On the Relations Between Liars' Dominating and Set-sized Dominating Parameters

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Abstract

We define the (i,j)-liars' domination number of G, denoted by $LR_{(i,j)}(G)$, to be the minimum cardinality of a set $L \subseteq V(G)$ such that detection devices placed at the vertices in L can precisely determine the set of intruder locations when there are between 1 and i intruders and at most j detection devices that might "lie".
We also define the $X(c_1, c_2, \ldots, c_t, \ldots)$-domination number, denoted by $\gamma_{X(c_1,c_2,\ldots,c_t,\ldots)}(G)$, to be the minimum cardinality of a set $D \subseteq V(G)$ such that , if $S \subseteq V(G)$ with $|S| = k$, then $|(\cup_{s\in S} N[s]) \cap D| \geq c_k$. Thus, D dominates each set of k vertices at least c_k times making $\gamma_{X(c_1,c_2,\ldots,c_t,\ldots)}(G)$ a set-sized dominating parameter. We consider the relations between these set-sized dominating parameters and the liars' dominating parameters.
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1 Introduction

In this paper we assume that each vertex of a graph G is the possible location for an "intruder" such as a thief, a saboteur, a fire in a facility or some possible processor malfunction in a multiprocessor network. Here a detection device at a vertex v is assumed to be able to detect any intruders situated at vertices in its closed neighborhood $N[v]$ *and to identify at which vertices in* $N[v]$ *the intruders are located.* When a detector at v functions properly and the intruders are located at $S \subseteq V(G)$, then v reports intruders at $S \cap N[v]$, denoted $v \mapsto S \cap N[v]$. Note that $S \cap N[v]$ is the empty set if there are no intruders in $N[v]$. Thus v reporting the empty set, denoted $v \mapsto \varnothing$, is equivalent to v reporting that there are no intruders in $N[v]$.

The reliability problem considered here involves the situation in which a device in the neighborhood of an intruder vertex can misidentify (lie about) location(s) of the intruder(s). If a detector at v might lie, when there is at least one intruder in $N[v]$, then v can report any $I_v \subseteq N[v]$ as the set of intruder locations in $N[v]$ including the $v \mapsto \varnothing$ possibility that $I_v = \varnothing$. We use the notation $v \mapsto I_v$ to indicate that v is reporting I_v as its set of intruder locations. We assume there are no false alarms, that is, a detection device at v, whether or not v is a liar, with no intruder in $N[v]$ will report $v \mapsto \varnothing$.

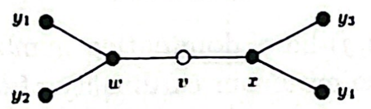

Figure 1: Graph T with $LR_{(1,1)}(T) = 6$.

Consider the graph T in Figure 1 and the set $L_6 = \{y_1, y_2, y_3, y_4, w, x\}$. As will be seen in Theorem 1, assuming at most one intruder and at most one liar, detection devices located at the vertices in L_6 will precisely determine the location of the single intruder. As will be illustrated, if it is possible to have more than one liar or intruder, there will be problems. First, consider L_6 still assuming at most one liar but possibly with multiple intruders, suppose $y_1 \mapsto \{y_1\}$, $w \mapsto \{v, y_1\}$, $x \mapsto \{y_4\}$, $y_4 \mapsto \{y_4\}$, $y_2 \mapsto \varnothing$, and $y_3 \mapsto \varnothing$. We can determine that there are intruders at y_1 and y_4, but we cannot decide if v is also an intruder location. Either w lies and there are intruders only at y_1 and y_4, or x lies and there are intruders at y_1, y_4, and v. That is, the possibility of at most three intruders and at most one liar leads to incomplete information. Likewise, if $w \mapsto \{w, y_1\}$, $y_2 \mapsto \{w\}$, $y_1 \mapsto \{w\}$, $y_3 \mapsto \varnothing$, $y_4 \mapsto \varnothing$, $x \mapsto \varnothing$, then there is an intruder at w, and it is unknown if there is one at y_1. Either w lies and there is an intruder only at w, or y_1 lies and there are intruders at w and y_1. That is, the possibility of at most two intruders and at most one liar leads to incomplete information. As a third example, assume that there can be at most two liars. If $y_1 \mapsto \varnothing$, $w \mapsto \{w, y_1\}$, $y_2 \mapsto \{w\}$, $x \mapsto \varnothing$, $y_3 \mapsto \varnothing$, and $y_4 \mapsto \varnothing$, then we cannot precisely locate any of the intruders. There are three disjoint possible sets of intruders. Either y_2 and w both lie and there is an intruder only at y_2, or only y_1 lies and there are intruders at y_1 and w, or y_1 and w lie and there is an intruder only at w. That is, the possibility of at most two intruders and at most two liars leads to incomplete information. That is, L_6 can precisely determine the location of a single intruder but cannot precisely determine the locations for a set of more than one intruder.

We call $L \subseteq V(G)$ an **(i,j)-liars' dominating set** or an **LR(i,j)-set** if, when it is known that there are at most i intruders, the detection devices placed at the vertices in L can correctly identify any set I of intruder locations with $|I| \leq i$, given that at most j devices are misreporting. In particular, when $v \mapsto I_v$ we can assume $|I_v| \leq i$. The **(i,j)-liars' domination number** $LR_{(i,j)}(G)$ is the minimum cardinality of an LR(i,j)-set for G. An LR$(1,1)$-set for G is called a **liar's dominating set**. In particular, the liar's domination number is $\gamma_{LR}(G) = LR_{(1,1)}(G)$. Note that for $i \geq 1$ we have $LR_{(i,0)}(G) = \gamma(G)$, the domination number. We also let $LR_{(\infty,j)}(G)$ be the minimum cardinality of a detection set $L \subseteq V(G)$ that can identify any intruder set $I \subseteq V(G)$ assuming at most j liars. LR$(1,1)$-sets were introduced in [10] and [8]. Cases with multiple liars and/or multiple intruders were first considered in [7]. Some related work appears in [1],[2] [4], [5], [6], and [11].

Theorem 1 [10]. *Vertex set $L \subseteq V(G)$ is a liar's dominating set if and only if (1) for each $v \in V(G)$ we have$|N[v] \cap L| \geq 2$ and (2) for every pair u,v of distinct vertices we have $|(N[u] \cup N[v]) \cap L| \geq 3$.*

This theorem motivated the definition of the following set-sized domination parameters. Set-sized domination parameters are considered in [7, 8, 9, 10]. By Theorem 1, $L \subseteq V(G)$ is a liar's dominating set if and only if L double dominates each vertex and triple dominates each pair of vertices, so we can also call a liar's dominating set a X(2,3)(G)-set. Then an alternate notation for $LR_{(1,1)}(G)$ is $\gamma_{X(2,3)}(G)$. For $S \subseteq V(G)$, let $N[S] = \cup_{s \in S} N[s]$. Given a sequence of nonnegative integers $(c_1, c_2, \ldots, c_t)$, a set $D \subseteq V(G)$ is a X$(c_1, c_2, \ldots, c_t)$-dominating set if, for $1 \leq i \leq t$, every $S \subseteq V(G)$ with $|S| = i$ has $|N[S] \cap D| \geq c_i$. The minimum cardinality of a X$(c_1, c_2, \ldots, c_t)$-dominating set is called the **X$(c_1, c_2, \ldots, c_t)$-domination number**, and it is denoted by $\gamma_{X(c_1,c_2,\ldots,c_t)}(G)$. This is the first time that the focus has been shifted from dominating single vertices to dominating sets of k vertices at least c_k times. Thus, $\gamma_{X(c_1,c_2,\ldots,c_t)}(G)$ is a **set-sized domination parameter**.

A similar definition for an infinite sequence $(c_1, c_2, c_3, \ldots)$ of nonnegative integers holds for $\gamma_{X(c_1,c_2,c_3,\ldots)}(G)$. Note that requiring $|N[S] \cap D| \geq c_i$

holds vacuously for $i \geq |V(G)|$. This infinite sequence gives us infinitely many domination parameters, and the infinite dimensional lattice of domination parameters is referred to as the **domination continuum**.

As defined by Harary and Haynes [3], vertex set $D \subseteq V(G)$ is a k-tuple dominating set if $|N[x] \cap D| \geq k$ for every $x \in V(G)$, and the minimum cardinality of a k-tuple dominating set for G is denoted by $\gamma_{\times k}(G)$. A 2-tuple dominating set is also called a double-dominating set, and $\gamma_{\times 2}(G)$ is also denoted by $dd(G)$. Obviously, a 3-tuple dominating set is also called a triple-dominating set. Note that for $\gamma_{\times k}(G)$ to be defined one requires the minimum degree of a vertex in G to satisfy $\delta(G) \geq k-1$.

Theorem 2 [10]. *For every connected graph G of order $n \geq 3$ we have $\gamma_{\times(2)}(G) \leq \gamma_{LR}(G)$, and, if $\delta(G) \geq 2$, then $\gamma_{\times(2)}(G) \leq \gamma_{LR}(G) \leq \gamma_{\times(3)}(G)$.*

Observation 3 [7]. *If $c_{i+1} < c_i$, then*

$$\gamma_{\times(c_1,c_2,\ldots,c_i,c_{i+1},c_{i+2},\ldots)}(G) = \gamma_{\times(c_1,c_2,\ldots,c_i,c_i,c_{i+2},\ldots)}(G).$$

That is, we can assume that $0 \leq c_1 \leq c_2 \leq \ldots \leq c_i \leq c_{i+1} \leq \ldots$.

Observation 4 [7]. $\gamma_{\times(c_1,c_2,\ldots,c_t)}(G) = \gamma_{\times(c_1,c_2,\ldots,c_t,c_t,c_t,\ldots)}(G)$.
In particular, $\gamma_{\times(k,k,\ldots,k)}(G) = \gamma_{\times k}(G)$.

Observation 5 [7]. *If $c_i \leq b_i$ for every i, then*

$$\gamma_{\times(c_1,c_2,\ldots)}(G) \leq \gamma_{\times(b_1,b_2,\ldots)}(G).$$

In this paper we consider the relations between the liars' dominating parameters and set-sized dominating parameters.

2 Relations Between Liars' Dominating and Set-sized Dominating Parameters

We will show that for some values of i and j, $LR_{(i,j)}(G)$ is equivalent to a "domination continuum" or set-sized domination parameter $\gamma_{\times(c_1,c_2,\ldots)}(G)$. Note that Theorem 1 can be restated as $LR_{(1,1)}(G) = \gamma_{\times(2,3)}(G)$. However, for other values of i and j we will see that $LR_{(i,j)}(G)$ is not equivalent to any set-sized domination parameter. First we consider relationships between different liars' domination numbers.

Observation 6 *If $i \leq h$ and $j \leq k$, then since an $LR(h,k)$-set for G can protect against as many as h intruders and as many as k liars, it is also an $LR(i,j)$-set for G. In particular, $i \leq h$ and $j \leq k$ implies that $LR_{(i,j)}(G) \leq LR_{(h,k)}(G)$.*

Next we compare the (i, j)-liars' domination numbers and the set-sized domination continuum parameters.

Theorem 7 *If an $LR(i, j)$-set L with $1 \leq i \leq j$ exists for graph G, then L is a $X(i + j, 2j + 1)$-set for G and $LR_{(i,j)}(G) \geq \gamma_{X(i+j,2j+1)}(G)$.*

Proof. Assume that L is an LR(i, j)-dominating set for G with $1 \leq i \leq j$. First, notice that if at most j devices protect a vertex x, then all of the devices protecting x could lie and we could not determine if there is an intruder at x. Thus $|N[x] \cap L| \geq j + 1$ for every $x \in V(G)$. Now we will show that $|N[x] \cap L| \geq i + j$ for every $x \in V(G)$. Consider a graph G and a set $L \subseteq V(G)$. Let $v \in V(G)$ with $|N[v] \cap L| \leq i + j - 1$. If $v \notin L$, let $N(v) \cap L = \{x_1, x_2, \ldots, x_j, x_{j+1}, \ldots, x_k\}$ and if $v \in L$, let $N(v) \cap L = \{v = x_1, x_2, \ldots, x_j, x_{j+1}, \ldots, x_k\}$. Note that $j+1 \leq k \leq i+j-1$. Suppose that each vertex x_t with $1 \leq t \leq j$ does not report v, and that each vertex x_t with $j + 1 \leq t \leq k$ reports an intruder at v and at some vertex $y_t \in N(x_t) - v$. Also every vertex in each $N[y_t] \cap L$ reports y_t. Then either the intruder set is $I = \{v, y_{j+1}, \ldots, y_k\}$ with the j liars x_1, x_2, ..., x_j, or $I = \{y_{j+1}, \ldots, y_k\}$ with the liars x_{j+1}, ..., x_k. Note that $k - j \leq i - 1 < j$. Thus L is not an LR(i, j)-set of G. Hence, $|N[x] \cap L| \geq i + j$ for every $x \in V(G)$.

Suppose $\{u, v\} \subseteq V(G)$ with $|(N[u] \cup N[v]) \cap L| \leq 2j$. Let $|(N[u] - N[v]) \cap L| = i$, $|(N[v] - N[u]) \cap L| = s$, and $|(N[u] \cap N[v]) \cap L| = t$. Since $|(N[u] \cup N[v]) \cap L| \leq 2j$, $|N[u] \cap L| \geq j + 1$ and $|N[v] \cap L| \geq j + 1$, then $i \leq j - 1$, $s \leq j - 1$, and we can let $t = t_1 + t_2$, where $t_1 \geq 1$ and $t_2 \geq 1$. If t_1 and t_2 are taken such that $i + t_1 \leq j$ and $s + t_2 \leq j$, the t_1 vertices report v, the t_2 vertices report u and the i and s vertices do not report, then the intruder's position cannot be determined. The intruder can either be at u with the $i + t_1$ vertices lying or at v with the $s + t_2$ vertices lying. Hence, we must have $|(N[u] \cup N[v]) \cap L| \geq 2j + 1$. Thus L is a $X(j + 1, 2j + 1)$-dominating set. Hence $LR_{(i,j)}(G) \geq \gamma_{X(i+j,2j+1)}(G)$. □

For completeness we include the next two proofs. The following theorem is a generalization for the single intruder, multiple liars scenarios of the characterization of a liar's dominating set in Theorem 1.

Theorem 8 [7]. *Vertex set $L \subseteq V(G)$ is an $LR(1, j)$-dominating set for G if and only if it is a $X(j + 1, 2j + 1)$-dominating set for G. In particular, $LR_{(1,j)}(G) = \gamma_{X(j+1,2j+1)}(G)$.*

Proof. Assume that L is an LR$(1, j)$-dominating set for G. By Theorem 7, L is a $X(j + 1, 2j + 1)$-dominating set.

Next assume L is a $X(j+1, 2j+1)$-dominating set for a graph G. We will show that, when considering a pair of vertices which includes the intruder, the vertex without the intruder can be eliminated. So, by considering all possible pairs of vertices, the intruder's location can be determined. Let $\{x, y\} \subseteq V(G)$ with $|(N[x] - N[y]) \cap L| = i$, $|(N[y] - N[x]) \cap L| = s$, and $|(N[x] \cap N[y]) \cap L| = t$. Then $i + s + t \geq 2j + 1$. Suppose the intruder is at x. Note that the s vertices in $(N[y] - N[x]) \cap L$ do not report any intruder. If some $v \in (N[x] - N[y]) \cap L$ reports the intruder at one of its neighbors, then the intruder is not at y since it is not adjacent to v and there are no false alarms. Assume the i and s vertices do not report, and $t = t_1 + t_2 + t_3$, where for the t vertices in $(N[x] \cap N[y]) \cap L$, we have that t_1 vertices report x, t_2 vertices report y, and t_3 vertices either do not report or report neighboring vertices other than x or y. Then since the intruder is at x and there are at most j liars, we have $i + t_2 + t_3 \leq j$. Hence $s + t_1 \geq j + 1$ with at least one of these vertices telling the truth that the intruder is not at y. Hence, it will follow that L is an LR$(1, j)$-dominating set for G. □

Next we consider the characterization of an LR(∞, j)-dominating set. For this case there can be arbitrarily many intruders and at most j liars.

Theorem 9 [7]. *Vertex set $L \subseteq V(G)$ is an LR(∞, j)-dominating set for G if and only if it is a X$(2j + 1)$-dominating set for G. In particular, $LR_{(\infty,j)}(G) = \gamma_{X(2j+1)}(G)$.*

Proof. Assume $L \subseteq V(G)$ is a X$(2j + 1)$-dominating set for G. Since there are at most j liars, then, if $j+1$ devices report $x \in V(G)$, one of them must be telling the truth that the intruder is at x. Since L is a X$(2j + 1)$-dominating set and there are at most j liars, then, if there is an intruder at $x \in V(G)$, at least $j + 1$ devices will report x. Thus there is an intruder at x if and only if at least $j + 1$ vertices in L report x. Hence L is an LR(∞, j)-dominating set for G.

Next assume that $L \subseteq V(G)$ is an LR(∞, j)-dominating set for G. Assume $|N[u] \cap L| \leq 2j$. Let $N[u] \cap L = L_1 \cup L_2$, where $L_1 \cap L_2 = \varnothing$ and $|L_i| \leq j$. If $x \mapsto N[x] - u$ for every $x \in L_1$ and $y \mapsto N[y]$ for every $y \in L - L_1$, then the intruder set, I, cannot be determined. Either $I = V(G)$ with L_1 being the set of liars or $I = V(G) - \{u\}$ with L_2 being the set of liars. Thus $|N[u] \cap L| \geq 2j + 1$. Hence L is a X$(2j + 1)$-dominating set for G. □

Clearly, if L is an LR(∞, j)-dominating set for G, then L is an LR(i, j)-set for G. Thus we have the following corollary to Theorem 9.

Corollary 10 *If an LR(i, j)-set exists for graph G, then $LR_{(i,j)}(G) \leq \gamma_{X(2j+1)}(G)$.*

It was shown in [7] that vertex set $L \subseteq V(G)$ is an LR$(2, 1)$-dominating set for G if and only if it is a X3-dominating set for G. That is, $LR_{(2,1)}(G) = \gamma_{X3}(G)$. This can be generalized to include any LR(i, j)-set with $i > j$ as follows.

Theorem 11 *Vertex set $L \subseteq V(G)$ is an $LR(i,j)$-dominating set with $i > j$ for G if and only if it is a $X(2j+1)$-dominating set for G. That is, if $i > j$, then $LR_{(i,j)}(G) = \gamma_{X(2j+1)}(G)$.*

Proof. Assume $L \subseteq V(G)$ is a $X(2j+1)$-dominating set for G. Since L is an $LR(\infty, j)$-dominating set for G by Theorem 9, then clearly L is an $LR(i,j)$-dominating set for G. Next assume that $L \subseteq V(G)$ is an $LR(i,j)$-dominating set for G with $i > j$. Assume $|N[u] \cap L| = 2j$ with $N[u] \cap L = \{x_1, x_2, \ldots, x_{2j}\}$. For Case 1, assume that $u \notin L$. If for $1 \leq i \leq j$, each vertex in $N[x_i] \cap L$ reports x_i, each vertex not adjacent to any x_i for $1 \leq i \leq j$ does not report, and for $1 \leq i \leq j$ each x_i also reports u then the intruder set, I, cannot be determined. Either the $I = \{u, x_1, x_2, \ldots, x_j\}$ with x_{j+1}, x_{j+2}, ..., x_{2j} lying or $I = \{x_1, x_2, \ldots, x_j\}$ with x_1, x_2, ..., x_j lying. For Case 2, assume that $u \in L$ and $N[u] \cap L = \{u, x_1, x_2, \ldots, x_{2j-1}\}$. If for $1 \leq i \leq j-1$, each vertex in $N[x_i] \cap L$ reports x_i, each vertex not adjacent to any x_i for $1 \leq i \leq j-1$ does not report, and for $1 \leq i \leq j-1$ each x_i also reports u then the intruder set, I, cannot be determined. Either $I = \{u, x_1, x_2, \ldots, x_{j-1}\}$ with u, x_j, x_{j+1}, ..., x_{2j-1} lying or $I = \{x_1, x_2, \ldots, x_{j-1}\}$ with x_1, x_2, ..., x_{j-1} lying. Thus for every $u \in V(G)$, $|N[u] \cap L| \geq 2j+1$. Hence L is a $X(2j+1)$-dominating set for G.□

However, $LR_{(i,j)}(G)$ is not always equivalent to a set-sized domination parameter for every i and j. We next show (as discussed in [7]) that there does not exist a sequence $(c_1, c_2, c_3, \ldots)$ such that $LR_{(2,2)}(G) = \gamma_{X(c_1,c_2,c_3,\ldots)}(G)$ for all G. By Corollary 10, we have that $LR_{(2,2)}(G) \leq \gamma_{X(5)}(G)$.

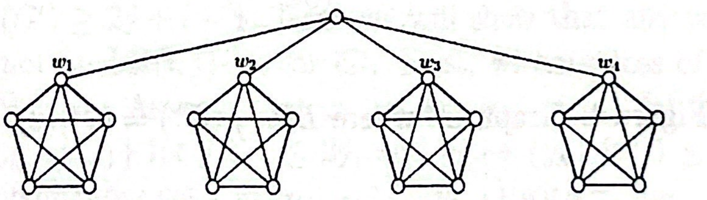

Figure 2: Graph $G^{\#}$.

Observation 12 *For the graph $G^{\#}$ which consists of a vertex of degree four adjacent to one vertex in each of four disjoint K_5's, say w_1, w_2, w_3, and w_4 as in Figure 2 we have $LR_{(2,2)}(G^{\#}) = 20 \leq \gamma_{X5}(G^{\#}) = 21$.*

By Observation 12 and Observation 5 we see that $LR_{(2,2)}(G^{\#}) < \gamma_{X5}(G^{\#}) \leq \gamma_{X(c_1,c_2,\ldots)}(G^{\#})$ whenever $c_1 \geq 5$.

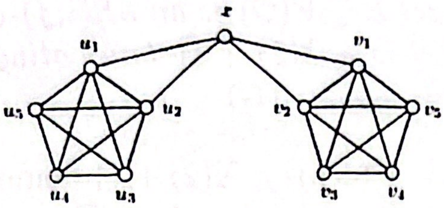

Figure 3: Graph G_p with $p = 5$.

By Theorem 7 we know that $LR_{(2,2)}(G) \geq \gamma_{\times(4,5)}(G)$. Consider G_p consisting of a vertex x adjacent to two vertices in each of two K_p's as in Figure 3. When $p \geq 5$, we can see that $LR_{(2,2)}(G_p) = 11$ and $\gamma_{\times(4,5)}(G_p) = 10$. Hence $LR_{(2,2)}(G_p) > \gamma_{\times(4,5)}(G_p)$. Now for any $(4, c_2, c_3, \ldots, c_t, \ldots)$ with $c_2 \geq 5$, some $c_t \geq 6$, and $p = max\{t+2, c_t\}$, we have $LR_{(2,2)}(G_p) = 11 < 12 \leq 2c_t \leq \gamma_{\times(4,c_2,c_3,\ldots,c_t,\ldots)}(G_p)$. Hence $LR_{(2,2)}(G_p) < \gamma_{\times(4,c_2,c_3,\ldots,c_t,\ldots)}(G_p)$ whenever $c_2 \geq 5$ and some $c_t \geq 6$. Thus there does not exist a sequence $(c_1, c_2, c_3, \ldots)$ for which $LR_{(2,2)}(G) = \gamma_{\times(c_1,c_2,c_3,\ldots)}(G)$ for all G.

Our principal theorem generalizes this for all liars' domination parameters with $2 \leq i \leq j$.

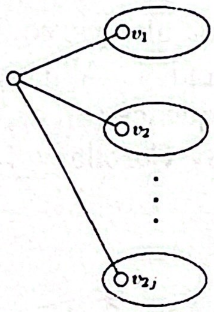

Figure 4: Graph G^* where $LR_{(i,j)}(G^*) = (2j)(2j+1)$

Theorem 13 *For $2 \leq i \leq j$ there does not exist a sequence $(c_1, c_2, c_3, \ldots)$ for which $LR_{(i,j)}(G) = \gamma_{\times(c_1,c_2,c_3,\ldots)}(G)$ for all G.*

Proof. By Theorem 7, $LR_{(i,j)}(G) \geq \gamma_{\times(i+j,2j+1)}(G)$ for $i \leq j$. First, we will show that there exists a graph G^* such that for $2 \leq i \leq j$, $LR_{(i,j)}(G^*) < \gamma_{\times(c_1,c_2,\ldots,c_t)}(G^*)$, where $i+j \leq c_1 \leq 2j+1$ and $c_t \geq 2j+2$. Let G^* consist of $2j$ K_p's, complete graphs on p vertices, each with one vertex v_i adjacent to x as seen in Figure 4. Let $p \geq max\{t+1, c_t\}$. Let $L \subseteq V(G^*)$ consist of each v_i and $2j$ other vertices from each of the K_p's. Thus $|L| = (2j)(2j+1)$. Notice that L is a $\gamma_{\times(i+j,2j+1)}(G^*)$-set and hence $LR_{(i,j)}(G^*) \geq (2j)(2j+1)$.

Now we will see that L is an $LR(i,j)$-set for G^*. Since each vertex v in any K_p is dominated by $2j+1$ vertices, then we can determine if there is an intruder at v. The only vertex not dominated $2j+1$ times is x. If more than j of the v_i agree about the presence of an intruder at x, then since there are at most j liars, they must be telling the truth about x. Suppose exactly j of the v_i's report that there is an intruder at x. If each of the K_p's corresponding to a v_i which reported x also contains an identified intruder, then since there are at most $i \leq j$ intruders, there is no intruder at x. If one of the K_p's corresponding to a v_i which reported x does not contain an identified intruder, then since there are no false alarms, there is no intruder at x. Thus, L is an $LR(i,j)$-set for G^* and $LR_{(i,j)}(G^*) \leq (2j)(2j+1)$. Hence, $LR_{(i,j)}(G^*) = (2j)(2j+1)$.

Now consider $\gamma_{X(c_1,c_2,\ldots,c_t)}(G^*)$, where $i+j \leq c_1 \leq 2j+1$ and $c_t \geq 2j+2$. Notice that since $c_t \geq 2j+2$, at least $2j+2$ vertices are needed in each K_p to c_t-dominate any set of size t in the K_p. Thus $\gamma_{X(c_1,c_2,\ldots,c_t)}(G^*) \geq (2j)c_t \geq (2j)(2j+2)$. Thus $LR_{(i,j)}(G^*) < \gamma_{X(c_1,c_2,\ldots,c_t)}(G^*)$.

Second, we will show that there exists a graph G' such that for $2 \leq i \leq j$, $LR_{(i,j)}(G') > \gamma_{X(c_1,2j+1)}(G')$, where $i+j \leq c_1 \leq 2j$. Let G be the join of a complete graph on $2j$ vertices and the complement of a complete graph on i vertices. Thus $G' = K_{2j} + \bar{K}_i$. Let $V(K_{2j}) = \{x_1, x_2, \ldots, x_{2j}\}$ and $V(\bar{K}_i) = \{y_1, y_2, \ldots, y_i\}$. Consider $\gamma_{X(c_1,2j+1)}(G')$, where $i+j \leq c_1 \leq 2j$. Notice that to $2j+1$-dominate any pair $\{y_p, y_q\}$ either all of $2j$ vertices from K_{2j} and at least one of y_p and y_q is needed, or all but one of the vertices from K_{2j} and both y_p and y_q are needed. Thus for a $X(c_1, 2j+1)$-set L for G', at most one vertex from $V(G)$ is not in L. Also notice that for any $v \in V(G')$, $V(G') - v$ is a $X(c_1, 2j+1)$-set for G'. Hence, $\gamma_{X(c_1,2j+1)}(G') = 2j+i-1$.

Now consider $LR_{(i,j)}(G')$. By Theorem 7, $LR_{(i,j)}(G') \geq \gamma_{X(i+j,2j+1)}(G')$. Thus, $LR_{(i,j)}(G') \geq 2j+i-1$. Next we will show that any set L of size $2j+i-1$ is not an $LR(i,j)$-set for G'. First, without loss of generality, let $L = V(G') - y_i$. Assume that $x_k \mapsto \{y_1, y_2, \ldots, y_i\}$ for $1 \leq k \leq j$, $x_k \mapsto \{y_1, y_2, \ldots, y_{i-1}\}$ for $j+1 \leq 2j$, and $y_k \mapsto \{y_k\}$ for $1 \leq k \leq i-1$. Then either the intruder set $I = \{y_1, y_2, \ldots, y_{i-1}\}$ with x_1, x_2, ..., x_j lying, or $I = \{y_1, y_2, \ldots, y_i\}$ with x_{j+1}, x_{j+2}, ..., x_{2j} lying. Thus, L is not an $LR(i,j)$-set for G'. Second, without loss of generality, let $L = V(G') - x_{2j}$. Assume that $x_k \mapsto \{y_1, y_2, \ldots, y_i\}$ for $1 \leq k \leq j$, $x_k \mapsto \{y_1, y_2, \ldots, y_{i-1}\}$ for $j+1 \leq k \leq 2j-1$, and $y_k \mapsto \{y_k\}$ for $1 \leq k \leq i-1$. Then either the intruder set is $I = \{y_1, y_2, \ldots, y_{i-1}\}$ with x_1, x_2, ..., x_j lying, or $I = \{y_1, y_2, \ldots, y_i\}$ with y_i, x_{j+1}, x_{j+2}, ..., x_{2j-1} lying. Thus, L is not an $LR(i,j)$-set for G'. Hence, $LR_{(i,j)}(G') \geq 2j+i$. By Corollary 10, $LR_{(i,j)}(G') \leq \gamma_{X(2j+1)}(G')$ and since $\gamma_{X(2j+1)}(G') = 2j+i$, then $LR_{(i,j)}(G') = 2j+i$. Thus, $LR_{(i,j)}(G') > \gamma_{X(c_1,2j+1)}(G')$, where $i+j \leq c_1 \leq 2j$.

Third, we will show that there exists a graph G^+ such that for $2 \leq i \leq j$,

$LR_{(i,j)}(G^+) < \gamma_{\times(2j+1)}(G^+)$. Let G^+ consist of $2j$ complete graphs on p vertices, each with one vertex v_i adjacent to x similar to the graph in Figure 4. Let $p \geq 2j+1$. Notice that to dominate x $2j+1$ times, x and each of the v_i's are needed. To dominate each vertex in any one of the K_p's, $2j$ more vertices are needed in that K_p. Thus $\gamma_{\times(2j+1)}(G') = (2j)(2j+1)+1$. However, as seen before with G^*, $LR_{(i,j)}(G^+) = (2j)(2j+1)$. Hence, $LR_{(i,j)}(G^+) < \gamma_{\times(2j+1)}(G^+)$. Therefore, for $2 \leq i \leq j$ there does not exist a sequence $(c_1, c_2, c_3, \ldots)$ with $LR_{(i,j)}(G) = \gamma_{\times(c_1,c_2,c_3,\ldots)}(G)$ for all G. □

3 Summary

In brief, $LR(i,0)(G) = \gamma(G)$; from Theorem 1 we have that $LR_{(1,1)}(G) = \gamma_{\times(2,3)}(G)$; more generally, from Theorem 8 we have that $LR_{(1,j)}(G) = \gamma_{\times(j+1,2j+1)}(G)$. Also we know that $LR_{(2,1)}(G) = \gamma_{\times(3)}(G)$; more generally, from Theorem 11 we have $LR_{(i,j)}(G) = \gamma_{\times(2j+1)}(G)$ for $i > j$. Rather surprisingly, for the remaining cases, by Theorem 13, if $2 \leq i \leq j$ then parameter $LR_{(i,j)}$ is not equivalent to any set-sized domination parameter $\gamma_{\times(c_1,c_2,c_3,\ldots)}(G)$. We can however, for example, bound $LR_{(2,2)}$. Specifically, by Theorem 7 and Corollary 10 we have $\gamma_{\times(4,5)}(G) \leq LR_{(2,2)} \leq \gamma_{\times(5)}(G)$.

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