Relating 2-rainbow domination to weak Roman domination

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Abstract

Addressing a problem posed by Chellali, Haynes, and Hedetniemi (Discrete Appl. Math. 178 (2014) 27-32) we prove $\gamma_{r2}(G) \leq 2\gamma_r(G)$ for every graph G, where $\gamma_{r2}(G)$ and $\gamma_r(G)$ denote the 2-rainbow domination number and the weak Roman domination number of G, respectively. We characterize the extremal graphs for this inequality that are $\{K_4, K_4 - e\}$-free, and show that the recognition of the K_5-free extremal graphs is NP-hard.

Keywords: 2-rainbow domination; Roman domination; weak Roman domination

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1 Introduction

We consider finite, simple, and undirected graphs, and use standard terminology and notation.

Rainbow domination was introduced in [1]. Here we consider the special case of 2-rainbow domination. A 2-*rainbow dominating function* of a graph

G is a function $f : V(G) \to 2^{\{1,2\}}$ such that $\bigcup_{v \in N_G(u)} f(v) = \{1,2\}$ for every vertex u of G with $f(u) = \emptyset$. The *weight of* f is $\sum_{u \in V(G)} |f(u)|$. The 2-*rainbow domination number* $\gamma_{r2}(G)$ *of* G is the minimum weight of a 2-rainbow dominating function of G, and a 2-rainbow dominating function of weight $\gamma_{r2}(G)$ is *minimum*. Weak Roman domination was introduced in [5]. For a graph G, a function $g : V(G) \to \mathbb{R}$, and two distinct vertices u and v of G, let

$$g_{v \to u} : V(G) \to \mathbb{R} : x \mapsto \begin{cases} g(u) + 1 & , x = u, \\ g(v) - 1 & , x = v, \text{ and} \\ g(x) & , x \in V(G) \setminus \{u, v\}. \end{cases}$$

A set D of vertices of G is *dominating* if every vertex in $V(G) \setminus D$ has a neighbor in D. A *weak Roman dominating function* of G is a function $g : V(G) \to \{0, 1, 2\}$ such that every vertex u of G with $g(u) = 0$ has a neighbor v with $g(v) \geq 1$ such that the set $\{x \in V(G) : g_{v \to u}(x) \geq 1\}$ is dominating. The *weight of* g is $\sum_{u \in V(G)} g(u)$. The *weak Roman domination number* $\gamma_r(G)$ of G is the minimum weight of a weak Roman dominating function of G, and a weak Roman dominating function of weight $\gamma_r(G)$ is *minimum*.

For a positive integer k, let $[k]$ be the set of positive integers at most k.

In [2] Chellali, Haynes, and Hedetniemi show that $\gamma_r(G) \leq \gamma_{r2}(G)$ for every graph G, and pose the problem to upper bound the ratio $\frac{\gamma_{r2}(G)}{\gamma_r(G)}$ (cf. Problem 17 in [2]). In the present paper we address this problem. As we shall see in Theorem 1 below, $\frac{\gamma_{r2}(G)}{\gamma_r(G)} \leq 2$ for every graph G. While the proof of this inequality is very simple, the extremal graphs are surprisingly complex. We collect some structural properties of these graphs in Theorem 1, and characterize all $\{K_4, K_4 - e\}$-free extremal graphs in Corollary 2, where K_n denotes the complete graph of order n, and $K_n - e$ arises by removing one edge from K_n. In contrast to this characterization, we show in Theorem 4 that the recognition of the K_5-free extremal graphs is algorithmically hard, which means that these graphs do not have a transparent structure. In our last result, Theorem 5, we consider graphs whose induced subgraphs are extremal.

The weak Roman domination number was introduced as a variant of the

Roman domination number $\gamma_R(G)$ of a graph G [6]. For results concerning the ratio $\frac{\gamma_{r2}(G)}{\gamma_R(G)}$ see [3, 4, 7].

2 Results

Theorem 1 *If G is a graph, then $\gamma_{r2}(G) \leq 2\gamma_r(G)$. Furthermore, if $\gamma_{r2}(G) = 2\gamma_r(G)$ and $g : V(G) \to \{0,1,2\}$ is a minimum weak Roman dominating function of G, then*

- *there is no vertex x of G with $g(x) = 2$, and*
- *if $V_1 = \{v_1, \ldots, v_k\}$ is the set of vertices x of G with $g(x) = 1$, then $V(G) \setminus V_1$ has a partition into $2k$ sets $P_1, \ldots, P_k, Q_1, \ldots, Q_k$ such that for every $i \in [k]$,*
  - *$P_i = \{u \in V(G) \setminus V_1 : N_G(u) \cap V_1 = \{v_i\}\}$ is non-empty and complete for $i \in [k]$, and*
  - *every vertex in the possibly empty set Q_i is adjacent to every vertex in $\{v_i\} \cup P_i$.*

Proof: Let $g : V(G) \to \{0,1,2\}$ is a minimum weak Roman dominating function of G. Clearly, $f : V(G) \to 2^{[2]}$ with

$$f(x) = \begin{cases} \emptyset, & g(x) = 0 \text{ and} \\ \{1,2\}, & g(x) > 0 \end{cases}$$

is a 2-rainbow dominating function of G, which immediately implies

$$\gamma_{r2}(G) \leq \sum_{u \in V(G)} |f(u)| \leq 2 \sum_{u \in V(G)} g(u) = 2\gamma_r(G). \tag{1}$$

Now, let $\gamma_{r2}(G) = 2\gamma_r(G)$, which implies that equality holds throughout (1). This implies that there is no vertex x of G with $g(x) = 2$. Let $V_1 = \{v_1, \ldots, v_k\}$ be the set of vertices x of G with $g(x) = 1$. For $i \in [k]$, let $P_i = \{u \in V(G) \setminus V_1 : N_G(u) \cap V_1 = \{v_i\}\}$, that is, for $u \in P_i$, the only neighbor v of u with $g(v) \geq 1$ is v_i. Therefore, the set $\{x \in V(G) : g_{v_i \to u}(x) \geq 1\}$ is dominating, which implies that P_i is complete. If $P_i = \emptyset$

for some $i \in [k]$, then $f' : V(G) \to 2^{[2]}$ with

$$f'(x) = \begin{cases} \emptyset, & g(x) = 0, \\ \{1,2\}, & x \in V_1 \setminus \{v_i\}, \text{ and} \\ \{1\}, & x = v_i \end{cases}$$

is a 2-rainbow dominating function of G of weight less than $2\gamma_r(G)$, which is a contradiction. Hence, for every $i \in [k]$, the set P_i is non-empty and complete.

For $u \in V(G) \setminus (V_1 \cup P_1 \cup \cdots \cup P_k)$, let $i(u)$ be the smallest integer in $[k]$ such that $v_{i(u)}$ is a neighbor of u and the set $\{x \in V(G) : g_{v_{i(u)} \to u}(x) \geq 1\}$ is dominating. Note that $i(u)$ is well-defined, because g is a weak Roman dominating function. For $i \in [k]$, let $Q_i = \{u \in V(G) \setminus (V_1 \cup P_1 \cup \cdots \cup P_k) : i(u) = i\}$. Since for every $u \in Q_i$, the set $\{x \in V(G) : g_{v_i \to u}(x) \geq 1\}$ is dominating, we obtain that every vertex in Q_i is adjacent to every vertex in $\{v_i\} \cup P_i$, which completes the proof. □

Corollary 2 *Let G be a connected $\{K_4, K_4 - e\}$-free graph.*
$\gamma_{r2}(G) = 2\gamma_r(G)$ if and only if

- *either G is K_2,*
- *or G arises by adding a matching containing two edges between two disjoint triangles,*
- *or G arises from the disjoint union of $k = \gamma_r(G)$ triangles*

$$v_1w_1u_1v_1,\ v_2w_2u_2v_2,\ \ldots,\ v_kw_ku_kv_k$$

by adding edges between the vertices in $\{v_1, \ldots, v_k\}$.

Proof: Since the sufficiency is straightforward, we only prove the necessity. Therefore, let G be a connected $\{K_4, K_4 - e\}$-free graph with $\gamma_{r2}(G) = 2\gamma_r(G)$. Let $g : V(G) \to \{0,1,2\}$ be a minimum weak Roman dominating function of G, and let $V_1, P_1, \ldots, P_k, Q_1, \ldots, Q_k$ be as in Theorem 1, that is, $k = \gamma_r(G)$. Since G is $\{K_4, K_4 - e\}$-free, we have $|Q_i| \leq 1$ and $|P_i| + |Q_i| \leq 2$ for every $i \in [k]$. This implies that G has a spanning subgraph H that is the union of ℓ triangles $v_1w_1u_1v_1$, $v_2w_2u_2v_2$, $\ldots$, $v_\ell w_\ell u_\ell v_\ell$ for some $\ell \leq k$, and $k-\ell$ complete graphs of order two $v_{\ell+1}u_{\ell+1}$, $v_{\ell+2}u_{\ell+2}$, $\ldots$, v_ku_k.

If a vertex u' in some component v_iu_i of H with $\ell+1 \leq i \leq k$ has a neighbor v' in some other component K of H, then $f : V(G) \to 2^{[2]}$ with

$$f(x) = \begin{cases} \{1,2\}, & x \in \{v_1,\ldots,v_k\} \setminus (\{v_i,u_i\} \cup V(K)), \\ \{1,2\}, & x = v', \\ \{1\}, & x \in \{v_i,u_i\} \setminus \{u'\}, \text{ and} \\ \emptyset, & \text{otherwise} \end{cases}$$

is a 2-rainbow dominating function of G of weight less than $2\gamma_r(G)$, which is a contradiction. Since G is connected, this implies that G is either K_2 or $\ell = k$. Hence, we may assume that $\ell = k$, that is, H is the union of k triangles.

If there are two edges $v'u'$ and $v''u''$ such that $u', u'' \in V(K)$ with $u' \neq u''$, $v' \in V(K')$, and $v'' \in V(K'')$ for three distinct components K, K', and K'' of H, then $f : V(G) \to 2^{[2]}$ with

$$f(x) = \begin{cases} \{1,2\}, & x \in \{v_1,\ldots,v_k\} \setminus (V(K) \cup V(K') \cup V(K'')), \\ \{1,2\}, & x \in \{v',v''\}, \\ \{1\}, & x \in V(K) \setminus \{u',u''\}, \text{ and} \\ \emptyset, & \text{otherwise} \end{cases}$$

is a 2-rainbow dominating function of G of weight less than $2\gamma_r(G)$, which is a contradiction. Hence, such a pair of edges does not exist. In view of the desired statement, we may now assume that there is some component K of H such that two vertices in K have neighbors in other components of H. By the previous observation and since G is connected, this implies that $k = 2$, and that G arises by adding a matching containing two or three edges between two disjoint triangles. If G arises by adding a matching containing three edges between two disjoint triangles, then $\gamma_{r2}(G) = 3 < 2\gamma_r(G)$, which is a contradiction. Hence, G arises by adding a matching containing two edges between two disjoint triangles, which completes the proof. □

The last result immediately implies the following.

Corollary 3 *Let G be a triangle-free graph.*
$\gamma_{r2}(G) = 2\gamma_r(G)$ *if and only if G is the disjoint union of copies of K_2.*

Theorem 4 *It is NP-hard to decide $\gamma_{r2}(G) = 2\gamma_r(G)$ for a given K_5-free graph G.*

Proof: We describe a reduction from 3SAT. Therefore, let F be a 3SAT instance with m clauses $C_1, \ldots, C_m$ over n boolean variables $x_1, \ldots, x_n$. Clearly, we may assume that $m \geq 2$. We will construct a K_5-free graph G whose order is polynomially bounded in terms of n and m such that F is satisfiable if and only if $\gamma_{r2}(G) = 2\gamma_r(G)$. For every variable x_i, create a copy $G(x_i)$ of K_4 and denote two distinct vertices of $G(x_i)$ by x_i and $\bar{x}_i$. For every clause C_j, create a vertex c_j. For every literal $x \in \{x_1, \ldots, x_n\} \cup \{\bar{x}_1, \ldots, \bar{x}_k\}$ and every clause C_j such that x appears in C_j, add the edge xc_j. Finally, add two further vertices a and b, the edge ab, and all possible edges between $\{a, b\}$ and $\{c_1, \ldots, c_m\}$. This completes the construction of G. Clearly, G is K_5-free and has order $4n + m + 2$.

Let f be a 2-rainbow dominating function of G.

Clearly, $\sum_{u \in \{a,b\} \cup \{c_1, \ldots, c_m\}} |f(u)| \geq 2$, and $\sum_{u \in V(G_i)} |f(u)| \geq 2$ for every $i \in [n]$, which implies $\gamma_{r2}(G) \geq 2n + 2$. Since

$$x \mapsto \begin{cases} \{1,2\}, & x \in \{a, x_1, \ldots, x_n\} \text{ and} \\ \emptyset, & \text{otherwise} \end{cases}$$

defines a 2-rainbow dominating function of weight $2n+2$, we obtain $\gamma_{r2}(G) = 2n + 2$. By Theorem 1, we have $\gamma_r(G) \geq n + 1$. In remains to show that F is satisfiable if and only if $\gamma_r(G) = n + 1$.

Let $\gamma_r(G) = n+1$. Let g be a minimum weak Roman dominating function of G. By Theorem 1, there is no vertex x of G with $g(x) = 2$. Let V_1 be the set of vertices x of G with $g(x) = 1$. Since, $\sum_{u \in \{a,b\} \cup \{c_1, \ldots, c_m\}} g(u) \geq 1$, and $\sum_{u \in V(G_i)} g(u) \geq 1$ for every $i \in [n]$, we obtain that $\{a, b\} \cup \{c_1, \ldots, c_m\}$ contains exactly one vertex, say y_0, from V_1, and that $V(G_i)$ contains exactly one vertex, say y_i, from V_1 for every $i \in [n]$. Since $m \geq 2$, we may assume, by symmetry, that $g(c_1) = 0$. If no neighbor v of c_1 with $g(v) \geq 1$ belongs to $\{a, b\} \cup \{c_1, \ldots, c_m\}$, then g is not a weak Roman dominating function. Hence $y_0 \in \{a, b\}$, and y_0 is the only neighbor of c_1 with positive g-value such that the set $\{x \in V(G) : g_{y_0 \to c_1}(x) \geq 1\}$ is dominating, which implies that for every $\ell \in [m] \setminus \{1\}$, the vertex c_ℓ is adjacent to a vertex in $\{y_1, \ldots, y_k\}$. Since $y_0 \in \{a, b\}$ and $m \geq 2$, this actually implies, by symmetry, that for every $\ell \in [m]$, the vertex c_ℓ is adjacent to a vertex in $\{y_1, \ldots, y_k\}$, that is, the intersection of $\{y_1, \ldots, y_k\}$ with $\{x_1, \ldots, x_n\} \cup \{\bar{x}_1, \ldots, \bar{x}_k\}$ indicates a satisfying truth assignment for F.

Conversely, if F has a satisfying truth assignment, then

$$x \mapsto \begin{cases} 1, & x = a, \\ 1, & x \in \{x_1, \ldots, x_n\} \cup \{\bar{x}_1, \ldots, \bar{x}_k\} \text{ and } x \text{ is true, and} \\ 0, & \text{otherwise} \end{cases}$$

defines a weak Roman dominating function of G of weight $n + 1$, which implies $\gamma_r(G) = n + 1$, and completes the proof. □

For a positive integer k, let

$$\mathcal{G}_k = \{G : \forall H \subseteq_{\text{ind}} G : \gamma_r(H) \geq k \Rightarrow \gamma_{r2}(H) = 2\gamma_r(H)\},$$

where $H \subseteq_{\text{ind}} G$ means that H is an induced subgraph of G. Since $\gamma_{r2}(K_1) = 1 = \gamma_r(K_1)$, the set $\mathcal{G}_1$ contains no graph of positive order. Since $\gamma_{r2}(\bar{K}_2) = 2 = \gamma_r(\bar{K}_2)$, where $\bar{H}$ denotes the complement of some graph H, the set $\mathcal{G}_2$ consists exactly of all complete graphs. The smallest value for k that leads to an interesting class of graphs is 3.

Theorem 5 $\mathcal{G}_3 = \text{Free}(\{\bar{K}_3, C_5\})$.

Proof: Since $\gamma_{r2}(\bar{K}_3) = 3 = \gamma_r(\bar{K}_3)$ and $\gamma_{r2}(C_5) = 3 = \gamma_r(C_5)$, it follows easily that $\bar{K}_3$ and C_5 are minimal forbidden induced subgraphs for $\mathcal{G}_3$. Now, let G be a minimal forbidden induced subgraphs for $\mathcal{G}_3$, which implies that $\gamma_r(G) \geq 3$ and $\gamma_{r2}(H) \neq 2\gamma_r(H)$. It remains to show that G is either $\bar{K}_3$ or C_5. For a contradiction, we assume that G is neither $\bar{K}_3$ nor C_5. Since G is a minimal forbidden induced subgraph, this implies that G is $\{\bar{K}_3, C_5\}$-free. Since $\gamma_r(G) \geq 3$, the graph G is not complete. Let u and v be two non-adjacent vertices of G. Since G is $\bar{K}_3$-free, we have $V(G) = \{u, v\} \cup N_u \cup N_v \cup N_{u,v}$, where $N_u = N_G(u) \setminus N_G(v)$, $N_v = N_G(v) \setminus N_G(u)$, and $N_{u,v} = N_G(u) \cap N_G(v)$. Since G is $\bar{K}_3$-free, the sets N_u and N_v are complete. If for every vertex w in $N_{u,v}$, we have $N_u \subseteq N_G(w)$ or $N_v \subseteq N_G(w)$, then

$$x \mapsto \begin{cases} 1, & x \in \{u, v\}, \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

defines a weak Roman dominating function of G of weight 2, which implies the contradiction $\gamma_r(G) < 3$. Hence, there are vertices $w_u \in N_u$, $w_v \in N_v$, and $w_{u,v} \in N_{u,v}$ such that $w_{u,v}$ is adjacent to neither w_u nor w_v. Since G is $\bar{K}_3$-free, this implies that w_u is adjacent to w_v, and $uw_uw_vvw_{u,v}u$ is an induced C_5 in G, which is a contradiction and completes the proof. □

Our results motivate several questions. Do the graphs G with $\gamma_{r2}(G) = 2\gamma_r(G)$ that are either K_4-free or $(K_4 - e)$-free have a simple structure? Can they at least be recognized efficiently? Can Theorem 4 be strengthened by restricting the input graphs even further? What are the minimal forbidden induced subgraphs for the classes $\mathcal{G}_k$ where $k \geq 4$?

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