\}$ and $L(u_i) = \{c(v_i), c(u)(=\beta_1), \beta_2, ..., \beta_{\lceil \frac{\Delta}{k-1} \rceil}\}$. We can first erease the colors of v_i and u_i, then color v_0 with $c(v_i)$, color u_0 with α, and recolor v_i with $c(u_i)$ and $u_i$with $c(v_i)$. It is easy to see that the obtained coloring is a k-frugal L-coloring of G, so we can also extend the coloring c to the whole graph G, a contradiction.

□

Now we apply a discharging procedure to get a contradiction. We assign an original weight $\omega(v) = 2d(v) - 6$ to each vertex v and a weight $\omega(f) = d(f) - 6$ to each face f. By Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we get

$$\sum_{x \in V \cup F} \omega(x) = -12.$$

If we obtain a new weight $\omega^*(x)$ for all $x \in V \cup F$ by transferring weights from one element to another, then we also have

$$\sum_{x \in V \cup F} \omega^*(x) = -12$$

If these transfers result in $\omega^*(x) \geq 0$ for all $x \in V \cup F$, then we get a contradiction and the theorem is proved.

Our procedure has two steps. In the first step, we discharge the weight by the following rules in order.

(R1). Each 4^+-vertex v transfers $\frac{2d(v)-6}{d(v)}$ to each adjacent 3^--vertex.

(R2). Each 3-vertex transfers $\frac{2}{4k-3}$ to each adjacent 2-vertex.

(R3). Each 3-vertex transfers $1 - \frac{3}{4k-3}$ to each adjacent Type-(III)-6-face.

(R4). For each heavy edge $e = uv$ and $e \in \partial(f)$, if u is a Δ^{ϵ}-vertex, then u transfers $\frac{3}{4}$ to the incident face f; if u is not a Δ^{ϵ}-vertex, then u transfers $\frac{1}{4}$ to f.

Let $\omega'(x)$ be the weight of $x \in V \cup F$ after the first step.

Claim 2.1. *$\omega'(v) \geq 0$ for all $v \in V(G)$ except light 2-vertices. If v is a light 2-vertex, then $\omega'(v) \geq -\frac{2}{4k-3}$.*

Proof. For a 4^+-vertex v, if v is not a Δ^{ϵ}-vertex, we assume that v is adjacent to x_0 3^--vertices and $(d(v) - x_0)$ 4^+-vertices. Then, by R1 and R4, we have that $\omega'(v) = 2d(v) - 6 - \frac{2d(v)-6}{d(v)} \times x_0 - (\frac{1}{4} + \frac{1}{4}) \times (d(v) - x_0) \geq (\frac{3}{2} - \frac{6}{d(v)})(d(v) - x_0) \geq 0$. If v is a Δ^{ϵ}-vertex, then $d(v) \geq 12k - 9$. Let v be adjacent to x_0 3^--vertices and $(d(v) - x_0)$ 4^+-vertices, then we have $\omega'(v) = 2d(v) - 6 - \frac{2d(v)-6}{d(v)} \times x_0 - (\frac{3}{4} + \frac{3}{4}) \times (d(v) - x_0) = 2d(v) - 6 - (2 - \frac{6}{d(v)}) \times x_0 - \frac{3}{2} \times (d(v) - x_0) \geq 2d(v) - 6 - (2 - \frac{6}{d(v)}) \times (x_0 + d(v) - x_0) = 0$ by R1 and R4.

Let $d(v) = 3$ and $N_G(v) = \{x, y, z\}$ with $d(x) \leq d(y) \leq d(z)$.

Case 1. Suppose v is not incident with any Type-(III)-6-face.

If $d(x) \geq 3$, then $\omega'(v) = \omega(v) = 0$.

If $d(x) = 2 < d(y)$, then $\lceil \frac{2}{k-1} \rceil + \lceil \frac{d(y)}{k-1} \rceil + \lceil \frac{d(z)}{k-1} \rceil \geq \lceil \frac{\Delta}{k-1} \rceil + 1$ by Lemma 2.2, which implies that $d(y) + d(z) \geq \lceil \frac{d(y)}{k-1} \rceil + \lceil \frac{d(z)}{k-1} \rceil \geq \lceil \frac{13k-11}{k-1} \rceil = 14$. Thus y and z cannot be 3-vertices at the same time.

By R1 and R2, we have $\omega'(v) \geq \omega(v) + \frac{2d(y)-6}{d(y)} + \frac{2d(z)-6}{d(z)} - \frac{2}{4k-3} \geq 4 - \frac{2}{16-3} - 6 \times (\frac{1}{d(y)} + \frac{1}{d(z)}) > 4 - \frac{2}{13} - 6 \times (\frac{1}{3} + \frac{1}{4}) > 0$.

If $d(x) = d(y) = 2$, then $\lceil \frac{2}{k-1} \rceil + \lceil \frac{2}{k-1} \rceil + \lceil \frac{d(z)}{k-1} \rceil \geq \lceil \frac{\Delta}{k-1} \rceil + 1$ by Lemma 2.2, which implies that $d(z) \geq \lceil \frac{d(z)}{k-1} \rceil \geq \lceil \frac{13k-11}{k-1} \rceil - 1 = 13$. Thus, we have $\omega'(v) \geq \omega(v) + \frac{2\times 13-6}{13} - 2 \times \frac{2}{4k-3} > 0$.

Case 2. If v is incident with some Type-(III)-6-face f, where $\partial(f) = vv_1v_2 \cdots v_5$, then, by Lemma 2.3, the number of Type-(III)-6-faces is at most 2. Morever, one of $\{v_1, v_5\}$ is Δ^{ϵ}-vertex. Without loss of generality, let v_5 be a Δ^{ϵ}-vertex, then v_1 is a 3^--vertex since f is Type-(III)-6-face. Thus we have $\omega'(v) \geq \omega(v) + \frac{2\times(12k-9)-6}{12k-9} - 2 \times \frac{2}{4k-3} - 2 \times (1 - \frac{3}{4k-3}) = 0$.

Now let $d(v) = 2$ and $N_G(v) = \{x, y\}$ with $d(x) \leq d(y)$.

If $d(x) = 2$, then v and x are light 2-vertices, which implies that $d(y) \geq 12k - 9$ by Lemma 2.2. By R1, we have that $\omega'(v) \geq -2 + \frac{2\times(12k-9)-6}{12k-9} = -\frac{2}{4k-3}$.

If $d(x) = 3$, then $d(y) \geq 12k-9$ and $\omega'(v) \geq -2 + \frac{2\times(12k-9)-6}{12k-9} + \frac{2}{4k-3} = 0$ by Lemma 2.2 , R1 and R2.

If $d(x) \geq 4$, then we have $\lceil\frac{d(x)}{k-1}\rceil + \lceil\frac{d(y)}{k-1}\rceil \geq \lceil\frac{\Delta}{k-1}\rceil + 1$ by Lemma 2.2, which implies that $\frac{d(x)+d(y)}{2} \geq \lceil\frac{d(x)}{k-1}\rceil + \lceil\frac{d(y)}{k-1}\rceil \geq \lceil\frac{13k-11}{k-1}\rceil + 1 = 15$. Thus, $\omega'(v) \geq -2 + \frac{2d(x)-6}{d(x)} + \frac{2d(y)-6}{d(y)} = 2 - 6 \times (\frac{1}{d(x)} + \frac{1}{d(y)}) > 0$.

□

Now we consider the faces.

Claim 2.2. *$\omega'(f) \geq 0$ for every face $f \in F(G)$. In detail,*

$$\omega'(f) \begin{cases} \geq 0 & \text{If } f \text{ is a Type-(I)-6 or (IV)-6-face} \\ \geq 1 & \text{If } f \text{ is a Type-(II)-6 or (II)-7 or (III)-7-face} \\ \geq 1 - \frac{3}{4k-3} & \text{If } f \text{ is a Type-(III)-6-face} \\ \geq 2 & \text{If } f \text{ is a Type-(I)-7-face} \\ \geq d(f) - 6 & \text{If } f \text{ is a } 8^{+}\text{-face} \end{cases}$$

Proof. If f is a 8^{+}-face, then $\omega'(f) \geq \omega(f) = d(f) - 6 \geq 2$.

Let $d(f) = 7$. If f is Type-(I)-7-face, then there must be a heavy-edge on $\partial(f)$ since $\partial(f)$ has two threads. Thus, by R4, we have $\omega'(f) \geq \omega(f) + \frac{1}{4} + \frac{3}{4} = 2$. If f is Type-(II) or Type-(III), then $\omega'(f) \geq \omega(f) = 1$.

Let $d(f) = 6$ now. If f is a Type-(I)-6-face, then $\omega'(f) = \omega(f) = 0$. If f is a Type-(II)-6-face, then by R4, $\omega'(f) \geq \omega(f) + \frac{1}{4} + \frac{3}{4} = 1$. If f is Type-(III), then there exists a 3-vertex incident with f. By R3, we have $\omega'(f) \geq \omega(f) + 1 - \frac{3}{4k-3} = 1 - \frac{3}{4k-3}$. If f is a Type-(IV), then $\omega'(f) \geq \omega(f) = 0$.

□

After the first step, we can see that only light 2-vertices have negative weights.

Now we will give the second step of the discharging.

We define a *thread-combine-transformation* of graph G. For each Type-(I)-6-face $f = [vv_1u_1uu_2v_2]$, where vv_1u_1u-path and vv_2u_2u-path are both uv-threads. We delete v_2 and add its charge to v_1, delete u_2 and add its charge to u_1. Note that the total weight does not change in the transformation since $\omega'(f) = 0$. Repeat this process until there are no Type-(I)-6-faces any more, then we get a new graph G' with $\sum_{x\in V(G')\cup F(G')} \omega'(x) = \sum_{x\in V(G)\cup F(G)} \omega'(x)$.

For the new graph G', we discharge the weights as follows.

(R5). Each face f of G' transfer $\frac{1}{2} - \frac{3}{8k-6}$ to each light 2-vetex which is incident with f.

For each $x \in V(G') \cup F(G')$, let $\omega^{*}(x)$ be the new weight of x after the second step.

Let v be a light 2-vetex of G'. By Claim 2.1 and Lemma 2.4, for any vertex a and vertex b, the number of ab-threads is at most $2k-3$, so $\omega'(v) \geq -\frac{2}{4k-3} \times (2k-3) = -(1-\frac{3}{4k-3})$. By R5, the new weight of v is $\omega^*(v) \geq -(1-\frac{3}{4k-3}) + 2 \times (\frac{1}{2} - \frac{3}{8k-6}) = 0$. By Claim 2.1, we know that all the new weights of vertices in G' are nonnegative now.

Now consider the weights of the faces in $F(G')$. We can see that $\omega'(f)$ will not change if there are no threads on the boundary of f, which means that we can only talk about the faces with threads on the boundary.

Let $d(f) = 6$, then f is not Type-(I). If f is a Type-(II)-6-face, then $\partial(f)$ has only 1 thread and $\omega'(f) \geq 1$ by Claim 2.2. Thus, $\omega^*(f) \geq 1 - 2 \times (\frac{1}{2} - \frac{3}{8k-6}) = \frac{3}{4k-3} \geq 0$. If f is a Type-(III)-6-face, then $\omega'(f) \geq 1 - \frac{3}{4k-3}$, and f has only 1 thread, which implies there at most two light 2-vertices on the boundary. Thus we have $\omega^*(f) \geq 1 - \frac{3}{4k-3} - 2 \times (\frac{1}{2} - \frac{3}{8k-6}) = 0$.

Let $d(f) = 7$. If f is a Type-(I)-7-face, then $\omega'(f) \geq 2$ by Claim 2.2, and $\partial(f)$ has at most two threads. Thus, $\omega^*(f) \geq 2 - 4 \times (\frac{1}{2} - \frac{3}{8k-6}) = \frac{6}{4k-3} \geq 0$. If f is a Type-(II)-7-face, then $\omega'(f) \geq 1$ by Claim 2.2, and $\partial(f)$ has only one thread. Thus, $\omega^*(f) \geq 1 - 2 \times (\frac{1}{2} - \frac{3}{8k-6}) = \frac{3}{4k-3} \geq 0$.

Let $d(f) = 8$, then, by Claim 2.2, we know that $\omega'(f) \geq 2$ and f has at most two threads. Thus $\omega^*(f) \geq 2 - 4 \times (\frac{1}{2} - \frac{3}{8k-6}) = \frac{6}{4k-3} \geq 0$.

If f is a 9^+-face, then $\omega'(f) \geq d(f) - 6$, and $\partial(f)$ has at most $\lfloor \frac{d(f)}{3} \rfloor$ threads. Thus $\omega^*(f) \geq d(f) - 6 - \lfloor \frac{d(f)}{3} \rfloor \times 2 \times (\frac{1}{2} - \frac{3}{8k-6}) \geq \frac{2}{3}d(f) - 6 \geq 0$.

Now we get a contradiction since

$$-12 = \sum_{x \in V(G) \cup F(G)} \omega(x) = \sum_{x \in V(G) \cup F(G)} \omega'(x) = \sum_{x \in V(G') \cup F(G')} \omega^*(x) \geq 0.$$

The proof of Theorem 1.1 is completed.

3 Proof of Theorem 1.2

Assume to the contrary, suppose that G is a minimal counterexample with the fewest vertices. Let L be a $(\lceil \frac{\Delta}{k-1} \rceil + 2)$-list such that G is not k-frugal L-colorable.

We first present some structures of the plane graph G, then apply a discharging procedure to get a contradiction.

Lemma 3.1. *G is connected and $\delta(G) \geq 2$.*

We omit the proof here since it is completely similar to that of Lemma 2.1.

Lemma 3.2. *G cotains no 2-vertex which is adjacent to a 3^--vertex.*

Proof. Suppose to the contrary that G has a 2-vertex v which is adjacent to a 3^--vertex u. By the choice of G, $G - v$ has a k-frugal L-coloring c. Let $N_G(v) = \{u, w\}$, then v can receive any color except for $c(u)$, $c(w)$ and those colors that appear $k-1$ times in $N_G(w)$ (i.e. $C_{k-1}(w)$), so the number of forbidden colors of v is at most

$$1 + 1 + \lfloor \frac{d(w)-1}{k-1} \rfloor = \lceil \frac{d(w)}{k-1} \rceil + 1 \leq \lceil \frac{\Delta}{k-1} \rceil + 1.$$

We can extend the coloring c to the whole graph G, a contraction.

□

Lemma 3.3. *G cotains no 4-vertex which is adjacent to three 2-vertices.*

Proof. Suppose to the contrary that G has a 4-vertex v which is adjacent to three 2-vertices x, y, z. Let x_1, y_1, z_1 be the other neighbor of x, y, z, respectively, and v_1 be the fourth neighbor of v. By the choice of G, $G - \{v, x, y, z\}$ has a k-frugal L-coloring c. Let the vertex $w \in \{v, x, y, z\}$, then w can receive any color except for $c(w_1)$ and those colors that appear $k-1$ times in $N_G(w_1)$ (i.e. $C_{k-1}(w_1)$), which implies that the number of forbidden colors of w is at most $1 + \lfloor \frac{d(w_1)-1}{k-1} \rfloor = \lceil \frac{d(w_1)}{k-1} \rceil \leq \lceil \frac{\Delta}{k-1} \rceil$.

Let $L^*(w) = L(w) \backslash (\{c(w_1)\} \cup C_{k-1}(w_1))$, then we get $|L^*(w)| \geq 2$, where $w \in \{v, x, y, z\}$.

We can first color x with a color $c(x) \in L^*(x) \setminus \{c(v_1)\}$, then color v with a color $c(v) \in L^*(v) \setminus \{c(x)\}$, color y with a color in $L^*(y) \setminus \{c(v)\}$ and z with a color in $L^*(z) \setminus \{c(v)\}$. This is just a k-frugal L-coloring of G, a contradiction.

□

Lemma 3.4. *G cotains no 5-vertex which is adjacent to five 2-vertices.*

Proof. Suppose to the contrary, $d_G(v) = 5$ and $N_G(v) = \{x_1, x_2, \cdots, x_5\}$, wehre $d_G(x_i) = 2$ and y_i is the other neighbor of x_i for all $i = 1, \cdots, 5$. By the choice of G, $G \backslash \{v, x_1, \cdots, x_5\}$ has a k-frugal L-coloring c. For $\forall 1 \leq i \leq 5$, the vertex x_i can receive any color except for $c(y_i)$ and those colors that appear $k-1$ times in $N_G(y_i)$ (i.e. $C_{k-1}(y_i)$). Let $L^*(x_i) = L(x_i) \backslash (\{c(y_i)\} \cup C_{k-1}(y_i))$, then we get $|L^*(x_i)| \geq \lceil \frac{\Delta}{k-1} \rceil + 2 - (1 + \lfloor \frac{d(y_i)-1}{k-1} \rfloor) \geq 2$. For the vertex v, we know that $|L(v)| \geq \lceil \frac{\Delta}{k-1} \rceil + 2 \geq 3$.

Now we extend the coloring c to the whole graph G as follows.

We first color x_1 with a color (say a) in $L^*(x_1)$, then color x_2 with a color $c(x_2)$ in $L^*(x_2) \setminus \{a\}$, color v with a color (say b) in $L(v) \setminus \{a, c(x_2)\}$, and then color x_i with a color in $L^*(x_i) \setminus \{b\}$ for $i = 3, 4, 5$. It is a k-frugal L-coloring of G as long as it isn't $c(x_3) = c(x_4) = c(x_5) = a$ or $c(x_3) = c(x_4) = c(x_5) = c(x_2)$. Without loss of generality, now assume that $L^*(x_i) = \{a, b\}$ and $c(x_i) = a$ for all $i = 3, 4, 5$.

If $L^*(x_1) \neq \{a,b\}$, then we can recolor x_1 by a color in $L^*(x_1) \setminus \{a\}$, which is a new k-frugal L-coloring of G. Now suppose $L^*(x_1) = \{a,b\}$. We can recolor x_1 and x_5 by the color b, and recolor v by a color $c(v) \in L(v) \setminus \{a,b\}$, and recolor x_2 by a color in $L^*(x_2) \setminus \{c(v)\}$, which is also a new k-frugal L-coloring of the whole graph G, a contradiction.

□

Now we assign an original weight $\omega(v) = 2d(v) - 6$ to each vertex v and $\omega(f) = d(f) - 6$ to each face f. By Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we get

$$\sum_{x \in V \cup F} \omega(x) = -12.$$

If we obtain a new weight $\omega^*(x)$ for all $x \in V \cup F$ by transferring weights from one element to another, and these transfers result in $\omega^*(x) \geq 0$ for all $x \in V \cup F$, then we get a contradiction and the theorem is proved. We transfer the weights by the following rule.

(R1). Each 4^+-vertex v transfers 1 to each adjacent 2-vertex.

Let $\omega^*(x)$ be the new weight of $x \in V \cup F$ after (R1).

For each 6^+-vertex v, it is easy to check that $\omega^*(v) \geq 2d(v) - 6 - d(v) \geq d(v) - 6 \geq 0$ by R1. For each 5-vertex v, $\omega^*(v) \geq 2d(v) - 6 - 4 \geq 0$ by Lemma 3.4 and (R1). For each 4-vertex v, $\omega^*(v) \geq 2d(v) - 6 - 2 \geq 0$ by Lemma 3.3 and (R1). For each 3-vertex v, $\omega^*(v) \geq 2d(v) - 6 \geq 0$ by Lemma 3.2 and (R1). For each 2-vertex v, $\omega^*(v) \geq 2d(v) - 6 + 1 + 1 \geq 0$ by Lemma 3.2 and (R1).

For each face f, we have $\omega^*(f) \geq 0$ since the girth $g \geq 6$.

Now we get a contradiction and the proof of Theorem 1.2 is completed.

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