Packing the crowns with cycles and stars

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Abstract

Let F, G and H be graphs. A (G, H)*-decomposition* of F is a partition of the edge set of F into copies of G and copies of H with at least one copy of G and at least one copy of H. For $L \subseteq F$, a (G, H)*-packing* of F with *leave* L is a (G, H)-decomposition of $F - E(L)$. A (G, H)-packing of F with the largest cardinality is a *maximum* (G, H)*-packing*. This paper gives the solution of finding the maximum (C_k, S_k)-packing of the crown $C_{n,n-1}$.

1 Introduction

Let F, G and H be graphs. A G*-decomposition* of F is a partition of the edge set of F into copies of G. If F has a G-decomposition, we say that F is G*-decomposable.* A (G, H)*-decomposition* of F is a partition of the edge set of F into copies of G and copies of H with at least one copy of G and at least one copy of H. If F has a (G, H)-decomposition, we say that F is (G, H)*-decomposable.* A (G, H)-decomposition of F may not exist, however, it is of interest to see just how "close" one can come to a (G, H)-decomposition. For $L \subseteq F$, a (G, H)*-packing* of F with *leave* L is a (G, H)-decomposition of $F - E(L)$. A (G, H)-packing of F with the largest cardinality is a *maximum* (G, H)*-packing.* Moreover, the cardinality of the maximum (G, H)-packing of F is called the (G, H)*-packing number* of F, denoted by $p(F; G, H)$.

The *degree* of a vertex x of G, denoted by $\deg_G x$, is the number of edges incident with x. As usual K_n denotes the complete graph with n vertices and $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. A *k-star*, denoted by S_k, is the complete bipartite graph $K_{1,k}$. The vertex of degree k in S_k is the *center* of S_k and any vertex of degree 1 is an *endvertex* of S_k. Let $(x; y_1, y_2, \ldots, y_k)$ denote the k-star with center x and endvertices $y_1, y_2, \ldots, y_k$. A *k-cycle* (respectively, *k-path* and *k-matching*), denoted by C_k (respectively, P_k and M_k), is a cycle (respectively, path and matching) with k edges. Let $(v_1, v_2, \ldots, v_k)$ and $v_1 v_2 \ldots, v_k$ denote the k-cycle and

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$(k-1)$-path through vertices $v_1, \ldots, v_k$ in order, respectively. A *spanning subgraph* H of a graph G is a subgraph of G with $V(H) = V(G)$. A *1-factor* of G is a spanning subgraph of G with each vertex incident with exactly one edge. For positive integers ℓ and n with $1 \leq \ell \leq n$, the *crown* $C_{n,\ell}$ is a bipartite graph with bipartition (A, B) where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$, and edge set $\{a_ib_j : i = 0, 1, \ldots, n-1,\ j \equiv i+1, i+2, \ldots, i+\ell \pmod{n}\}$. Hereafter (A, B) always means the bipartition of $C_{n,\ell}$ defined here. Note that $C_{n,n-1}$ is the graph obtained from the complete bipartite graph $K_{n,n}$ with a 1-factor removed.

The existence problems for (C_k, S_k)-decomposition of $K_{m,n}$ and $C_{n,n-1}$ have been completely settled by Lee [4] and Lee and Lin [5], respectively. Abueida and Daven [2] obtained the maximum packing of the complete graph K_n with (K_k, S_k). Abueida and Daven [1] and Abueida, Daven and Roblee [3] gave the maximum packing of K_n and λK_n with G and H, respectively, where (G, H) is a graph-pair of order 4 or 5. This paper gives the solution of finding the maximum (C_k, S_k)-packing of the crown $C_{n,n-1}$.

2 Preliminaries

We first collect some needed terminology and notation. Let $G = (V, E)$ be a graph. For sets $A \subseteq V(G)$ and $B \subseteq E(G)$, we use $G[A]$ to denote the subgraph of G induced by A and $G - B$ (respectively, $G + B$) to denote the subgraph obtained from G by deleting (respectively, adding) the edges in B. When $G_1, \ldots, G_t$ are graphs, not necessarily disjoint, we write $G_1 \cup \cdots \cup G_t$ or $\bigcup_{i=1}^{t} G_i$ for the graph with vertex set $\bigcup_{i=1}^{t} V(G_i)$ and edge set $\bigcup_{i=1}^{t} E(G_i)$. When the edge sets are disjoint, $G = \bigcup_{i=1}^{t} G_i$ expresses the decomposition of G into $G_1, \ldots, G_t$. For a graph G and a positive integer $\lambda \geq 2$, we use λG to denote the multigraph obtained from G by replacing each edge e by λ edges, each of which has the same ends as e.

The following results are essential to our proof.

Lemma 2.1. ([9]) *For integers m and n with $m \geq n \geq 1$, the graph $K_{m,n}$ is S_k-decomposable if and only if $m \geq k$ and*

$$\begin{cases} m \equiv 0 \pmod{k} & \text{if } n < k \\ mn \equiv 0 \pmod{k} & \text{if } n \geq k. \end{cases}$$

Lemma 2.2. ([6]) *$\lambda C_{n,\ell}$ is S_k-decomposable if and only if $k \leq \ell$ and $\lambda n \ell \equiv 0 \pmod{k}$.*

Lemma 2.3. ([6]) *Let $\{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$ be the vertex set of the multicrown $\lambda C_{n,\ell}$. Suppose that p and q are positive integers such that $q < p \leq \ell$. If $\lambda n q \equiv 0 \pmod{p}$, then there exists a spanning subgraph*

G of $\lambda C_{n,\ell}$ such that $\deg_G b_j = \lambda q$ for $0 \le j \le n-1$ and G has an S_p-decomposition.

Lemma 2.4. ([8]) *For positive integers m, n, and k, the graph $K_{m,n}$ is C_k-decomposable if and only if m, n, and k are even, $k \ge 4$, $\min\{m,n\} \ge k/2$, and $mn \equiv 0 \pmod{k}$.*

Lemma 2.5. ([7]) *For positive integers k and n, $C_{n,n-1}$ is C_k-decomposable if and only if n is odd, k is even, $4 \le k \le 2n$, and $n(n-1) \equiv 0 \pmod{k}$.*

3 Packing numbers

In this section a complete solution to the maximum (C_k, S_k)-packing problem of $C_{n,n-1}$ is given.

Lemma 3.1. ([5]) *If k is an even integer with $k \ge 4$, then there exist $k/2-1$ edge-disjoint k-cycles in $C_{k/2,k/2-1} \cup K_{k/2,k/2}$.*

Lemma 3.2. ([5]) *If k is an even integer with $k \ge 4$, then $C_{k+1,k}$ is not (C_k, S_k)-decomposable.*

Lemma 3.3. *If k is an even integer with $k \ge 4$, then $C_{k+1,k}$ has a (C_k, S_k)-packing with leave $S_{k/2} \cup M_{k/2}$.*

Proof. By Lemma 3.2, we have that $C_{k+1,k}$ is not (C_k, S_k)-decomposable. Let $H_0 = C_{k+1,k}[\{a_0, a_1, \ldots, a_{k/2-1}\} \cup \{b_0, b_1, \ldots, b_{k-1}\}]$, $H_1 = C_{k+1,k}[\{a_0, a_1, \ldots, a_{k/2-1}\} \cup \{b_k\}]$ and $H_2 = C_{k+1,k}[\{a_{k/2}, a_{k/2+1}, \ldots, a_k\} \cup \{b_0, b_1, \ldots, b_k\}]$. Clearly, $C_{k+1,k} = \cup_{i=0}^{2} H_i$. Note that H_0 is isomorphic to $C_{k/2,k/2-1} \cup K_{k/2,k/2}$, H_1 is isomorphic to $S_{k/2}$ and H_2 is isomorphic to $K_{k/2+1,k/2} \cup C_{k/2+1,k/2}$. By Lemma 3.1, we have H_0 can be decomposed into $k/2-1$ edge-disjoint k-cycles and a $k/2$-matching. In addition, since $\deg_{H_2} a_i = k$ for $k/2 \le i \le k$, it follows that H_2 is S_k-decomposable. Hence $C_{k+1,k}$ has a (C_k, S_k)-packing with leave $S_{k/2} \cup M_{k/2}$. □

Therefore, with the results of Lemmas 3.2 and 3.3, we have the following.

Corollary 3.4. $p(C_{k+1,k}; C_k, S_k) = k$.

Lemma 3.5. ([5]) *If k is an even integer with $k \ge 4$, then $C_{2k,2k-1}$ is (C_k, S_k)-decomposable.*

Lemma 3.6. *Let k be a positive even integer and let n be a positive integer with $4 \le k < n-1 < 2k-1$. If $(n-k)(n-k-1) < k$, then $C_{n,n-1}$ has a (C_k, S_k)-packing with leave $C_{n-k,n-k-1}$.*

Proof. Let $n-1=k+r$. The assumption $k<n-1<2k-1$ implies $0<r<k-1$. Let $H_0'=C_{k+r+1,k+r}[\{a_0,a_1,\ldots,a_k\}\cup\{b_0,b_1,\ldots,b_k\}]$, $H_1'=C_{k+r+1,k+r}[\{a_0,a_1,\ldots,a_{k-1}\}\cup\{b_{k+1},b_{k+2}\ldots,b_{k+r}\}]$, $H_2'=C_{k+r+1,k+r}[\{a_{k+1},a_{k+2}\ldots,a_{k+r}\}\cup\{b_0,b_1,\ldots,b_{k-1}\}\}]$ and $H_3'=C_{k+1,k}[\{a_k,a_{k+1},\ldots,a_{k+r}\}\cup\{b_k,b_{k+1},\ldots,b_{k+r}\}]$. Clearly, $C_{k+r+1,k+r}=\cup_{i=0}^{3}H_i'$. Note that H_0' is isomorphic to $C_{k+1,k}$, H_1' and H_2' are isomorphic to $K_{k,r}$ and H_3' is isomorphic to $C_{r+1,r}$. By Lemma 2.5, H_0' is C_k-decomposable. In addition, by Lemma 2.1, H_1' and H_2' are S_k-decomposable. Note that $|E(H_3')|=(n-k)(n-k-1)<k$, $C_{n,n-1}$ has a (C_k,S_k)-packing with leave H_3', that is, $C_{n-k,n-k-1}$. □

Lemma 3.7. *Let k be a positive even integer and let n be a positive integer with $4\le k<n-1<2k-1$. If $(n-k)(n-k-1)\ge k$, then $C_{n,n-1}$ has a (C_k,S_k)-packing $\mathscr{P}$ with $|\mathscr{P}|=\lfloor n(n-1)/k\rfloor$.*

Proof. Let $n-1=k+r$. From the assumption $k<n-1<2k$, we have $0<r<k-1$. Since $(n-k)(n-k-1)\ge k$, let $m=(n-k)(n-k-1)=r(r+1)=tk+s$, where $t\ge 1$ and $0\le s\le k-1$. The proof is divided into two parts according to the value of t.

Case 1. $t=1$. Since $4\le k\le r(r+1)$, we have $r\ge 2$. We distinguish two subcases.

Subcase 1.1. $r=2$, then $k=4$ or 6. For the case that $k=4$, then $n=7$. Let $E_0=C_{7,6}[\{a_0,a_1\}\cup B]$, $E_1=C_{7,6}[\{a_2,a_3\}\cup B]$, $E_2=C_{7,6}[\{a_4,a_5\}\cup B]$ and $E_3=C_{7,6}[\{a_6\}\cup B]$. Clearly, $C_{7,6}=\cup_{i=0}^{3}E_i$. Note that E_i is isomorphic to $C_{2,1}\cup K_{2,5}$ for $i\in\{0,1,2\}$ and $E_3=S_4\cup S_2$. Since E_i can be decomposed into two copies of S_4 and one copy of C_4 for $i\in\{0,1,2\}$, it follows that $C_{7,6}$ has a (C_4,S_4)-packing $\mathscr{P}_1$ with leave S_2 and $|\mathscr{P}_1|=3\cdot(2+1)+1=10$.

For the other case that $k=6$, then $n=9$. We have that $C_{9,8}=\cup_{i=0}^{2}C_{9,8}[\{a_{3i},a_{3i+1},a_{3i+2}\}\cup B]=3C_{3,2}\cup 3K_{3,6}=3C_6\cup 9S_6$ for $i\in\{0,1,2\}$. Hence, $C_{9,8}$ is (C_6,S_6)-decomposable, that is, $C_{9,8}$ has a (C_6,S_6)-packing $\mathscr{P}_2$ with leave $\varnothing$ and $|\mathscr{P}_2|=3\cdot(1+3)=12$.

Subcase 1.2. $r\ge 3$. Let $A_0=\{a_0,a_1,\ldots,a_{\frac{m}{2}-1}\}$, $B_0=\{b_0,b_1,\ldots,b_{\frac{m}{2}-1}\}$, $D_0=C_{n,n-1}[A_0\cup B_0]$, $D_1=C_{n,n-1}[(A\setminus A_0)\cup B_0]$, and $D_2=C_{n,n-1}[A\cup(B\setminus B_0)]$. Clearly $C_{n,n-1}=D_0\cup D_1\cup D_2$. Note that D_0 is isomorphic to $C_{\frac{m}{2},\frac{m}{2}-1}$, D_1 is isomorphic to $K_{k+r+1-\frac{m}{2},\frac{m}{2}}$, and D_2 is isomorphic to $C_{k+r+1-\frac{m}{2},k+r-\frac{m}{2}}\cup K_{\frac{m}{2},k+r+1-\frac{m}{2}}$.

Claim. $C_{n,n-1}$ can be decomposed into k-stars together with a m-cycle.

Check. Let $C=(b_1,a_0,b_2,a_1,b_3,a_2,\ldots,b_{\frac{m}{2}-1},a_{\frac{m}{2}-2},b_0,a_{\frac{m}{2}-1})$ and $D=D_0-E(C)$. Trivially, C is a m-cycle in D_0 and $D=C_{\frac{m}{2},\frac{m}{2}-3}$. Note that $r-2<\frac{m}{2}-r-1$ for $r\ge 3$ and $\frac{m}{2}(r-2)=r(r+1)(r-2)/2=r(\frac{m}{2}-r-1)$. Thus there exists a spanning subgraph X of D such that $\deg_X b_j=r-2$ for $0\le j\le\frac{m}{2}-1$ and X has an $S_{\frac{m}{2}-r-1}$-decomposition $\mathscr{D}$ with $|\mathscr{D}|=r$ by

Lemma 2.3. Furthermore, each $S_{\frac{m}{2}-r-1}$ has its center in A_0 since $\deg_X b_j = r-2 < \frac{m}{2}-r-1$. Suppose that the centers of the $(\frac{m}{2}-r-1)$-stars in $\mathscr{D}$ are $a_{i_1},\ldots,a_{i_r}$. Let $S(w)$ be the $(\frac{m}{2}-r-1)$-star with center a_{i_w} in $\mathscr{D}$, and let $Y = D - E(X) \cup D_1$. Note that $\deg_Y b_j = (\frac{m}{2}-3-(r-2))+(k+r+1-\frac{m}{2}) = k$ for $0 \le j \le \frac{m}{2}-1$. Hence Y is S_k-decomposable. For $w \in \{1,\ldots,r\}$, define $S'(w) = D_2[\{a_{i_w}\} \cup (B \setminus B_0)]$ and $Z = D_2 - E(\bigcup_{w=1}^{r} S'(w))$. Clearly $S'(w)$ is a $(k+r+1-\frac{m}{2})$-star with center a_{i_w} in D_2, and $S(w) \cup S'(w)$ is a k-star. Moreover, $\deg_Z b_j = k+r-r = k$ for $\frac{m}{2} \le j \le k+r$. Thus Z is S_k-decomposable. This completes the check of Claim.

Let S be the k-star containing the edge $a_{\frac{k}{2}-1}b_1$ in the decomposition in Claim. We can see that

$$C \cup S = (C - e + a_{\frac{k}{2}-1}b_1) \cup (S - a_{\frac{k}{2}-1}b_1 + e),$$

where $e \in E(C)$ and $e = \begin{cases} a_{\frac{k}{2}-1}b_{\frac{k}{2}+1}, & \text{if the center of } S \text{ in } A_0, \\ a_{\frac{m}{2}-1}b_1, & \text{if the center of } S \text{ in } B_0. \end{cases}$

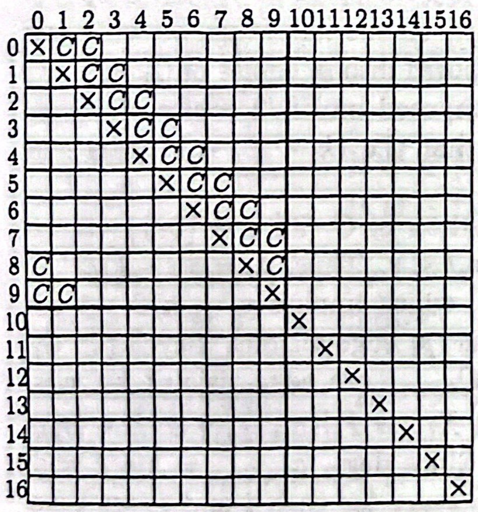
Then the graph $C-e+a_{\frac{k}{2}-1}b_1$ is the union of a k-cycle $C' : (b_1, a_0, b_2, a_1, b_3, a_2, \ldots, b_{\frac{k}{2}-1}, a_{\frac{k}{2}-2}, b_{\frac{k}{2}}, a_{\frac{k}{2}-1})$ and a $(m-k)$-path P', where

$$P' = \begin{cases} b_{\frac{k}{2}+1}a_{\frac{k}{2}}b_{\frac{k}{2}+2}a_{\frac{k}{2}+1}\cdots b_{\frac{m}{2}-1}a_{\frac{m}{2}-2}b_0a_{\frac{m}{2}-1}b_1, & \text{if the center of } S \text{ in } A_0, \\ a_{\frac{k}{2}-1}b_{\frac{k}{2}+1}a_{\frac{k}{2}}b_{\frac{k}{2}+2}\cdots b_{\frac{m}{2}-1}a_{\frac{m}{2}-2}b_0a_{\frac{m}{2}-1}, & \text{if the center of } S \text{ in } B_0. \end{cases}$$

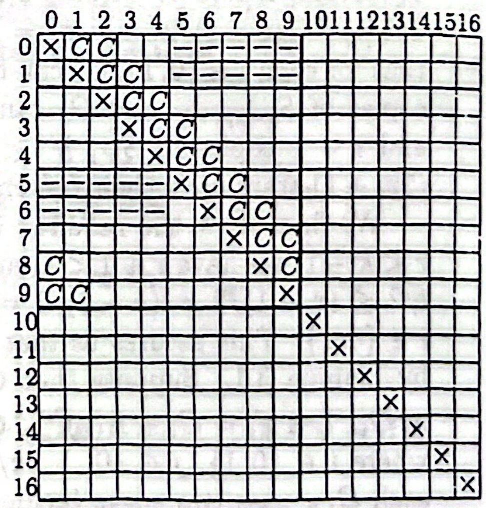
On the other hand, the graph $S - a_{\frac{k}{2}-1}b_1 + e$ is still a k-star. Hence $C_{n,n-1}$ has a (C_k, S_k)-packing $\mathscr{P}_3$ with leave P' and $|\mathscr{P}_3| = r + (k+r+1) + 1 = k + 2r + 2 = \lfloor n(n-1)/k \rfloor$, $|P'| = s$. This settles Case 1.

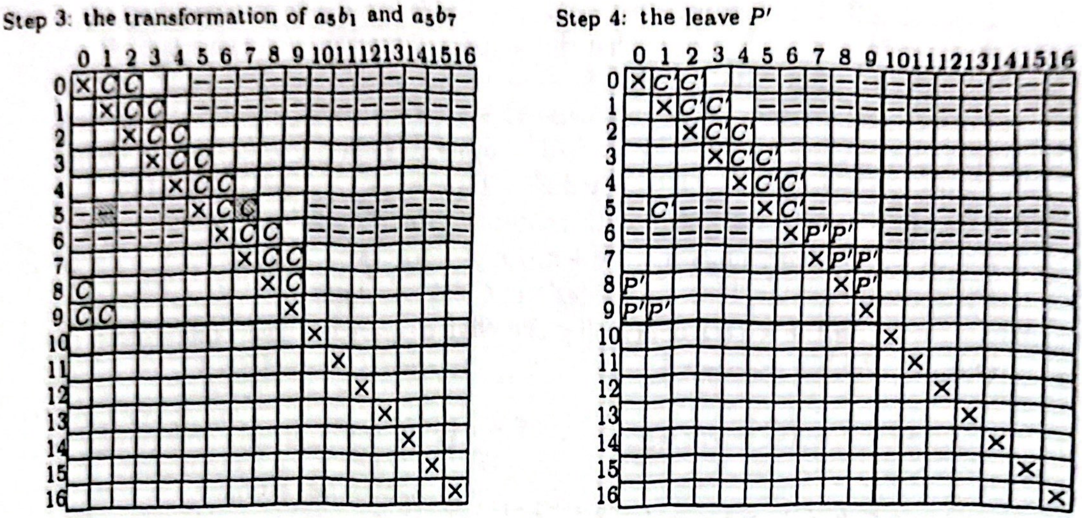
To illustrate the decomposition in Subcase 1.2 of Lemma 3.7, in Figure 1 we give the maximum (C_{12}, S_{12})-packing of $C_{17,16}$.

Step 1: C_{20} $(m = 20)$



Step 2: 4 copies of S_5 $(r = 4,\ \frac{m}{2} - r - 1 = 5)$





(The rows are numbered $0, 1, 2, \cdots, 16$ from top to bottom and the columns are numbered $0, 1, 2, \cdots, 16$ from left to right. The mark $\times$ means no edge. The (i, j)-position is marked "C", "C'", "P'", "$-$", or " " according to a edge of the m-cycle, k-cycle, leave, star a_ib_j with center a_i, or b_j.)

Figure 1: The maximum (C_{12}, S_{12})-packing of $C_{17,16}$

Case 2. $t \geq 2$.

Let $A'_0 = \{a_0, a_1, \ldots, a_{\frac{k}{2}-1}\}$, $A'_1 = \{a_{\frac{k}{2}}, a_{\frac{k}{2}+1}, \ldots, a_{k-1}\}$, $A'_2 = A \setminus (A'_0 \cup A'_1)$, $B'_0 = \{b_0, b_1, \ldots, b_{\frac{k}{2}-1}\}$, $B'_1 = \{b_{\frac{k}{2}}, b_{\frac{k}{2}+1}, \ldots, b_{k-1}\}$, $B'_2 = B \setminus (B'_0 \cup B'_1)$. Let $G_i = C_{n,n-1}[A'_i \cup B'_0 \cup B'_1]$ for $i \in \{0, 1, 2\}$ and $G_3 = C_{n,n-1}[A \cup B'_2]$. Clearly $C_{n,n-1} = \cup_{i=0}^{3} G_i$. Note that G_0 and G_1 are isomorphic to $C_{k/2,k/2-1} \cup K_{k/2,k/2}$, G_2 is isomorphic to $K_{r+1,k}$, which is S_k-decomposable by Lemma 2.1 and G_3 is isomorphic to $K_{k,r+1} \cup C_{r+1,r}$. Let $p_0 = \lceil (t-1)/2 \rceil$ and $p_1 = \lfloor (t-1)/2 \rfloor$. In the following, we will show that for each $i \in \{0, 1\}$, G_i can be decomposed into p_i copies of C_k and $k/2$ copies of S_{k-2p_i-1}, and G_3 can be decomposed into $k/2$ copies of S_{2p_i+1} and $r+1$ copies of $S_{k'}$, $k' \geq k$, such that the $(k-2p_i-1)$-stars and $(2p_i+1)$-stars have their centers in A'_i.

We first show the required decomposition of G_i for $i \in \{0, 1\}$. Since $r < k-1$, we have $r+1 < k$, and in turn $t < r$. Thus, $p_0 = \lceil (t-1)/2 \rceil \leq t/2 \leq (r-1)/2 < (k-2)/2 = k/2 - 1$, which implies $p_i < k/2 - 1$ for $i \in \{0, 1\}$. This assures us that there exist p_i edge-disjoint k-cycles in G_i by Lemma 3.1. Suppose that $Q_{i,0}, \ldots, Q_{i,p_i-1}$ are edge-disjoint k-cycles in G_i. Let $F_i = G_i - E(\bigcup_{h=0}^{p_i-1} Q_{i,h})$ and $X_{i,j} = F_i[\{a_{ik/2+j}\} \cup (B - B'_2)]$ where $i \in \{0, 1\}$, $j \in \{0, \ldots, k/2-1\}$. Since $\deg_{G_i} a_{ik/2+j} = k-1$ and each $Q_{i,h}$ uses two edges incident with $a_{ik/2+j}$ for each i and j, we have

$\deg_{F_i} a_{ik/2+j} = k - 2p_i - 1$. Hence $X_{i,j}$ is a $(k-2p_i-1)$-star with center $a_{ik/2+j}$.

Next we show the required star decomposition of G_3. For $j \in \{0,\dots,k/2-1\}$, let

$$X'_{i,j} = \begin{cases} (a_j; b_{k+(2p_0+1)j}, b_{k+(2p_0+1)j+1}, \dots, b_{k+(2p_0+1)j+2p_0}), & \text{if } i = 0, \\ (a_{\frac{k}{2}+j}; b_{(p_0+\frac{3}{2})k+(2p_1+1)j}, b_{(p_0+\frac{3}{2})k+(2p_1+1)j+1}, \dots, b_{(p_0+\frac{3}{2})k+(2p_1+1)j+2p_1}), & \text{if } i = 1, \end{cases}$$

where the subscripts of b's are taken modulo $r+1$ in the set of numbers $\{k, k+1, \dots, k+r\}$. Since $2p_1+1 \le 2p_0+1 \le t+1 \le r$, this assures us that there are enough edges for the construction of $X'_{0,j}$ and $X'_{1,j}$. Note that $X'_{i,j}$ is a $(2p_i+1)$-star and $X_{i,j} \cup X'_{i,j}$ is a k-star for $i \in \{0,1\}$, $j \in \{0,\dots,k/2-1\}$.

On the other hand, let $s = \alpha(r+1)+\beta$ where $\alpha \ge 0$ and $0 \le \beta \le r$, we have that

$$\begin{aligned} & |E(G_3)| - |E(\cup_{i\in\{0,1\}} \cup_{j\in\{0,\dots,k/2-1\}} X'_{i,j})| \\ = \; & (k+r)(r+1) - (2p_0+2p_1+2)(k/2) \\ = \; & (k+r)(r+1) - tk \\ = \; & (k+r)(r+1) - r(r+1) + s \\ = \; & k(r+1) + \alpha(r+1) + \beta \\ = \; & (k+\alpha)(r+1) + \beta. \end{aligned}$$

Hence there exists a decomposition $\mathscr{D}$ of $G_3 - E(\cup_{i\in\{0,1\}}\cup_{j\in\{0,\dots,k/2-1\}} X'_{i,j})$ into $r+1-\beta$ copies of $(k+\alpha)$-star with center b_w for $w = k, k+1, \dots, k+r-\beta$ and β copies of $(k+\alpha+1)$-star with center b_w for $w = k+r-\beta+1, k+r-\beta+2, \dots k+r$. Let

$$Y_w = \begin{cases} S_{k+\alpha}, & \text{if } w \in \{k, k+1, \dots, k+r-\beta\}, \\ S_{k+\alpha+1}, & \text{if } w \in \{k+r-\beta+1, k+r-\beta+2, \dots k+r\} \end{cases}$$

in $\mathscr{D}$. Note that any endvertex a_i of Y_w, we have that $i \le k-1$.

Define a star Y'_w as follows:

$$Y'_w = \begin{cases} S(b_w; a_{w+1}, a_{w+2}, \dots, a_{w+\alpha}), & \text{if } w \in \{k, k+1, \dots, k+r-\beta\}, \\ S(b_w; a_{w+1}, a_{w+2}, \dots, a_{w+\alpha+1}), & \text{if } w \in \{k+r-\beta+1, k+r-\beta+2, \dots k+r\}, \end{cases}$$

where the subscripts of a's are taken modulo $r+1$ in the set of numbers $\{k, k+1, \dots, k+r\}$. Since $\alpha(r+1)+\beta = s = r(r+1) - tk < r(r+1)$, it follows that $\alpha < r$. This assures us that there are enough edges for the construction of Y'_w. It is easy to see that $Y_w - E(Y'_w)$ is a k-star. Hence $C_{n,n-1}$ has a (C_k, S_k)-packing $\mathscr{P}_4$ with leave $\cup_{w\in\{k,k+1,\ldots,k+r\}} Y'_w$ and $|\mathscr{P}_4| = (k+r+1)+(r+1)+(t-1) = k+2r+1+t = \lfloor n(n-1)/k \rfloor$. This completes the proof. □

To illustrate the decomposition in Case 2 of Lemma 3.7, in Figure 2 we give the maximum (C_{10}, S_{10})-packing of $C_{17,16}$. Note that $n = 17$, $k = 10$, $r = 6$, $t = 4$, $p_0 = 2$ and $p_1 = 1$. The number s in the row i, column j indicates that the cycle C_{10} contains the edge $a_i b_j$.

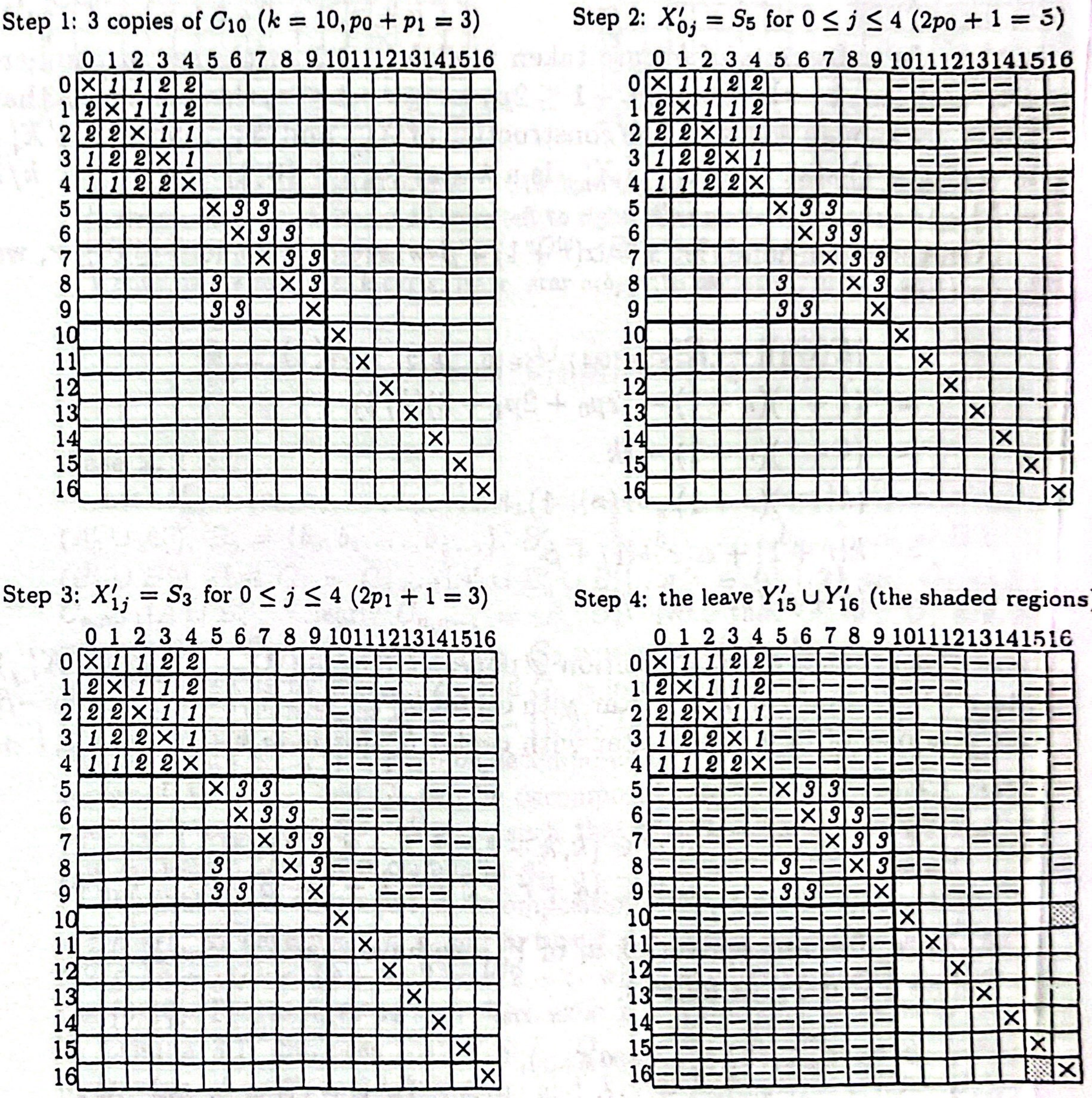


Figure 2: The maximum (C_{10}, S_{10})-packing of $C_{17,16}$

Theorem 3.8. *Let k be a positive even integer and let n be a positive integer with $4 \leq k \leq n-1$, then*

$$p(C_{n,n-1}; C_k, S_k) = \begin{cases} \lfloor n(n-1)/k \rfloor, & \text{if } k < n-1, \\ k, & \text{if } k = n-1. \end{cases}$$

Proof. Obviously, $p(C_{n,n-1}; C_k, S_k) \leq \lfloor n(n-1)/k \rfloor$. Let $n-1 = qk + r$, where q and r are integers with $q \geq 1$, $0 \leq r \leq k-1$. We consider the following two cases.

Case 1. $r = 0$.

For $q = 1$, the result follows from Corollary 3.4. If $q \geq 2$, then

$$C_{n,n-1} = C_{(q-1)k+1,(q-1)k} \cup K_{(q-1)k,k} \cup K_{k,(q-1)k} \cup C_{k+1,k}.$$

Trivially, $|E(C_{(q-1)k+1,(q-1)k})|$, $|E(K_{(q-1)k,k})|$ and $|E(K_{k,(q-1)k})|$ are multiples of k. By Lemmas 2.1 and 2.2, we have that $C_{(q-1)k+1,(q-1)k}$, $K_{(q-1)k,k}$ and $K_{k,(q-1)k}$ have S_k-decompositions $\mathscr{F}$, $\mathscr{F}'$ and $\mathscr{F}''$ with $|\mathscr{F}| = (q-1)((q-1)k+1)$, $|\mathscr{F}'| = |\mathscr{F}''| = k(q-1)$. In addition, by Lemma 2.5, $C_{k+1,k}$ has a C_k-decomposition $\mathscr{C}$ with $|\mathscr{C}| = k+1$. Hence $C_{n,n-1}$ is (C_k, S_k)-decomposable, that is, $C_{n,n-1}$ has a (C_k, S_k)-packing $\mathscr{F} \cup \mathscr{F}' \cup \mathscr{F}'' \cup \mathscr{C}$ with cardinality $(q-1)((q-1)k+1) + k(q-1) + k(q-1) + k + 1 = q(qk+1) = n(n-1)/k$.

Case 2. $r > 0$.

For $q = 1$, the result follows from Lemmas 3.5, 3.6 and 3.7. If $q \geq 2$, then

$$C_{n,n-1} = C_{(q-1)k+1,(q-1)k} \cup K_{(q-1)k,k+r} \cup K_{k+r,(q-1)k} \cup C_{k+r+1,k+r}.$$

Note that $C_{k+r+1,k+r}$ has a (C_k, S_k)-packing $\mathscr{P}$ with $|\mathscr{P}| = \lfloor (k+r+1)(k+r)/k \rfloor$. Trivially, $C_{(q-1)k+1,(q-1)k}$, $K_{(q-1)k,k+r}$ and $K_{k+r,(q-1)k}$ have S_k-decompositions $\mathscr{D}$, $\mathscr{D}'$ and $\mathscr{D}''$ with $|\mathscr{D}| = (q-1)((q-1)k+1)$, $|\mathscr{D}'| = |\mathscr{D}''| = (q-1)(k+r)$. Hence $\mathscr{P} \cup \mathscr{D} \cup \mathscr{D}' \cup \mathscr{D}''$ is a (C_k, S_k)-packing of $C_{n,n-1}$ with cardinality $\lfloor (k+r+1)(k+r)/k \rfloor + (q-1)((q-1)k+1) + (q-1)(k+r) + (q-1)(k+r) = \lfloor (qk+r+1)(qk+r)/k \rfloor = \lfloor n(n-1)/k \rfloor$. This completes the proof. □

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