On the k-domination, k-tuple domination and Roman k-domination numbers in graphs

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Abstract

Rautenbach and Volkmann [Appl. Math. Lett. 20 (2007), 98–102] gave an upper bound for the k-domination number and k-tuple domination number of a graph. Hansberg and Volkmann, [Discrete Appl, Math. 157 (2009), 1634–1639] gave upper bounds for the k-domination number and Roman k-domination number of a graph. In this note, using the probabilistic method and the known Caro-Wei Theorem on the size of the independence number of a graph, we improve the above bounds on the k-domination number, the k-tuple domination number and the Roman k-domination number in a graph for any integer $k \geq 1$. The special case $k = 1$ of our bounds improve the known bounds of Arnautov and Payan [V.I. Arnautov, Prikl. Mat. Programm. 11 (1974), 3–8 (in Russian); C. Payan, Cahiers Centre Études Recherche Opér. 17 (1975) 307–317] and Cockayne et al. [Discrete Math. 278 (2004), 11–22].

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1 Introduction

For notation and graph theory terminology not given here we refer to [9]. We consider finite, undirected and simple graphs G with vertex set $V =$

$V(G)$ and edge set $E(G)$. The number of vertices of G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to G), is the cardinality of its open neighborhood. We denote by $\delta(G)$ and $\Delta(G)$, the minimum and maximum degrees among all vertices of G, respectively. For a subset S of vertices of G the subgraph of G induced by S is denoted by $G[S]$. A subset S of vertices is an *independent set* if $G[S]$ has no edge. The *independence number*, $\alpha(G)$ of G, is the maximum cardinality of an independent set. A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S. The *domination number*, $\gamma(G)$, is the minimum cardinality of a dominating set of G.

Fink and Jacobson [5, 6] introduced the concept of k-domination in graphs for a positive integer k. A subset $D \subseteq V(G)$ is a *k-dominating set* of G, if $|N(v) \cap D| \geq k$ for every $v \in V(G) - D$. The *k-domination number*, $\gamma_k(G)$, is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ coincides with the usual domination number $\gamma(G)$. A set $S \subseteq V(G)$ is called a *k-tuple dominating set* in G if for every vertex $v \in V(G)$, $|N[v] \cap S| \geq k$. The minimum cardinality of a k-tuple dominating set in G is the *k-tuple domination number* $\gamma_{\times k}(G)$ of G. The concept of k-tuple domination number was introduced by Harary and Haynes [8].

A *Roman dominating function* (or just RDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v of G for which $f(v) = 2$. The weight of f is $w(f) = f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight over all such functions f is called the *Roman domination number*, $\gamma_R(G)$. Roman domination was introduced by Cockayne et al. [4]. This concept has been extended to *Roman k-domination* by Kammerling and Volkmann [10]. A *Roman k-dominating function* on G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least k vertices $v_1, v_2, ..., v_k$ with $f(v_i) = 2$ for $i = 1, 2, ..., k$. The weight of a Roman k-dominating function is defined as expected. The *Roman k-domination number* $\gamma_{kR}(G)$ is the minimum weight of a Roman k-dominating function on G. Note that γ_{1R} corresponds to γ_R. Hansberg, Rautenbach and Volkmann proved the following.

Theorem 1 (Hansberg and Volkmann [7]) *Let G be a graph of order n and with minimum degree $\delta \geq 1$, and let k be a positive integer. If*

$\frac{\delta+1}{\ln(\delta+1)} \geq 2k$ *then*

$$\text{(1)} \quad \gamma_k(G) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1) + \sum_{i=0}^{k-1} \frac{\delta^i}{i!(\delta+1)^{k-1}}\right).$$

$$\text{(2)} \quad \gamma_k(G) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1) - \ln(2) + 2\sum_{i=0}^{k-1} \frac{\delta^i}{i!(\delta+1)^{k-1}}\right).$$

$$\text{(3)} \quad \gamma_{kR}(G) \leq \frac{2n}{\delta+1}\left(k\ln(\delta+1) - \ln(2) + \sum_{i=0}^{k-1} \frac{\delta^i}{i!(\delta+1)^{k-1}}\right).$$

Theorem 2 (Rautenbach and Volkmann [12]) *Let G be a graph of order n and with minimum degree $\delta \geq 1$, and let k be a positive integer. If $\frac{\delta+1}{\ln(\delta+1)} \geq 2k$ then*

$$\gamma_{\times k}(G) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1) + \sum_{i=0}^{k-1} \frac{k-i}{i!(\delta+1)^{k-1-i}}\right).$$

In this note, following the same probabilistic method as in [7, 12] (also [2]), we improve the bounds of Theorems 1 and 2, for any integer $k \geq 1$. Our results for the special case $k = 1$ improve the following known bounds.

Theorem 3 (Arnautov [1], Payan [11]) *If G is a graph on n vertices with minimum degree $\delta \geq 1$, then $\gamma(G) \leq \frac{n}{\delta+1}(\ln(\delta+1)+1)$.*

Theorem 4 (Cockayne et al. [4]) *If G is a graph on n vertices with minimum degree $\delta \geq 1$, then $\gamma_R(G) \leq \frac{2n}{\delta+1}(\ln(\delta+1) - \ln(2) + 1)$.*

Our proofs are along similar lines to those presented in the proof of Theorem 1 (given in [7]) and Theorem 2 (given in [12]), and we do not state (repeat) details. The following theorem plays a fundamental role in this paper.

Theorem 5 (Caro [3] and Wei [13]) *For any graph G,*

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1+\deg(v)}.$$

2 Main Results

Theorem 6 *Let G be a graph of order n, with minimum degree $\delta \geq 1$ and maximum degree Δ, and let k be a positive integer. If $\frac{\delta+1}{\ln(\delta+1)} \geq 2k$, then*

$$(i) \quad \gamma_k(G) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1) + \sum_{i=0}^{k-1} \frac{\delta^i}{i!(\delta+1)^{k-1}}\right) - \frac{n}{1+\Delta}\left(\frac{k\ln(\delta+1)}{\delta+1}\right)^{1+\Delta}.$$

$$(ii) \quad \gamma_k(G) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1) - \ln(2) + 2\sum_{i=0}^{k-1} \frac{\delta^i}{i!(\delta+1)^{k-1}}\right) - \frac{n}{1+\Delta}\left(\frac{k\ln(\delta+1)-\ln(2)}{\delta+1}\right)^{1+\Delta}.$$

$$(iii) \quad \gamma_{kR}(G) \leq \frac{2n}{\delta+1}\left(k\ln(\delta+1) - \ln(2) + \sum_{i=0}^{k-1} \frac{\delta^i}{i!(\delta+1)^{k-1}}\right) - \frac{2n}{1+\Delta}\left(\frac{k\ln(\delta+1)-\ln(2)}{\delta+1}\right)^{1+\Delta}.$$

$$(iv) \quad \gamma_{\times k}(G) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1) + \sum_{i=0}^{k-1} \frac{k-i}{i!(\delta+1)^{k-1-i}}\right) - \frac{n}{1+\Delta}\left(\frac{k\ln(\delta+1)}{\delta+1}\right)^{1+\Delta^2}.$$

Proof. (i). Let $p = \frac{k\ln(\delta+1)}{\delta+1}$. The assumption $\frac{\delta+1}{\ln(\delta+1)} \geq 2k$ implies that $p \leq \frac{1}{2}$. Moreover, $\delta \geq k$, as it is shown in [7]. We form a set A by picking every vertex v of G independently at random with $P[v \in A] = p$. Let $A' = \{v \in V(G) : N[v] \subseteq A\}$, and I be a maximum independent set in $G[A']$. Let B be the set of vertices of $V(G) - A$ with fewer than k neighbors in A. Then $(A - I) \cup B$ is a k-dominating set for G. Thus $\gamma_k(G) \leq E(|(A-I) \cup B|) = E(|A| + |B| - |I|) = E(|A|) + E(|B|) - E(|I|)$. As it is shown in [7],

$$E(|A|) + E(|B|) \leq \frac{n}{\delta+1}\left(k\ln(\delta+1) + \sum_{i=0}^{k-1} \frac{\delta^i}{i!(\delta+1)^{k-1}}\right).$$

We calculate the expectation of $|I|$. By Theorem 5,

$$\begin{aligned} E(|I|) &\geq E\left(\sum_{v\in A'} \frac{1}{1+\deg_{G[A']}(v)}\right) \\ &\geq \sum_{v\in V} \frac{1}{1+\deg_G(v)} Pr(v\in A') \\ &= \sum_{v\in V} \frac{1}{1+\deg_G(v)} p^{1+\deg_G(v)} \\ &\geq \sum_{v\in V} \frac{1}{1+\deg_G(v)} p^{1+\Delta} \\ &\geq \frac{n}{1+\Delta}\left(\frac{k\ln(\delta+1)}{\delta+1}\right)^{1+\Delta}. \end{aligned}$$

We conclude that

$$\begin{aligned} \gamma_k(G) &\leq E(|A|)+E(|B|)-E(|I|) \\ &\leq \frac{n}{\delta+1}\left(k\ln(\delta+1)+\sum_{i=0}^{k-1}\frac{\delta^i}{i!(\delta+1)^{k-1}}\right) \\ &- \frac{n}{1+\Delta}\left(\frac{k\ln(\delta+1)}{\delta+1}\right)^{1+\Delta}. \end{aligned}$$

(ii) Let $p=\frac{k\ln(\delta+1)-\ln(2)}{\delta+1}$, and follow the proof of (i).

(iii). Let $p=\frac{k\ln(\delta+1)-\ln(2)}{\delta+1}$, and A, B, and I be defined as in the proof of (i). Then $f: V(G)\to\{0,1,2\}$ defined by $f(x)=2$ if $x\in(A-I)\cup B$, and $f(x)=0$ otherwise, is an RDF for G. Thus $\gamma_{kR}(G)\leq E(w(f))= 2E(|(A-I)\cup B|)\leq 2E(|A|)+2E(|B|)-2E(|I|)$. Now putting the bounds for $E(|B|)$ and $E(|I|)$ that are presented in the proof of (i) completes the proof of (iii).

(iv) Let p, A and A' be defined as in the proof of (i). Let $A''=\{v: N[v]\subseteq A'\}$, and let I be a maximum independent set in $G[A'']$. Clearly, $\deg_{G[A']}(v)=\deg(v)$ for every vertex $v\in A''$. For $0\leq i\leq k-1$ let $B_i=\{v\in V(G)\mid |N[v]\cap A|=i\}$, and let $B=\cup_{i=0}^{k-1}B_i$. Clearly $B\cap A'=\emptyset$. For $1\leq i\leq k-1$, let $B_i'\subseteq V-A$ be the union of sets containing $k-i$ neighbors of $v\in B_i$ that do not lie in A. Then $|B_i'|\leq(k-i)|B_i|$ and $|N_G[v]\cap B_i'|\geq k-i$ for all $v\in B_i$. Let $B'=\cup_{i=0}^{k-1}B_i'$. Then $(A-I)\cup B'$

is a k-tuple dominating set for G. Thus $\gamma\times k(G) \le E(|(A-I)\cup B'|) = E(|A|+|B'|-|I|) = E(|A|)+E(|B'|)-E(|I|)$. As it is shown in [12], $E(|A|)+E(|B'|) \le \frac{n}{\delta+1}\left(k\ln(\delta+1)+\sum_{i=0}^{k-1}\frac{k-i}{i!(\delta+1)^{k-1-i}}\right)$. For a vertex v, if $N(v)=\{v_1,\dots,v_d\}$, then

$$Pr(v\in A'') = pp^{\deg(v_1)}\dots p^{\deg(v_d)} \ge p^{1+d\Delta} \ge p^{1+\Delta^2}.$$

Thus, the expectation of $|I|$ is bounded bellow as:

$$\begin{aligned}
E(|I|) &\ge E\left(\sum_{v\in A''}\frac{1}{1+\deg_{G[A'']}(v)}\right)\\
&\ge \sum_{v\in V}\frac{1}{1+\deg_{G[A']}(v)}Pr(v\in A'')\\
&\ge \sum_{v\in V}\frac{1}{1+\deg_G(v)}Pr(v\in A'')\\
&= \sum_{v\in V}\frac{1}{1+\deg_G(v)}p^{1+\Delta^2}\\
&\ge \sum_{v\in V}\frac{1}{1+\deg_G(v)}p^{1+\Delta^2}\\
&\ge \frac{n}{1+\Delta}\left(\frac{k\ln(\delta+1)}{\delta+1}\right)^{1+\Delta^2}.
\end{aligned}$$

Now the result follows from $\gamma\times k(G) \le E(|A|)+E(|B'|)-E(|I|)$. ■

Note that if $k=1$ then we have the following improvements of Theorems 3 and 4.

Corollary 7 *If G is a graph on n vertices with minimum degree $\delta \ge 1$, then* $\gamma(G) \le \frac{n}{\delta+1}(\ln(\delta+1)+1) - \frac{n}{1+\Delta}\left(\frac{\ln(\delta+1)}{\delta+1}\right)^{1+\Delta}$.

Corollary 8 *If G is a graph on n vertices with minimum degree $\delta \ge 1$, then*

$$\gamma_R(G) \le \frac{2n}{\delta+1}(\ln(\delta+1)-\ln(2)+1) - \frac{2n}{1+\Delta}\left(\frac{\ln(\delta+1)-\ln(2)}{\delta+1}\right)^{1+\Delta}.$$

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