Note on p-competition graphs of double stars

Kyohei NAKADA, Kenjiro OGAWA, Satoshi TAGUSARI
and
Morimasa TSUCHIYA*
Department of Mathematical Sciences, Tokai University
Hiratsuka 259-1292, JAPAN
e-mail: morimasa@keyaki.cc.u-tokai.ac.jp

Abstract

The p-competition graph $C_p(D)$ of a digraph $D = (V, A)$ is a graph with $V(C_p(D)) = V(D)$, where an edge between distinct vertices x and y if and only if there exist p distinct vertices $v_1, v_2, ..., v_p \in V$ such that $x \to v_i$, $y \to v_i$ are arcs of the digraph D for each $i = 1, 2, ..., p$. In this paper, we prove that double stars DS_m $(m \geq 2)$ are p-competition graphs. We also show that full regular m-ary trees $T_{m,n}$ with height n are p-competition graphs, where $p \leq \frac{m-1}{2}$.

Keywords: p-competition graph, double star, full regular m-ary tree

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* corresponding author

1 Introduction

In this paper we consider finite simple graphs and finite digraphs. For a digraph D and $v \in V(D)$, $\mathrm{Out}_D(v) = \{u \,;\, v \to u \text{ in } A(D)\}$. For a digraph D, the p-competition graph $C_p(D)$ of D is the graph satisfying the following:

1. $V(C_p(D)) = V(D)$,
2. $xy \in E(C_p(D))$ if and only if there exist p distinct vertices $v_1, v_2, \ldots, v_p \in V(D)$ such that $x \to v_i$, $y \to v_i$ in $A(D)$ for each $i = 1, 2, ..., p$.

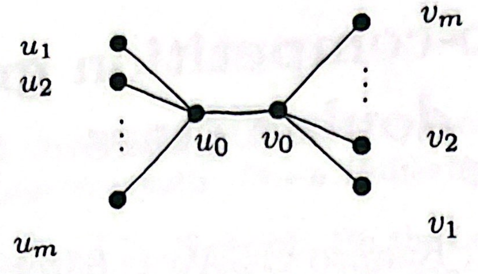

Figure 1: A double star DS_m

A graph G is called a *p-competition graph* if there exists a digraph D such that $C_p(D) \cong G$. In [4] McKee and McMorris surveyed various properties of intersection graphs, which contain p-competition graphs. In [2] and [3] Kim et al. dealt with cycles and chordal graphs in terms of p-competition graphs.

Theorem 1.1 (Kim et al. [2]) *A chordal graph is a 2-competition graph.*

Theorem 1.2 (Kim et al. [3]) *Let C_n be a cycle with $n \geq 5$ vertices and p be a positive integer. Then C_n is a p-competition graph if and only if $n \geq p+3$.*

These results dealt with digraphs which allow loops and symmetric arcs. In [1] Kidokoro et al. dealt with paths, cycles and wheels in terms of p-competition graphs of loopless digraphs without symmetric arcs.

Theorem 1.3 (Kidokoro et al. [1]) *For a positive integer p and $n \geq 3$, P_n and C_n are p-competition graphs of loopless digraphs without symmetric arcs if and only if $n \geq 2p+3$.*

In this paper we deal with some kinds of trees, that is, double trees and full regular m-ary trees in terms of p-completion graphs of loopless digraphs without symmetric arcs.

2 Double stars and 1-competition graphs

In this section we deal with double stars. The double star DS_m is the following graph: $V(\mathrm{DS}_m) = \{u_0, u_1, ..., u_m, v_0, v_1, ..., v_m\}$ and $E(\mathrm{DS}_m) = \{\{u_0, u_i\}\ ;\ i = 1, 2, ..., m\} \cup \{\{v_0, v_i\}\ ;\ i = 1, 2, ..., m\} \cup \{u_0, v_0\}$. We have the following result.

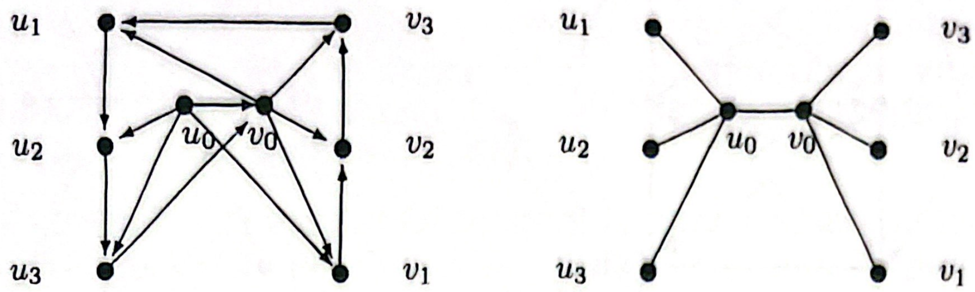

Figure 2: A digraph D and the 1-competition graph $C_1(D) \cong \mathrm{DS}_3$

Theorem 2.1 *For $m \geq 2$, DS_m is a 1-competition graph of a loopless digraph without symmetric arcs.*

Proof. The digraph D is the digraph as follows: $V(D) = V(\mathrm{DS}_m)$ and $A(D) = \{u_0 \to u_i\ ;\ i = 2,3,...,m\} \cup \{u_0 \to v_0, u_0 \to v_1\} \cup \{u_i \to u_{i+1}\ ;\ i = 1,2,...,m-1\} \cup \{u_m \to v_0\} \cup \{v_0 \to v_i\ ;\ i = 1,2,...,m\} \cup \{v_0 \to u_1\} \cup \{v_i \to v_{i+1}\ ;\ i = 1,2,...,m-1\} \cup \{v_m \to u_1\}$. Then the digraph D is a loopless digraph without symmetric arcs. In the following, we show $C_1(D) \cong \mathrm{DS}_m$. Note that $\mathrm{Out}_D(u_0) = \{u_i\ ;\ i = 2,3,...,m\} \cup \{v_0, v_1\}$, $\mathrm{Out}_D(u_i) = \{u_{i+1}\}$ for $i = 1,2,...,m-1$, $\mathrm{Out}_D(u_m) = \{v_0\}$, $\mathrm{Out}_D(v_0) = \{v_i\ ;\ i = 1,2,...,m\} \cup \{u_1\}$, $\mathrm{Out}_D(v_i) = \{v_{i+1}\}$ for $i = 1,2,...,m-1$, $\mathrm{Out}_D(v_m) = \{u_1\}$.

First, we consider adjacency relations. For $i = 1,2,...,m$, $\{u_0, u_i\} \in E(C_1(D))$, because for $i = 1,2,...,m-1$, $\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(u_i) = \{u_{i+1}\}$ and $\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(u_m) = \{v_0\}$. Since $\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(v_0) = \{v_1\}$, $\{u_0, v_0\} \in E(C_1(D))$. For $i = 1,2,...,m$, $\{v_0, v_i\} \in E(C_1(D))$, because for $i = 1,2,...,m-1$, $\mathrm{Out}_D(v_0) \cap \mathrm{Out}_D(v_i) = \{v_{i+1}\}$ and $\mathrm{Out}_D(v_0) \cap \mathrm{Out}_D(v_m) = \{u_1\}$.

Next, we consider non-adjacency relations. For $i = 1,2,...,m$, $\{u_0, v_i\} \notin E(C_1(D))$, because $\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(v_i) = \emptyset$. For $i,j = 1,2,...,m$ and $i \neq j$, $\{u_i, u_j\} \notin E(C_1(D))$, because $\mathrm{Out}_D(u_i) \cap \mathrm{Out}_D(u_j) = \emptyset$. For $i = 1,2,...,m$, $\{u_i, v_0\} \notin E(C_1(D))$, because $\mathrm{Out}_D(u_i) \cap \mathrm{Out}_D(v_0) = \emptyset$. For $i,j = 1,2,...,m$, $\{u_i, v_j\} \notin E(C_1(D))$, because $\mathrm{Out}_D(u_i) \cap \mathrm{Out}_D(v_j) = \emptyset$. For $i,j = 1,2,...,m$ and $i \neq j$, $\{v_i, v_j\} \notin E(C_1(D))$, because $\mathrm{Out}_D(v_i) \cap \mathrm{Out}_D(v_j) = \emptyset$.

Therefore $C_1(D) \cong \mathrm{DS}_m$. □

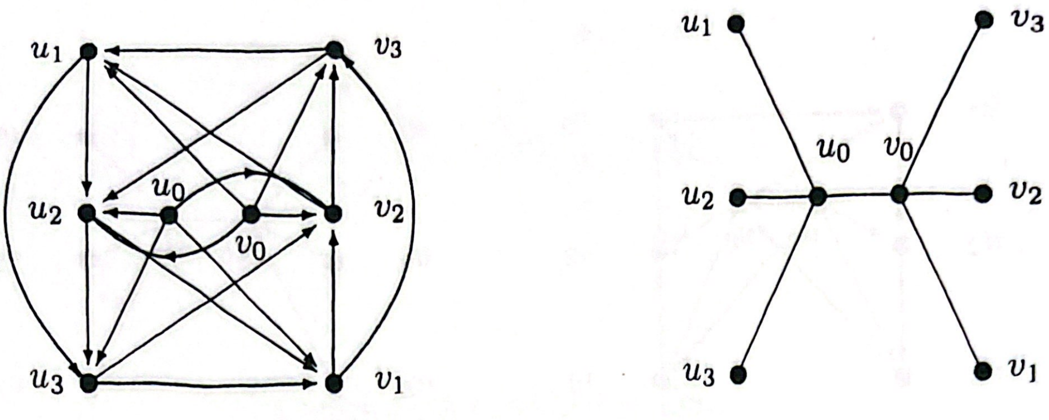

Figure 3: A digraph D and the 2-competition graph $C_2(D)$, DS_3

3 Double stars and p-competition graphs of loopless digraphs without symmetric arcs

In this section we deal with double stars DS_m in terms of p-competition graphs on loopless digraphs without symmetric arcs. We have the following result.

Theorem 3.1 *Let p be a positive integer and $2 \le p \le m-1$. Then DS_m $(m \ge 2)$ are p-competition graphs of loopless digraphs without symmetric arcs.*

Proof. The digraph D is the digraph as follows: $V(D) = V(DS_m)$ and $A(D) = \{u_0 \to u_i; i = 2,3,...,m\} \cup \{u_0 \to v_i; i = 1,2,...,p\} \cup \{u_i \to u_{i+1}, u_i \to u_{i+2}, ..., u_i \to u_{i+p};$ if $i \le m-p$ with $i = 1,2,...,m\} \cup \{u_i \to u_{i+1}, u_i \to u_{i+2}, ..., u_i \to u_m, u_i \to v_1, u_i \to v_2, ..., u_i \to v_{i+p-m}; i = 1,2,...,m-1$ if $i > m-p\} \cup \{u_m \to v_1, u_m \to v_2, ..., u_m \to v_p\} \cup \{v_0 \to v_i; i = 2,3,...,m\} \cup \{v_0 \to u_i; i = 1,2,...,p\} \cup \{v_i \to v_{i+1}, v_i \to v_{i+2}, ..., v_i \to v_{i+p}\ ;\ i = 1,2,...,m$ if $i \le m-p\} \cup \{v_i \to v_{i+1}, v_i \to v_{i+2}, ..., v_i \to v_m, v_i \to u_1, v_i \to u_2, ..., v_i \to u_{i+p-m}$; if $i > m-p$ with $i = 1,2,...,m-1\} \cup \{v_m \to u_1, v_m \to u_2, ..., v_m \to u_p\}$. The digraph D is a loopless digraph without symmetric arcs.

In the following we show that $C_p(D) \cong DS_m$. We consider adjacency relations of u_0 and other vertices. Then $Out_D(u_0) = \{u_i; i = 2,3,...,m\} \cup \{v_i; i = 1,2,...,p\}$. For $i = 1,2,...,m$, $Out_D(u_i) = \{u_{i+1}, u_{i+2}, ..., u_{i+p}\}$ if $i \le m-p$ and $Out_D(u_i) = \{u_{i+1}, u_{i+2}, ..., u_m, v_1, v_2, ..., v_{i+p-m}\}$ if $i > m-p$.

Since $2 \le p \le m-1$ and $Out_D(v_0) = \{v_i; i = 2,3,...,m\} \cup \{u_i; i = 1,2,...,p\}$, $Out_D(u_0) \cap Out_D(v_0) = \{u_2, u_3, ..., u_p, v_2, v_3, ..., , v_p\}$, $|Out_D(u_0) \cap Out_D(v_0)| \ge p$ and $\{u_0, v_0\} \in E(C_p(D))$.

If $i \le m-p$, then $\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(u_i) = \{u_{i+1}, u_{i+2}, ..., u_{i+p}\}$ and $|\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(u_i)| = p$ for $i = 1, 2, ..., m-p$. If $i > m-p$, then $\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(u_i) = \{u_{i+1}, u_{i+2}, ..., u_m, v_1, v_2, ..., v_{i+p-m}\}$ and $|\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(u_i)| = p$ for $i = m-p+1, m-p+2, ..., m-1$. Moreover $\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(u_m) = \{v_1, v_2, ..., v_p\}$ and $|\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(u_m)| = p$. Thus $\{u_0, u_i\} \in E(C_p(D))$ for $i = 1, 2, ..., m$.

For $i = 1, 2, ..., m$, we consider the following cases on u_0 and v_i. Then for $i = 1, 2, ..., m$, $\mathrm{Out}_D(v_i) = \{v_{i+1}, v_{i+2}, ..., v_{i+p}\}$ if $i \le m-p$ and $\mathrm{Out}_D(v_i) = \{v_{i+1}, v_{i+2}, ..., v_m, u_1, u_2, ..., u_{i+p-m}\}$ if $i > m-p$.
Case 1: $i \le m-p$. Then $\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(v_i) = \emptyset$ if $i \ge p$ and $\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(v_i) = \{v_{i+1}, v_{i+2}, ..., v_p\}$ if $i < p$. Thus for $i = 1, 2, ..., m$, $|\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(v_i)| \le p-1 < p$.
Case 2: $i > m-p$. Since $p \le m-1$, $\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(v_i) \subseteq \{u_2, u_3, ..., u_{i+p-m}, v_{i+1}, v_{i+2}, ..., v_{m-1}\}$ and $|\mathrm{Out}_D(u_0) \cap \mathrm{Out}_D(v_i)| \le ((i+p-m)-1) + ((m-1)-i) = p-2 < p$.

Thus $\{u_0, v_i\} \notin E(C_p(D))$ for $i = 1, 2, ..., m$.

By similar way of u_0, we can show that for $i = 1, 2, ..., m$, $\{v_0, v_i\} \in E(C_p(D))$ and $\{v_0, u_i\} \notin E(C_p(D))$.

Next, we consider adjacency relations of u_i and v_j. Note that $|\mathrm{Out}_D(u_i)| = p$, $|\mathrm{Out}_D(v_j)| = p$, $\mathrm{Out}_D(u_i) \ne \mathrm{Out}_D(u_j)$ $(i \ne j)$, $\mathrm{Out}_D(v_i) \ne \mathrm{Out}_D(v_j)$ $(i \ne j)$ and $\mathrm{Out}_D(u_i) \ne \mathrm{Out}_D(v_j)$ for $i, j = 1, 2, ..., m$. Then $|\mathrm{Out}_D(u_i) \cap \mathrm{Out}_D(u_j)| < p$ $(i \ne j)$, $|\mathrm{Out}_D(v_i) \cap \mathrm{Out}_D(v_j)| < p$ $(i \ne j)$ and $|\mathrm{Out}_D(u_i) \cap \mathrm{Out}_D(v_j)| < p$ for $i, j = 1, 2, ..., m$. Thus $\{u_i, u_j\} \notin E(C_p(D))$, $\{v_i, v_j\} \notin E(C_p(D))$ and $\{u_i, v_j\} \notin E(C_p(D))$ for $i, j = 1, 2, ..., m$.

Therefore $C_p(D) \cong \mathrm{DS}_m$. □

4 Trees and p-competition graphs of loopless digraphs without symmetric arcs

We already knew the following result.

Theorem 4.1 (Kidokoro et al. [1]) *Let p be a positive integer and $p \le \frac{m-1}{2}$. Then $K_{1,m}$ is a p-competition graph of a loopless digraph without symmetric arcs.*

In [1] Kidokoro et al. gave the digraph $D_{m,1}$ whose p-competition graph is $K_{1,m}$. The digraph $D_{m,1}$ is the following loopless digraph without symmetric arcs, all subscript arithmetic is taken modulo m.

1. $V(D_{m,1}) = \{v_0, v_1, ..., v_{m-1}\} \cup \{u\}$,

2. $A(D_{m,1}) = \{v_i \to v_{i+j}$; $i = 0, 1, ..., m-1$ and $j = 1, 2, ..., p\} \cup \{u \to v_i$; $i = 0, 1, ..., m-1\}$.

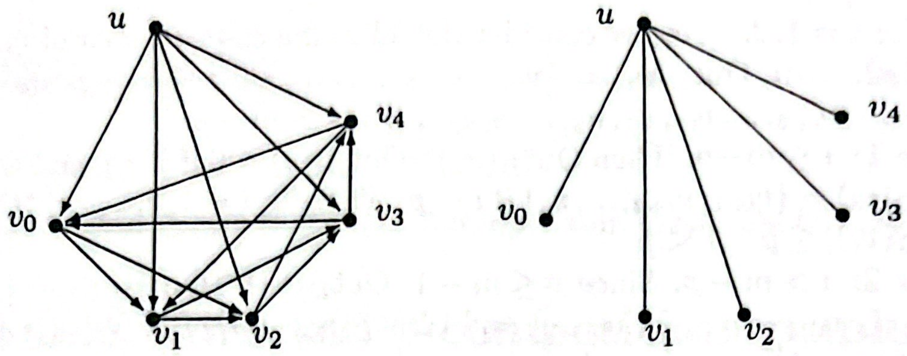

Figure 4: The digraph $D_{5,1}$ and $C_2(D_{5,1}) \cong K_{1,5}$

Let $\overrightarrow{T_{m,n}}$ be a full regular m-ary tree with height n, that is, every non-leaf has exactly m children and the leaves being equidistance n from the root. Let u be the root of $\overrightarrow{T_{m,n}}$ and $v_0, v_1, ..., v_{m-1}$ be children of the root u and $v_{i_1,i_2,...,i_k,0}, v_{i_1,i_2,...,i_k,1}, ..., v_{i_1,i_2,...,i_k,m-1}$ be children of a non-leaf $v_{i_1,i_2,...,i_k}$. Let $T_{m,n}$ be the graph from $\overrightarrow{T_{m,n}}$ without direction. Then $T_{m,n}$ is a tree.

The graph $K_{1,m}$ is a full regular m-ary tree with height 1, $T_{m,1}$. And each subtree of $T_{m,n}$ induced by a non-leaf and its children is also a full regular m-ary tree with height 1.

For graphs G and H, the union of G and H is the graph $G \cup H$ such that

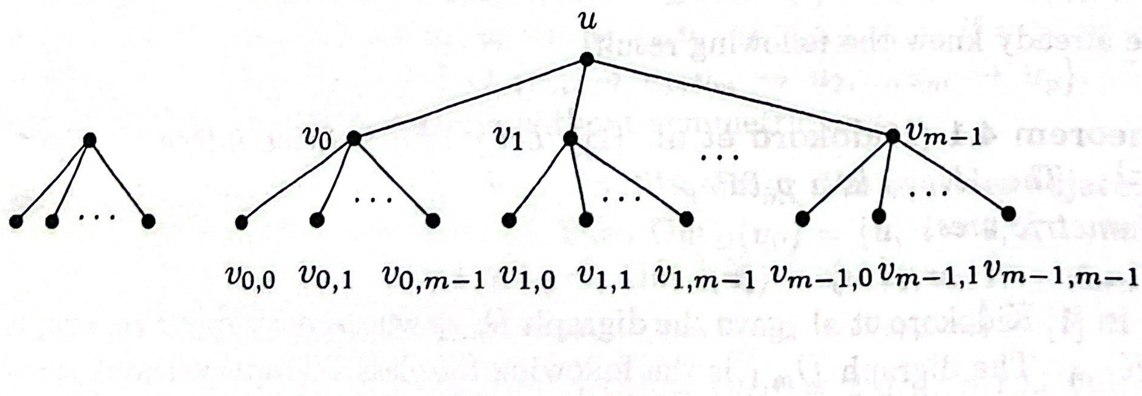

Figure 5: A regular m-ary $T_{m,1}$ and $T_{m,2}$

$V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. For digraphs D and F, the union of D and F is the graph $D \cup F$ such that $V(D \cup F) = V(D) \cup V(F)$ and $A(D \cup F) = A(D) \cup A(F)$.

We obtain the following result from Theorem 4.1.

Proposition 4.2 *Let p be a positive integer and $p \leq \frac{m-1}{2}$. Then $T_{m,n}$ is a p-competition graph of a loopless digraph without symmetric arcs.*

Proof. For a non-leaf v in $T_{m,n}$, let $V(v) = \{v\} \cup \{u; u \text{ is a child of } v\}$. For a $T_{m,n}$, let $\mathrm{Nl}(T_{m,n}) = \{v \in V(T_{m,n}); v \text{ is a non-leaf of } \overrightarrow{T_{m,n}}\}$. Let T_v be the subgraph induced by $V(v)$. Then $T_v \cong K_{1,m}$, $C_p(D_{m,1}) \cong K_{1,m}$ and $\bigcup_{v \in \mathrm{Nl}(T_{m,n})} T_v = T_{m,n}$. Thus $C_p(D) \cong T_{m,n}$, where $D = \bigcup_{v \in \mathrm{Nl}(T_{m,n})} D_{m,1}$. □

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