A NOTE ON THE INVERSE PROBLEMS ASSOCIATED WITH SUBSEQUENCE SUMS

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ABSTRACT. Let $G = C_n \oplus C_n$ with $n \geq 3$ and S be a sequence with elements of G. Let $\Sigma(S) \subset G$ denote the set of group elements which can be expressed as a sum of a nonempty subsequence of S. In this note, we show that if S contains $2n-3$ elements of G, then either $0 \in \Sigma(S)$ or $|\Sigma(S)| \geq n^2 - n - 1$. Moreover, we determine the structures of the sequence S over G with length $|S| = 2n-3$ such that $0 \notin \Sigma(S)$ and $|\Sigma(S)| = n^2 - n - 1$.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of positive integers and integers respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a, b \in \mathbb{Z}$ we set $[a, b] = \{x \in \mathbb{Z} | a \leq x \leq b\}$.

Let G be an additive finite abelian group and let C_n denote the cyclic group of order n. Let $\mathrm{ord}(g)$ denote the order of $g \in G$. Every sequence S over G (i.e. S is a sequence with elements of G) can be written in the form

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $\mathsf{v}_g(S) \in \mathbb{N}_0$ denotes the *multiplicity* of g in S. We call $|S| = \ell = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length* of S, $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in G\} \in \mathbb{N}_0$ the *maximum of the multiplicities* of S, $\mathrm{supp}(S) = \{g \mid \mathsf{v}_g(S) \geq 1\} \subset G$ the *support* of S, and $\sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S.

A sequence T is called a *subsequence* of S if $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for all $g \in G$. If S_1 and S_2 are two subsequences of S such that $\mathsf{v}_g(S_1) + \mathsf{v}_g(S_2) \leq \mathsf{v}_g(S)$ for all $g \in G$, let $S_1 S_2$ denote the subsequence of S satisfying that $\mathsf{v}_g(S_1 S_2) = \mathsf{v}_g(S_1) + \mathsf{v}_g(S_2)$ for all $g \in G$. Let

$$\Sigma(S) = \{\sigma(T) \mid T \text{ is a subsequence of } S \text{ with } 1 \leq |T| \leq |S|\}.$$

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The sequence S is called *zero-sum* if $\sigma(S) = 0 \in G$, *zero-sum free* if $0 \notin \Sigma(S)$, and *minimal zero-sum* if $\sigma(S) = 0$ and $\sigma(T) \neq 0$ for every subsequence T of S with $1 \leq |T| < |S|$.

The problem of determining the minimal cardinality of $\Sigma(S)$ for zero-sum free sequences S of a finite abelian group attracts many authors such as R.B. Eggleton and P. Erdös [2], J.E. Olson [7], B. Bollobás and I. Leader [1], W. Gao et al. [4], A. Pixton [10], P. Yuan and X. Zeng [15]. In 1999, B. Bollobás and I. Leader [1] stated the following conjecture.

Conjecture 1.1. [1, Conjecture 6] *Let $G = C_n \oplus C_n$ with $n \geq 2$ and $0 \leq k \leq n-2$ be an integer. Let $\{e_1, e_2\}$ be a basis of G and $T = e_1^{n-1}e_2^{k+1}$. Let S be a zero-sum free sequence over G with length $|S| = n+k$. Then $|\Sigma(S)| \geq |\Sigma(T)| = (k+2)n-1$.*

Conjecture 1.1 was confirmed for the cases when $k = 0, 1, 2, n-2$ by several authors (see [12, 4, 14]).

The inverse problem associated with $|\Sigma(S)|$ is to determine the structure of the sequence S over G with the given length such that $|\Sigma(S)|$ archives the the minimal cardinality (see [6, 8, 13] for more known results). Recently, J. Peng et al. [9] stated the following conjecture.

Conjecture 1.2. [9, Conjecture 2.4] *Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ be a finite abelian group with $1 < n_1 \mid \ldots \mid n_r$. Let $k \in [0, n_{r-1} - 2]$ be an integer and S be a zero-sum free sequence over G of length $|S| = n_r + k$. Then $|\Sigma(S)| \geq (k+2)n_r - 1$, and the equality holds if and only if S has one of the following forms.*

(1) $\langle S \rangle \cong C_{k+2} \oplus C_{n_r}$, *where* $k+2 \mid n_r$;

(2) $S = g^{n_r-1} \cdot (h + t_1 g) \cdot \ldots \cdot (h + t_{k+1} g)$, *where* $g, h \in G$ *with* $\mathrm{ord}(g) = n_r$, $ih \notin \langle g \rangle$ *for every* $i \in [1, k+1]$, *and* $t_1, \ldots, t_{k+1} \in [0, n_r - 1]$ *are integers.*

In this note we give a positive answer to Conjecture 1.1 and Conjecture 1.2 for the case when $G = C_n \oplus C_n$ with $n \geq 3$ and $k = n-3$. Our main result is as follows.

Theorem 1.3. *Let $G = C_n \oplus C_n$ with $n \geq 3$. If W is a zero-sum free sequence over G with $|W| = 2n-3$, then $|\Sigma(W)| \geq n^2 - n - 1$. Furthermore the equality holds if and only if there exist a basis (g_1, g_2) of G and integers $x_1, \ldots, x_{n-2} \in [0, n-1]$ such that $W = g_1^{n-1} \prod_{\nu=1}^{n-2}(x_\nu g_1 + g_2)$.*

2. Proof of Theorem 1.3

We need the following technical result.

Lemma 2.1. [4, Lemma 3.1] *Let G be a finite abelian group and A be a finite nonempty subset of G. Let $r \in \mathbb{N}$, $y_1, \dots y_r \in G$ and $k = \min\{\mathrm{ord}(g_i) \mid i \in [1,r]\}$. Then $|\Sigma(0y_1 \cdot \ldots \cdot y_r) + A| \geq \min\{k, r + |A|\}$.*

Let $G = C_n \oplus C_n$ with $n \geq 2$. We say that G has Property B if every minimal zero-sum sequence S over G of length $|S| = 2n - 1$ contains some element with multiplicity $n - 1$. It was proved that G has Property B for every positive integer $n \geq 2$ (see contributions in [3, 11]). Therefore, we have the following conclusion.

Lemma 2.2. [5, Theorem 5.8.7] *Let $G = C_n \oplus C_n$ with $n \geq 2$ and S be a minimal zero-sum sequence over G of length $|S| = 2n - 1$. Then there exist a basis (e_1, e_2) of G and integers $x_1, \dots, x_n \in [0, n-1]$ with $x_1 + \ldots + x_n \equiv 1 \pmod n$ such that $S = e_1^{n-1} \prod_{\nu=1}^{n} (x_\nu e_1 + e_2)$.*

In 2008, W. Gao et al. [4] proved the following result.

Lemma 2.3. [4, Lemma 4.3] *Suppose $G = C_n \oplus C_n$ with $n \geq 3$ satisfies Property B. If W is a zero-sum free sequence over G with $|W| = 2n-3$ then $|\Sigma(W)| \geq n^2 - n - 1$.*

Proof of Theorem 1.3

Proof. Note that $G = C_n \oplus C_n$ has Property B for every positive integer $n \geq 2$. If W is a zero-sum free sequence over G with $|W| = 2n - 3$, it follows from Lemma 2.3 that $|\Sigma(W)| \geq n^2 - n - 1$.

Suppose that there exist a basis (g_1, g_2) of G and integers $x_1, \dots, x_{n-2} \in [0, n-1]$ such that $W = g_1^{n-1} \prod_{\nu=1}^{n-2} (x_\nu g_1 + g_2)$. Then $|\Sigma(W) \cap \langle g_1 \rangle| = |\{g_1, \dots, (n-1)g_1\}| = n-1$, $|\Sigma(W) \cap (jg_2 + \langle g_1 \rangle)| = |jg_2 + (\sum_{\nu=1}^{j} x_\nu) g_1 + \{0, g_1, \dots, (n-1)g_1\}| = n$ for every $j \in [1, n-2]$, and $|\Sigma(W) \cap ((n-1)g_2 + \langle g_1 \rangle)| = 0$. Therefore, $|\Sigma(W)| = \sum_{j=0}^{n-1} |\Sigma(W) \cap (jg_2 + \langle g_1 \rangle)| = n - 1 + n(n-2) = n^2 - n - 1$.

Next we assume that W is a zero-sum free sequence over G such that $|W| = 2n - 3$ and $|\Sigma(W)| = n^2 - n - 1 < n^2 - 1 = |G| - 1$. Then $\Sigma(W) \neq G \setminus \{0\}$ and thus there exists $h \in G$ such that Wh is zero-sum free. So $Wh(-h - \sigma(W))$ is a minimal zero-sum sequence of length $2n - 1$. It follows from Lemma 2.2 that there exist a basis (e_1, e_2) of G and integers $x_1, \dots, x_n \in [0, n-1]$ with $x_1 + \ldots + x_n \equiv 1 \pmod n$ such that $Wh(-h - \sigma(W)) = e_1^{n-1} \prod_{\nu=1}^{n} (x_\nu e_1 + e_2)$.

Suppose $W = e_1^{2n-3-\ell} \prod_{\nu=1}^{\ell} (x_\nu e_1 + e_2)$, where $\ell \in [n-2, n]$. If $\ell = n - 2$, let $g_1 = e_1$ and $g_2 = e_2$. Then W is of the form as desired So we may assume that $\ell \in [n-1, n]$ and we divide the rest of the proof into two cases according to the values of ℓ.

Case 1. $\ell = n - 1$. Let $W_1 = e_1^{n-2}$ and $W_2 = \prod_{\nu=1}^{n-1} (x_\nu e_1 + e_2)$.

We first show that $x_1 = \ldots = x_{n-1}$. Otherwise, we may assume that $x_{n-2} \neq x_{n-1}$. Then $je_2 + (\sum_{\nu=1}^{j-1} x_\nu)e_1 + \{x_{n-2}e_1, x_{n-1}e_1\} \subset \Sigma(W_2) \cap (je_2 + \langle e_1 \rangle)$ for every $j \in [1, n-2]$. It follows from Lemma 2.1 that

$$|\Sigma(W) \cap (je_2 + \langle e_1 \rangle)| \geq |(\Sigma(W_2) \cap (je_2 + \langle e_1 \rangle)) + \Sigma(0W_1)|$$
$$\geq |je_2 + (\sum_{\nu=1}^{j-1} x_\nu)e_1 + \Sigma(0W_1)| \geq n$$

for every $j \in [1, n-2]$ and

$$|\Sigma(W) \cap ((n-1)e_2 + \langle e_1 \rangle)|$$
$$= |(n-1)e_2 + (\sum_{\nu=1}^{n-1} x_\nu)e_1 + \Sigma(0W_1)| = n-1.$$

Note that $|\Sigma(W) \cap \langle e_1 \rangle| = |\{e_1, \ldots, (n-2)e_1\}| = n-2$. Therefore,

$$|\Sigma(W)| = \sum_{j=0}^{n-1} |\Sigma(W) \cap (je_2 + \langle e_1 \rangle)| \geq (n-2) + n(n-2) + (n-1)$$
$$= n^2 - 3 > n^2 - n - 1,$$

yielding a contradiction. Therefore $x_1 = \ldots = x_{n-1}$.

Let $g_1 = x_1e_1 + e_2$ and $g_2 = e_1$. Then $\{g_1, g_2\}$ is a basis of G and $W = g_1^{n-1}g_2^{n-2}$ as desired.

Case 2. $\ell = n$. Let $W_1 = e_1^{n-3}$ and $W_2 = \prod_{\nu=1}^{n}(x_\nu e_1 + e_2)$.

We first show that $\mathsf{h}(W_2) = n-1$. Assume to the contrary that $\mathsf{h}(W_2) \leq n-2$. Suppose that $x_1 \neq x_2$. Similar to Case 1, we obtain that $|\Sigma(W) \cap (je_2 + \langle e_1 \rangle)| \geq n-1$ for every $j \in [1, n-1]$. Since $x_1 + \ldots + x_n \equiv 1 \pmod{n}$, we infer that $\sigma(W_2) = e_1$ and thus $|\Sigma(W) \cap \langle e_1 \rangle| = |\{e_1, \ldots, (n-2)e_1\}| = n-2$. Then

$$n^2 - n - 1 = |\Sigma(W)| = \sum_{j=0}^{n-1} |\Sigma(W) \cap (je_2 + \langle e_1 \rangle)|$$
$$\geq n - 2 + (n-1)(n-1) = n^2 - n - 1.$$

So $|\Sigma(W) \cap (je_2 + \langle e_1 \rangle)| = n-1$ for every $j \in [1, n-1]$.

By Lemma 2.1, we obtain that

$$n - 1 \leq |e_2 + \{x_1e_1, x_2e_1\} + \{0, e_1, \ldots, (n-3)e_1\}|$$
$$\leq |\Sigma(W) \cap (e_2 + \langle e_1 \rangle)| = n-1.$$

So $|\Sigma(W) \cap (e_2 + \langle e_1 \rangle)| = |e_2 + \{x_1e_1, x_2e_1\} + \{0, e_1, \ldots, (n-3)e_1\}| = n-1$. This forces that $\text{supp}(x_1 \cdot \ldots \cdot x_n) = \{x_1, x_2\}$ and $x_1 = x_2 \pm 1$. Suppose $x_1 \cdot \ldots \cdot x_n = x_1^{n-t}x_2^t$ and $x_2 = x_1 + 1$. Since $\mathsf{h}(S) \leq n-2$, we infer that

$t \in [2, n-2]$. Then $x_1 + \ldots + x_n = (n-t)x_1 + t(x_1+1) = nx_1 + t \equiv t \pmod n$, yielding a contradiction to that $x_1 + \ldots + x_n \equiv 1 \pmod n$. Therefore $\mathsf{h}(W_2) = n-1$.

Next we assume that $x_1 = \ldots = x_{n-1}$. Since $x_1 + \ldots + x_n \equiv 1 \pmod n$, we infer that $x_n = x_1 + 1$. Let $g_1 = x_1e_1 + e_2$ and $g_2 = e_1$. Then $\{g_1, g_2\}$ is a basis of G and $W = g_1^{n-1}g_2^{n-3}(g_1+g_2)$ as desired. □

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