On the Loebl-Komlós-Sós Conjecture, lopsided trees, and certain caterpillars

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Abstract

Let G be a graph with at least half of the vertices having degree at least k. For a tree T with k edges, Loebl, Komlós, and Sós conjectured that G contains T. It is known that if the length of a longest path in T (i.e., the diameter of T) is at most 5, then G contains T. Since T is a bipartite graph, let ℓ be the number of vertices in the smaller (or equal) part. Clearly $1 \leq \ell \leq \frac{1}{2}(k+1)$. In our main theorem, we prove that if $1 \leq \ell \leq \frac{1}{6}k+1$, then the graph G contains T. Notice that this includes certain trees of diameter up to $\frac{1}{3}k+2$.

If a tree T consists of only a path and vertices that are connected to the path by an edge, then the tree T is a *caterpillar*. Let P be the path obtained from the caterpillar T by removing each leaf of T, where $P = a_1, \ldots, a_r$. The path P is the *spine* of the caterpillar T, and each vertex on the spine of T with degree at least 3 in T is a *joint*. It is known that the graph G contains certain caterpillars having at most two joints. If only odd-indexed vertices on the spine P are joints, then the caterpillar T is a an *odd* caterpillar. If the spine P has at most $\lceil \frac{1}{2}k \rceil$ vertices, then T is a *short* caterpillar. We prove that the graph G contains every short, odd caterpillar with k edges.

1 Introduction

Let G be a simple graph, and let k be a natural number. If at least one-half of the vertices of G have degree at least k, then the graph G is a *Loebl-Komlós-Sós(k)-graph*, or simply an *LKS(k)-graph.* Loebl, Komlós and Sós conjectured the following.

Conjecture 1 (LKS Conjecture). *If G is an $LKS(k)$-graph, then G contains every tree with k edges.*

Various special cases of the conjecture have been proven. Many place restrictions on the graph G. Soffer [12] proved the case where the graph G has girth at least 7. The complement of the graph G is $\overline{G}$. Dobson [3] proved the case where $\overline{G}$ does not contain $K_{2,3}$. As stated in [6], Matsumoto and Sakamoto [10] extended Dobson's result by replacing $K_{2,3}$ with a slightly larger graph.

Other proven cases place restrictions on the class of trees. The *diameter* of a tree T is the number of edges in a longest path in T. Barr and Johannson [1], and Sun [13], independently proved that the LKS conjecture holds for all trees of diameter at most 4. Piguet and Stein [11] improved this to include trees of diameter 5.

In a series of four papers, [6, 7, 8, 9], the authors Hladký, Komlós, Piguet, Simonovits, Stein, and Szemerédi proved an approximate version of the conjecture for large values of k.

A tree is a bipartite graph, and the set of trees having ℓ and r as the sizes of the two parts is $\mathcal{T}_{\ell,r}$. Unless otherwise stated, the bipartition of T is L, R, where ℓ and r are the numbers of vertices in L and R, respectively. Let $T \in \mathcal{T}_{\ell,r}$ be a tree with k edges, where $\ell \leq r$. Clearly $1 \leq \ell \leq \frac{1}{2}(k+1)$. In our main theorem, Theorem 1, we prove that if $1 \leq \ell \leq \frac{1}{6}k+1$, then the LKS Conjecture holds for the tree T. Notice that this collection of trees includes certain trees having diameter up to $\frac{1}{3}k+2$.

Theorem 1. *Let G be an $LKS(k)$-graph, and let $T \in \mathcal{T}_{\ell,r}$ be a tree with k edges. If $1 \leq \ell \leq \frac{1}{6}k+1$, then the graph G contains T.*

If a tree consists of only a path and vertices that are connected to the path by an edge, then the tree is a *caterpillar*. Let P be the path obtained from a k-edge caterpillar T by removing each leaf of T. The path P is the *spine* of the caterpillar T, and each vertex on the spine P with degree at least 3 in T is a *joint*. Bazgan, Li and Woźniak [2] proved the conjecture holds for caterpillars that contain at most one joint. Piguet and Stein [11] improved this result to include caterpillars with up to two joints, however the distance between two joints is restricted.

Let P be the spine of a k-edge caterpillar T, where $P = a_1, \ldots, a_t$. If t is at most $\lceil \frac{1}{2}k \rceil$, then T is a *short* caterpillar. If only odd-indexed vertices in P are joints, then the caterpillar T is a an *odd* caterpillar. In Theorem 2, we prove that the LKS Conjecture holds for short, odd caterpillars.

Theorem 2. *If G is an $LKS(k)$-graph, and T is a short, odd caterpillar with k edges, then the graph G contains T.*

2 Terminology and tree lemmas

For standard notation and terminology in graph theory, see [14]. Let G be a graph. The degree of a vertex v in G is $d_G(v)$ or simply $d(v)$. For two subgraphs $C, D \subseteq G$, the set of edges with one end-point in C and one in D is $E(C, D)$; the number of edges in $E(C, D)$ is $e(C, D)$. The bipartite subgraph with biparition C, D and edge set $E(C, D)$ is (C, D). The subgraph induced by $V(C)$ is $G[C]$, its edge-set is $E(C)$, and the number of edges in $E(C)$ is $e(C)$. The subgraph $G - V(C)$, or simply $G - C$, is obtained from G by deleting $V(C)$ and the set of edges with an endpoint in $V(C)$.

Let T be a tree. For a vertex t in T, the set of leaf neighbors of t is $L(t)$. Let $T' \subseteq T$ be a tree that is a subgraph of T. Let f be an embedding of T' into a graph G. If the embedding f can be extended to an embedding of T into G, then the embedding f is *T-extensible* (and the graph G contains the tree T). The next lemma states a sufficient condition for an embedding of T' to be T-extensible.

Lemma 3. *Let G be a graph and let T be a tree with k vertices. Let L be a subset of vertices in T each of which has a leaf neighbor, where $L = \{a_1, \ldots, a_r\}$. Let T' be the tree obtained from T by removing at least one leaf neighbor of each vertex in L. For a graph G, let f be an embedding of T' into G. If $d_G(f(a_i)) \geq k - i$ for each $i \in [r]$, then the embedding f is T-extensible.*

Proof. We use induction on r. If $r = 1$, then the tree T' was obtained by removing at least one vertex from $L(a_1)$. Since $d(f(a_1)) \geq k - 1$, the embedding f is T-extensible.

Suppose that the statement holds for all natural numbers less than r. Let $\hat{T} = T - L(a_1)$, let $k' = k - 1$, and notice that $\hat{T}$ has at most k' vertices. Let $\hat{L} = L - a_1$ and notice that $\hat{L}$ consists of exactly $r - 1$ vertices. In addition, we see that $d(f(a_i)) \geq k' - i$ for each $i \in \{2, \ldots, r\}$. By the induction hypothesis, the embedding f can be extended to an embedding of $\hat{T}$ into G. Finally, since $d(f(a_1)) \geq k - 1$, the embedding f is T-extensible. □

In the following lemma, we establish a lower bound for the number of leaves in a tree with $k + 1$ vertices.

Lemma 4. *Let $T \in \mathcal{T}_{\ell,r}$ be a tree, where $\ell \leq r$, and $\ell + r = k + 1$. Let R_1 be vertices in R that are leaves in T, that is, let $R_1 = \{w \in R : d_T(w) = 1\}$. If $R_2 = R - R_1$, then*

(i) $|R_1| \geq k - 2\ell + 2$, *and*

(ii) $\ell \geq |R_2| + 1$.

Proof. If $R_1 \leq k - 2\ell + 1$, then

$$k = e(T) \geq (k - 2\ell + 1)(1) + (r - (k - 2\ell + 1))(2) = k + 1,$$

a contradiction. Thus $|R_1| \geq k - 2\ell + 2$, and this proves (i).

If $\ell \leq |R_2|$, then

$$k + 1 = |V(T)| = \ell + |R_1| + |R_2| \leq| R_1| + 2|R_2| \leq e(T) = k,$$

a contradiction. Otherwise $\ell \geq |R_2| + 1$, and this proves (ii). □

In Lemma 4(i), if $\ell \leq \frac{1}{6}k + 1$, then we have the following corollary.

Corollary 5. *Let $T \in \mathcal{T}_{\ell,r}$ be a tree, where $\ell \leq r$, and $\ell + r = k + 1$. Let R_1 be vertices in R that are leaves in T, that is, let $R_1 = \{w \in R : d_T(w) = 1\}$. If $\ell \leq \frac{1}{6}k + 1$, then $|R_1| \geq \frac{2}{3}k$.*

For a graph G, the minimum degree among the vertices of G is $\delta(G)$; for a subset $A \subseteq V(G)$, the minimum degree (in G) among the vertices in A is $\delta_A(G)$. The average degree of G is $\bar{d}(G)$, where $\bar{d}(G) = 2e(G)/|V(G)|$. A proof of the following lemma is in [4].

Lemma 6. *If a graph G is minimal with $\bar{d}(G) > k - 2$, then $\delta(G) \geq \lfloor \frac{k}{2} \rfloor$.*

The following are two well-known lemmas that we state without proof.

Lemma 7. *Let G be a graph with $\delta(G) \geq m$. If T is a tree with at most $m + 1$ vertices, then G contains T. Moreover, for a tree $T' \subseteq T$, if f is an embedding of T' into G, then the embedding f is T-extensible.*

Lemma 8. *Let $T \in \mathcal{T}_{\ell,r}$, and let H be a bipartite graph with bipartition U, W. If $\delta_U(H) \geq r$, and $\delta_W(H) \geq \ell$, then H contains T. Moreover, for a tree $T' \subseteq T$, if f is an embedding of T' into H, where a vertex from $V(T') \cap L$ is mapped to a vertex in U, or a vertex from $V(T') \cap R$ is mapped to a vertex in W, then the embedding f is T-extensible.*

3 The LKS Conjecture and short odd caterpillars

In this section, we prove Theorem 2 which states that the LKS Conjecture holds for short, odd caterpillars. First we prove several lemmas that will be used in the proof of Theorem 2, as well as in the proof of Theorem 1. We believe these lemmas may prove to be useful in further studies of the LKS conjecture as well.

For the remainder of the paper, if G is an LKS(k)-graph, then the set of vertices in G with minimum degree k is B, and the set of vertices in G with degree less than k is S, that is, $S = V(G) - B$. Lemma 9 compares the average degrees of the subgraphs $G[B]$ and (B, S) in an LKS(k)-graph G.

Lemma 9. *Let G be an $LKS(k)$-graph, and let m be a natural number. If $\bar{d}(G[B]) \leq m$, then $\bar{d}((B,S)) \geq k - m$.*

Proof. If $\bar{d}(G[B]) \leq m$, then $e(B) \leq \frac{1}{2}m|B|$. Let σ_B be the degree sum (in G) of the vertices in B, and notice that $\sigma_B \geq k|B|$. Since $\sigma_B = 2e(B) + e(B, S)$, we see that

$$e(B,S) \geq \sigma_B - 2e(B) \geq k|B| - m|B| = |B|(k-m)$$

and $\bar{d}((B,S)) \geq k - m$. □

Erdős and Gallai [5] proved the following which is central to the proofs of Lemmas 11 and 12 which immediately follow.

Theorem 10 (Erdős-Gallai Theorem). *If G is a graph with $\bar{d}(G) > k - 2$, then G contains a path on k vertices.*

Lemma 11. *Let G be an $LKS(k)$-graph, and let d be a natural number. If $\bar{d}(G[B]) > d$, then the graph G contains*

(i) every caterpillar T with $k+1$ vertices whose spine has at most $d+2$ vertices, and

(ii) every tree T on $k+1$ vertices that has at least $(k-1) - \lfloor \frac{1}{2}d \rfloor$ leaves.

Proof. (*i*). Let P be the spine of the caterpillar T, where $P = a_1, \ldots, a_t$, and t is at most $d + 2$. By Theorem 10, there is an embedding f of P into $G[B]$ (and into G). Since each vertex on the path $f(a_1), \ldots, f(a_t)$ has degree at least k in G, the embedding f is T-extensible (by Lemma 3), and this proves (*i*).

(*ii*). Let T' be the tree obtained from T by removing every leaf. Notice that T' has at most $\lfloor \frac{1}{2}d \rfloor + 2$ vertices. Let $H \subseteq G[B]$ be a subgraph that is minimal with $\bar{d}(H) > d$, and notice that $\delta(H) \geq \lfloor \frac{1}{2}d \rfloor + 1$ (by Lemma 6). By Lemma 7, there is an embedding f of T' into H. Since each vertex in $f(T')$ has degree at least k, the embedding is T-extensible, and this completes the proof. □

Lemma 12. *Let G be an $LKS(k)$-graph, and let $d \in \mathbb{N}$ be at most $k-1$. Let T be an odd caterpillar with $k+1$ vertices whose spine has at most d vertices. If $\bar{d}((B,S)) > d$, then the graph G contains T.*

Proof. Let $a_0, \ldots, a_{t+1}$ be a longest path in T. It follows that the spine P of T is $a_1, \ldots, a_t$, where t is at most d.

By Theorem 10, the graph (B,S) contains a $(d+2)$-path Q, where $Q = v_1, \ldots, v_{d+2}$. Since (B,S) is bipartite, either the even-indexed or the odd-indexed vertices of Q are members of B.

If the vertex v_1 is in B, then let $f(a_i) = v_i$ for each $i \in [t+1]$. Otherwise $v_1 \in S$, and let $f(a_i) = v_{i+1}$ for each $i \in [t+1]$. In either case, we see that f is an embedding of $a_1, \ldots, a_{t+1}$ into G. Since each odd-indexed vertex in the path $f(a_1), \ldots, f(a_t)$ has degree at least k in G, the embedding f is T-extensible (by Lemma 3). □

In Theorem 2, we prove that the LKS Conjecture holds for short, odd caterpillars.

Theorem 2. *If G is an $LKS(k)$-graph, and T is a short, odd caterpillar with k edges, then the graph G contains T.*

Proof. Let $t = \lceil \frac{1}{2}k \rceil$. Since T is a short caterpillar, its spine has at most t vertices. If $\bar{d}(G[B]) > t-2$, then the graph G contains T (by Lemma 11(*i*)).

Otherwise $\bar{d}(G[B]) \leq t-2$. By Lemma 9, we see that

$$\bar{d}((B,S)) \geq k - (t-2) = \lfloor \frac{1}{2}k \rfloor + 2 > t + 1.$$

Therefore, the graph G contains T (by Lemma 12). □

4 The LKS Conjecture and lopsided trees

In this section, we prove our main theorem which states that the LKS Conjecture holds for a collection of lopsided trees, namely, the set of k-edge trees where the smaller (or equal) partition has at most $\frac{1}{6}k+1$ vertices.

Theorem 1. *Let G be an $LKS(k)$-graph, and let $T \in \mathcal{T}_{\ell,r}$ be a tree with k edges. If $1 \leq \ell \leq \frac{1}{6}k+1$, then the graph G contains T.*

Proof. Let R_1 be the subset of vertices of R that have degree 1 in T, and let $R_2 = R - R_1$. Since $\ell \leq \frac{1}{6}k+1$, we see that $|R_1| \geq \frac{2}{3}k$ (by Corollary 5).

Let $T' = T - R_1 = (L, R_2)$. By Lemma 4(*ii*), we see that

$$\frac{1}{6}k+1 \geq \ell \geq |R_2|+1, \tag{1}$$

and

$$|V(T')| = \ell + |R_2| \leq 2(\frac{1}{6}k+1) - 1 = \frac{1}{3}k+1.$$

Let $d = \lceil \frac{2}{3}k \rceil - 2$. Since the tree T has at least $|R_1|$ leaves, and

$$(k-1) - \lfloor \frac{1}{2}d \rfloor \leq (k-1) - \frac{1}{2}\lfloor \lceil \frac{2}{3}k \rceil - 2 \rfloor = k - \lfloor \frac{k+1}{3} \rfloor \leq \lceil \frac{2}{3}k \rceil \leq |R_1|,$$

we see that the tree T has at least $(k+1) - \lfloor \frac{1}{2}d \rfloor$ leaves.

If $G[B]$ has average degree greater than d, then the graph G contains T (by Lemma 11(*ii*)).

Otherwise $G[B]$ has average degree at most d, and

$$\bar{d}((B,S)) \geq k - d = k - (\lceil \frac{2}{3}k \rceil - 2) = \lfloor \frac{1}{3}k \rfloor + 2, \tag{2}$$

(by Lemma 9).

Let $\hat{k} = \lfloor \frac{1}{3}k \rfloor + 3$. By Inequality (2), we see that $\bar{d}((B,S)) > \hat{k}-2$. Let (B', S') be a subgraph of (B,S) that is minimal with $\bar{d}((B',S')) > \hat{k}-2$, where $B' \subseteq B$ and $S' \subseteq S$.

By Lemma 6 and Inequality (1) we have

$$\delta((B',S')) \geq \lfloor \frac{1}{2}\hat{k} \rfloor = \lfloor \frac{1}{2}\lfloor \frac{1}{3}k+3 \rfloor \rfloor \geq \lfloor \frac{1}{6}k \rfloor + 1 \geq \ell > |R_2|.$$

Since $\delta_{B'}(B',S') \geq |R_2|$ and $\delta_{S'}(B',S') \geq \ell$, there is an embedding f of T' into (B',S'), where the vertices of L and R_2 are mapped into B' and S', respectively (by Lemma 8). Since every vertex in B' has degree at least k in G, the embedding f is T-extensible (by Lemma 3). □

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