The existence of regular and quasi-regular bipartite self-complementary 3-uniform hypergraphs

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Abstract

A hypergraph H with vertex set V and edge set E is called bipartite if V can be partitioned into two subsets V_1 and V_2 such that $e \cap V_1 \neq \phi$ and $e \cap V_2 \neq \phi$ for any $e \in E$. A bipartite self-complementary 3-uniform hypergraph H with partition (V_1, V_2) of a vertex set V such that $|V_1| = m$ and $|V_2| = n$ exists if and only if either (i) $m = n$ or (ii) $m \neq n$ and either m or n is congruent to 0 modulo 4 or (iii) $m \neq n$ and both m and n are congruent to 1 or 2 modulo 4.
In this paper we prove that, there exists a regular bipartite self-complementary 3-uniform hypergraph $H(V_1, V_2)$ with $|V_1| = m, |V_2| = n, m + n > 3$ if and only if $m = n$ and n is congruent to 0 or 1 modulo 4. Further we prove that, there exists a quasi-regular bipartite self-complementary 3-uniform hypergraph $H(V_1, V_2)$ with $|V_1| = m$, $|V_2| = n$, $m + n > 3$ if and only if either $m = 3, n = 4$ or $m = n$ and n is congruent to 2 or 3 modulo 4.

Keywords: bipartite hypergraph, bipartite self-complementary 3-uniform hypergraph, regular hypergraph, quasi-regular hypergraph

1. Introduction

A. Symański, A. P. Wojda ([7],[8],[9]) and S. Gosselin [2], independently characterized n and k for which there exist k-uniform self-complementary hypergraphs of order n and gave the structure of corresponding complementing permutations. P. Potočnik and M. Šajana [6] proved the existence

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of regular self-complementary 3-uniform hypergraphs. S. Gosselin [3] has characterized all n and k for which there exists a regular k-uniform self-complementary hypergraph of order n. T. Gangopadhyay and S. P. Rao Hebbare [1] studied bipartite self-complementary graphs. They characterized structural properties of r-partite complementing permutations.

In [5] a bipartite self-complementary 3-uniform hypergraph H with partition (V_1, V_2) of a vertex set V such that $|V_1| = m$ and $|V_2| = n$ is defined. And characterized m and n for a bipartite 3-uniform hypergraph $H^3(V_1, V_2)$ to be self-complementary. Structure of complementing permutation of bipartite self-complementary 3-uniform hypergraphs is also analyzed.

In this paper we find conditions on m and n for a bipartite self-complementary 3-uniform hypergraph $H^3(V_1, V_2)$ to be regular and quasi-regular.

2. Preliminary definitions and results

Definition 2.1. **(Bipartite Hypergraph)** *A hypergraph H with vertex set V and edge set E is called bipartite if V can be partitioned into two subsets V_1 and V_2 such that $e \cap V_1 \neq \phi$ and $e \cap V_2 \neq \phi$ for any $e \in E$.*

Furthermore if $|e| = k$ for every $e \in E$ then we call H, a bipartite k-uniform hypergraph, and denote it as $H^k(V_1, V_2)$. If $|V_1| = m$ and $|V_2| = n$ then $H^k(V_1, V_2) = H^k_{(m,n)}$.

If $H^3(V_1, V_2)$ is a bipartite 3-uniform hypergraph then every edge of $H^3(V_1, V_2)$ contains one vertex from one part and two vertices from the other part of the partition V_1 and V_2 of V. Thus any triple of vertices $\{x, y, z\}$ such that x, y, z belong to a single part of the partition of V is not an edge of $H^3(V_1, V_2)$.

Definition 2.2. **(Complete Bipartite 3-uniform Hypergraph)** *A 3-uniform hypergraph H with the vertex set $V = V_1 \cup V_2, V_1 \cap V_2 = \phi$ and the edge set $E = \{e : e \subset V,\ |e| = 3 \text{ and } e \cap V_i \neq \phi, \text{ for } i = 1, 2\}$ is called the complete bipartite 3-uniform hypergraph. It is denoted as $K^3(V_1, V_2)$ or $K^3_{(m,n)}$.*

Clearly, the total number of edges in $K^3_{(m,n)}$ is $m\binom{n}{2} + n\binom{m}{2} = \frac{mn(m+n-2)}{2}$.

Definition 2.3. **(Complement of bipartite 3-uniform hypergraph)** *Given a bipartite 3-uniform hypergraph $H = H^3(V_1, V_2)$, we define its bipartite complement to be the 3-uniform hypergraph $\bar{H} = \bar{H}^3(V_1, V_2)$ where $V(\bar{H}) = V(H)$ and $E(\bar{H}) = E(K^3(V_1, V_2)) - E(H)$.*

Definition 2.4. **(Bipartite self-complementary 3-uniform hypergraph)** *A bipartite 3-uniform hypergraph $H = H^3(V_1, V_2)$ is said to be self-complementary* *if it is isomorphic to its bipartite complement $\bar{H} = \bar{H}^3(V_1, V_2)$, that is there exists a bijection $\sigma : V \to V$ such that e is an edge in H if and only if $\sigma(e)$ is an edge in $\bar{H}$.*

That is there exists a bijection $\sigma : V \to V$ such that $e = \{x, y, z\}$ is an edge in H if and only if $\sigma(e) = \{\sigma(x), \sigma(y), \sigma(z)\}$ is an edge of $\bar{H}$. Such a σ is called as a complementing permutation.

Definition 2.5. (Regular hypergraph) *A hypergraph H is said to be regular if all vertices have the same degree.*

Definition 2.6. (Quasi-regular hypergraph) *A hypergraph H is said to be quasi-regular if the degree of each vertex is either r or $r-1$ for some positive integer r.*

Following theorem gives necessary and sufficient condition on the order of bipartite 3-uniform hypergraph $H^3_{(m,n)}$ to be self-complementary which is proved in [5].

Theorem 2.7. *There exists a bipartite self-complementary 3-uniform hypergraph $H^3_{(m,n)}$ if and only if*
either (i) $m = n$
or (ii) $m \neq n$ and either m or n is congruent to 0 modulo 4
or (iii) $m \neq n$ and both m and n are congruent to 1 or 2 modulo 4.

3. Existence of regular bipartite self-complementary 3-uniform hypergraphs

It is known that[6] a regular self-complementary 3-uniform hypergraph exists if and only if n is congruent to 1 or 2 modulo 4. In this section we find necessary and sufficient condition on order of bipartite self-complementary 3-uniform hypergraph to be regular.
Observe that for any $u \in V_1$ and $v \in V_2$, in $K^3(V_1, V_2) = K^3_{(m,n)}$, the degree of u is $n(m-1) + \binom{n}{2}$ and the degree of v is $m(n-1) + \binom{m}{2}$.
Following theorem gives necessary and sufficient condition on order of bipartite self-complementary 3-uniform hypergraph to be regular.

Theorem 3.1. *There exists a regular bipartite self-complementary 3-uniform hypergraph $H(V_1, V_2)$ with $|V_1| = m, |V_2| = n, m + n > 3$ if and only if $m = n$ and n is congruent to 0 or 1 modulo 4.*

Proof. Suppose there exists a regular bipartite self-complementary 3-uniform hypergraph $H(V_1, V_2)$ such that $|V_1| = m, |V_2| = n, m + n > 3$ with complementing permutation σ. Let r be its regular degree.

For any vertex $u \in V_1$,
$d_H(u) + d_H(\sigma(u)) = n(m-1) + \binom{n}{2}$. That is $r + r = n(m-1) + \binom{n}{2}$. That is $2r = \frac{2n(m-1)+n(n-1)}{2}$
Similarly, for $v \in V_2$,
$d_H(v) + d_H(\sigma(v)) = m(n-1) + \binom{m}{2}$. That is $r + r = m(n-1) + \binom{m}{2}$. That is $2r = \frac{2m(n-1)+m(m-1)}{2}$.
Hence, $2n(m-1) + n(n-1) = 2m(n-1) + m(m-1)$. That is $m^2 - n^2 = 3(m-n)$. That is $(m-n)(m+n-3) = 0$. That is either $m - n = 0$ or $m + n - 3 = 0$ which is not possible as $m + n > 3$. Hence $m = n$. Further, $2r = \frac{3n(n-1)}{2}$ that is $r = \frac{3n(n-1)}{4}$.
Since r is an integer we must have either 4 divides n or 4 divides $n-1$. Hence n is congruent to 0 or 1 modulo 4.

Conversely, suppose $m = n$ is congruent to 0 or 1 modulo 4. We prove there exists a regular bipartite self-complementary 3-uniform hypergraph $H(V_1, V_2)$ such that $|V_1| = |V_2| = n$. To prove this we construct a regular self-complementary bipartite 3-uniform hypergraph $H(V_1, V_2)$ such that $|V_1| = |V_2| = n$.

Case i) Suppose n is congruent to 0 modulo 4. Therefore $n = 4k$ for some positive integer k.
Let $V_1 = A_0 \cup A_1 \cup A_2 \cup A_3$ where $A_i = \{u_j^i | j \in \mathbb{Z}_k\}$ for all $i \in \mathbb{Z}_4$ and $V_2 = B_0 \cup B_1 \cup B_2 \cup B_3$ where $B_i = \{v_j^i | j \in \mathbb{Z}_k\}$ for all $i \in \mathbb{Z}_4$.
We construct the quasi-regular self-complementary graphs G_1 and G_2 with vertex sets V_1 and V_2 respectively as follows.
For pairwise distinct $i, i' \in \mathbb{Z}_4$ define the following subsets of $V_1^{(2)}$ where $V_1^{(2)}$ denotes the set of all 2-subsets of V_1.

$$E_i = A_i^{(2)}$$
$$E_{(i,i')} = \{\{u_{j_1}^i, u_{j_2}^{i'}\} : j_1, j_2 \in \mathbb{Z}_k\}$$
$$E_{G_1} = \bigcup_{i=0,1} (E_i) \cup E_{(0,3)} \cup E_{(2,3)} \cup E_{(1,2)}$$

Let G_1 be a graph with vertex set V_1 and edge set E_{G_1} as defined above having $n = 4k$ vertices.
First we show that G_1 is quasi-regular. Take any vertex u_j^i. Then, for fixed i, the vertex u_j^i lies in $k-1$ subsets of E_i and k subsets of $E_{(i,i')}$. Hence, for every vertex u_j^i in G_1 with $i \in \{0,1\}$, we have

$$\deg(u_j^i) = k - 1 + k = 2k - 1,$$

and for every vertex u_j^i in G_1 with $i \in \{2,3\}$, we have

$$\deg(u_j^i) = k + k = 2k.$$

Therefore there are $2k$ vertices having degree $2k-1$ and $2k$ vertices of degree $2k$. We conclude that G_1 is quasi-regular.

To prove G_1 is self-complementary we define a bijection $\sigma : V \to V$ by $\sigma(u_j^0) = u_j^3, \sigma(u_j^1) = u_j^2, \sigma(u_j^2) = u_j^0$, and $\sigma(u_j^3) = u_j^1$, for all $j \in \mathbb{Z}_k$. Similarly, we construct a quasi-regular self-complementary graph G_2 with vertex set V_2 wth complementing permutation $\sigma_2 = \prod_{j=1}^{k}(v_j^0\ v_j^3\ v_j^1\ v_j^2)$. Observe that, $d_{G_1}(u_j^0) = d_{G_1}(u_j^1) = d_{G_2}(v_j^0) = d_{G_2}(v_j^1) = 2k-1$ and $d_{G_1}(u_j^2) = d_{G_1}(u_j^3) = d_{G_2}(v_j^2) = d_{G_2}(v_j^3) = 2k$ for $j \in \mathbb{Z}_k$.

Consider a bipartite 3-uniform hypergraph $H_1(V_1, V_2)$ with edge set
$E_1 = \{e \cup \{v_j^i\} | i \in \mathbb{Z}_4,\ j \in \mathbb{Z}_k$, and e is an edge in $G_1\}$
$\cup \{e' \cup \{u_j^i\} | i \in \mathbb{Z}_4,\ j \in \mathbb{Z}_k$, and e' is an edge in $G_2\}$.
Clearly, H_1 is self-complementary with complementing permutation $\sigma = \sigma_1\sigma_2$.
Observe that, for $i = 0, 1$ and for $j \in \mathbb{Z}_k$,
$d_{H_1}(u_j^i) = 4k(2k-1) + k(4k-1) = 8k^2 - 4k + 4k^2 - k = 12k^2 - 5k$
$d_{H_2}(v_j^i) = 4k(2k-1)+k(4k-1) = 8k^2-4k+4k^2-k = 12k^2-5k$. Similarly, for $i = 2, 3$ and for $j \in \mathbb{Z}_k$,
$d_{H_1}(u_j^i) = 4k(2k) + k(4k-1) = 8k^2 + 4k^2 - k = 12k^2 - k$
$d_{H_1}(v_j^i) = 4k(2k) + k(4k-1) = 8k^2 + 4k^2 - k = 12k^2 - k$.

To obtain a regular bipartite self-complementary 3-uniform hypergraph we reduce the degrees of u_j^i and v_j^i for $i = 2, 3$ and $j \in \mathbb{Z}_k$ by $2k$ and increase degrees of u_j^i and v_j^i for $i = 0, 1$ and $j \in \mathbb{Z}_k$ by $2k$ by using the process of edge exchange as explained below.

We have, for each $j \in \mathbb{Z}_k$, $\{v_j^0, v_j^1\}$ is not an edge in G_2 and hence $\{v_j^0, v_j^1, u_{j'}^0\}$ is not an edge in H_1 for each $j' \in \mathbb{Z}_k$. Therefore $\sigma\{v_j^0, v_j^1, u_{j'}^0\}$ is an edge in H_1, $\sigma^2\{v_j^0, v_j^1, u_{j'}^0\}$ is not an edge in H_1 and $\sigma^3\{v_j^0, v_j^1, u_{j'}^0\}$ is an edge in H_1.

We exchange the edges $\sigma\{v_j^0, v_j^1, u_{j'}^0\}$ and $\sigma^3\{v_j^0, v_j^1, u_{j'}^0\}$ by the nonedges $\sigma^2\{v_j^0, v_j^1, u_{j'}^0\}$ and $\sigma^4\{v_j^0, v_j^1, u_{j'}^0\} = \{v_j^0, v_j^1, u_{j'}^0\}$ respectively. That is we exchange the edge $\{v_j^3, v_j^2, u_{j'}^3\}$ with the edge $\{v_j^1, v_j^0, u_{j'}^1\}$ and the edge $\{v_j^2, v_j^3, u_{j'}^2\}$ with the edge $\{v_j^0, v_j^1, u_{j'}^0\}$ for $j, j' \in \mathbb{Z}_k$. Similarly, we exchange the edge $\{v_j^2, v_j^1, u_{j'}^3\}$ with $\{v_j^0, v_j^2, u_{j'}^1\}$ and exchange the edge $\{v_j^3, v_j^0, u_{j'}^2\}$ with $\{v_j^1, v_j^3, u_{j'}^3\}$ for $j, j' \in \mathbb{Z}_k$. Call this new hypergraph as H.

In this process of edge exchange, for fixed j and j', the degrees of v_j^0, v_j^1, $u_{j'}^0$ and $u_{j'}^1$ are increased by 2 while the degrees of v_j^2, v_j^3, $u_{j'}^2$ and $u_{j'}^3$ are decreased by 2.

Thus after k such exchanges, degrees of v_j^0, v_j^1, $u_{j'}^0$ and $u_{j'}^1$ are increased by $2k$ and the degrees of v_j^2, v_j^3, $u_{j'}^2$ and $u_{j'}^3$ are decreased by $2k$ for $j, j' \in \mathbb{Z}_k$.

Thus, for $i = 0, 1, 2, 3$ and for $j \in \mathbb{Z}_k$,
$d_H(u_j^i) = d_H(v_j^i) = 12k^2 - 3k = 3k(4k-1)$.

Clearly H is bipartite self-complementary 3-uniform hypergraph with complementing permutation $\sigma = \sigma_1\sigma_2 = \prod_{j=1}^{k} (u_j^0\ u_j^3\ u_j^1\ u_j^2)(v_j^0\ v_j^3\ v_j^1\ v_j^2)$ and is $3k(4k-1)$ regular.

Case ii) Suppose n is congruent to 1 modulo 4. That is $n = 4k+1$ for some positive integer k.

Let $V_1 = \{u_1, u_2, ..., u_{4k+1}\}$ and $V_2 = \{v_1, v_2, ..., v_{4k+1}\}$. Let G_1 and G_2 be any regular self-complementary graphs with vertex set V_1 and V_2 respectively. Let σ_1 and σ_2 be complementing permutation of G_1 and G_2 respectively. Observe further that degree of every vertex in G_1 and G_2 is $2k$.

Let H be the 3-uniform hypergraph with the vertex set V and the edge set $E = \{e \cup \{v_i\}, i = 1, 2, ..., n \mid e \text{ is an edge in } G_1\}$

$\bigcup \{e' \cup \{u_i\}, i = 1, 2, ..., n \mid e' \text{ is an edge in } G_2\}$

For any $u_i \in V_1$. The degree of u_i in graph G_1 is $2k$, since G_1 is a regular self-complementary graph on $4k+1$ vertices having $k(4k+1)$ edges. Therefore the degree of u_i in H is $2k(4k+1) + k(4k+1) = 12k^2 + 3k$. Similarly, for any $v_i \in V_2$ the degree of v_i in H is $2k(4k+1) + k(4k+1) = 12k^2 + 3k$. Hence H is a regular bipartite 3-uniform hypergraph.

It can be easily checked that H is self-complementary with complementing permutation $\sigma = \sigma_1\sigma_2$. Therefore H is a regular bipartite self-complementary 3-uniform hypergraph. □

4. Existence of a quasi-regular bipartite self-complementary 3-uniform hypergraph

In [4] following result about existence of quasi-regular self-complementary 3-uniform hypergraph is proved.

Theorem 4.1. *There exists a quasi-regular self-complementary 3-uniform hypergraph of order n if and only if $n \geq 4$ and $n \equiv 0 \pmod 4$.*

The following theorem gives necessary and sufficient conditions on the order of a bipartite self-complementary 3-uniform hypergraph to be quasi-regular.

Theorem 4.2. *There exists a quasi-regular bipartite self-comple- mentary 3-uniform hypergraph $H(V_1, V_2)$ with $|V_1| = m$, $|V_2| = n$, $m + n > 3$ if and only if either $m = 3, n = 4$ or $m = n$ and n is congruent to 2 or 3 modulo 4.*

Proof. Suppose that there exists a quasi-regular bipartite self-complementary 3-uniform hypergraph $H(V_1, V_2)$ with $|V_1| = m$, $|V_2| = n$. Let r and $r-1$ be degrees of vertices of H. Let σ be a complementing permutation of $H(V_1, V_2)$.

For any $u \in V_1$ and for any $v \in V_2$, we have

$$d_H(u) + d_H(\sigma(u)) = n(m-1) + \binom{n}{2} \qquad (1)$$

$$d_H(v) + d_H(\sigma(v)) = m(n-1) + \binom{m}{2} \tag{2}$$

As H is quasi-regular, there exist $u, v \in V = V_1 \cup V_2$ such that $d_H(u) = r$ and $d_H(v) = r - 1$.
Note that $d_H(\sigma(u))$ and $d_H(\sigma(v))$ is either r or $r-1$. As u is in either V_1 or V_2, $d_H(u)$ and $d_H(\sigma(u))$ satisfy either equation 1 or 2. Similarly, $d_H(v)$ and $d_H(\sigma(v))$ satisfy either equation 1 or 2. That is

$$d_H(u) + d_H(\sigma(u)) = n(m-1) + \binom{n}{2} \text{ or } m(n-1) + \binom{m}{2} \tag{3}$$

$$d_H(v) + d_H(\sigma(v)) = n(m-1) + \binom{n}{2} \text{ or } m(n-1) + \binom{m}{2} \tag{4}$$

By considering all possible values of $d_H(\sigma(u))$ and $d_H(\sigma(v))$ we have the following cases.

Case i) If $d_H(\sigma(u)) = r - 1$ and $d_H(\sigma(v)) = r$. From equations 3 and 4 we get that
$r + r - 1 = n(m-1) + \binom{n}{2}$ *or* $m(n-1) + \binom{m}{2}$ and
$r - 1 + r = n(m-1) + \binom{n}{2}$ *or* $m(n-1) + \binom{m}{2}$.
This implies that $n(m-1) + \binom{n}{2} = m(n-1) + \binom{m}{2}$. Solving this equation we get, $(m-n)(m+n-3) = 0$. As $m + n > 3$, we get, $m = n$. Hence $2r - 1 = n(n-1) + \binom{n}{2}$ that is $r = \frac{3n(n-1)+2}{4}$.
Since r is an integer we must have, $3n(n-1) + 2 \equiv 0 \pmod 4$. That is $3n(n-1) \equiv -2 \pmod 4$. That is $n(n-1) \equiv 2 \pmod 4$. That is $n^2 - n - 2$ is a multiple of 4. That is $(n-2)(n+1)$ is a multiple of 4. Since both $(n-2)$ and $(n+1)$ cannot be even simultaneously, either $(n-2)$ or $(n+1)$ must be a multiple of 4. That is either $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$.

Case ii) If $d_H(\sigma(u)) = r$ and $d_H(\sigma(v)) = r$ then from equations 3 and 4 we get that
$r + r = n(m-1) + \binom{n}{2}$ *or* $m(n-1) + \binom{m}{2}$ and
$r - 1 + r = n(m-1) + \binom{n}{2}$ *or* $m(n-1) + \binom{m}{2}$.
That is, $\left|(n(m-1) + \binom{n}{2}) - (m(n-1) + \binom{m}{2})\right| = 1$. Solving this absolute value equation we get that either $m = 3, n = 1$ (or vice versa) or $m = 3, n = 2$ (or vice versa). For both these pairs of values of m and n there clearly does not exist a bipartite self-complementary 3-uniform hypergraph 3-uniform hypergraph.

Case iii) If $d_H(\sigma(u)) = r$ and $d_H(\sigma(v)) = r - 1$, then from equations 3 and 4 we get that,

$r+r=n(m-1)+\binom{n}{2}$ *or* $m(n-1)+\binom{m}{2}$ and
$r-1+r-1=n(m-1)+\binom{n}{2}$ *or* $m(n-1)+\binom{m}{2}$.
That is, $\left|\left(n(m-1)+\binom{n}{2}\right)-\left(m(n-1)+\binom{m}{2}\right)\right|=2$. Solving this absolute value equation we get that $m=4, n=3$.

Case iv) If $d_H(\sigma(u))=r-1$ and $d_H(\sigma(v))=r-1$. Then this case is same as Case ii).
From all the above cases we conclude the following.
If there exists a quasi-regular bipartite self-complementary 3-uniform hypergraph $H(V_1,V_2)$ with $|V_1|=m, |V_2|=n, m+n>3$ then either $m=3, n=4$ or $m=n$ and n is congruent to 2 or 3 modulo 4.

We prove the converse by constructing a quasi-regular bipartite self-complementary 3-uniform hypergraph 3-uniform hypergraph for each possible pair (m,n).

Case i) Let $m=3$ and $n=4$. Consider a hypergraph H with vertex set $V=V_1\cup V_2$ where $V_1=\{u_1,u_2,u_3\}$, $V_2=\{v_1,v_2,v_3,v_4\}$ and edge set
$E=\{\{v_1,v_2,u_1\},\{v_1,v_2,u_2\},\{v_1,v_2,u_3\},\{v_1,v_3,u_1\},\{v_1,v_3,u_2\},$
$\{v_1,v_3,u_3\},\{v_3,v_4,u_1\},\{v_3,v_4,u_2\},\{v_3,v_4,u_3\},\{u_1,u_2,v_2\},$
$\{u_1,u_2,v_4\},\{u_1,u_3,v_2\},\{u_1,u_3,v_4\},\{u_2,u_3,v_2\},\{u_2,u_3,v_4\}\}$
Observe that, $d_H(v_1)=d_H(v_2)=d_H(v_3)=d_H(v_4)=6$ and $d_H(u_1)=d_H(u_2)=d_H(u_3)=7$. Hence H is quasi-regular. To prove that H is self-complementary, define a bijection $\sigma: V(H)\rightarrow V(H)$ as $\sigma=(u_1\ u_2\ u_3)(v_1\ v_2\ v_3$ ...
σ is a complementing permutation of H. Hence H is quasi-regular bipartite self-complementary 3-uniform hypergraph.

Case ii) Suppose $m=n$ and n is congruent to 2 modulo 4, that is $n=4k+2$ for some positive integer k.

Let $V_1=\{u_1,u_2,...,u_{4k+1},x\}$. Let G be a regular self-complementary graph with vertex set $\{u_1,u_2,...,u_{4k+1}\}$. Let $\bar{G}$ be its complement. Denote the vertices $u_1,u_2,...,u_{4k+1}$ of $\bar{G}$ by $v_1,v_2,...,v_{4k+1}$ respectively. Observe that both G and $\bar{G}$ have $k(4k+1)$ edges such that degree of each vertex is $2k$ and $\{u_i,u_j\}$ is an edge in G if and only if $\{v_i,v_j\}$ is not an edge in $\bar{G}$.

Let $V_2=\{v_1,v_2,...,v_{4k+1},y\}$. For $l,s=1,2,...4k+1$, consider the following partition of the edge set of $K^3_{(m,n)}$.
$E_1=\{e\cup\{v\}, v\in V_2 \mid e \text{ is an edge in } G\}$
$\bar{E}_1=\{e\cup\{v\}, v\in V_2 \mid e \text{ is not an edge in } G\}$
$E_2=\{e'\cup\{u\}, u\in V_1 \mid e' \text{ is an edge in } \bar{G}\}$
$\bar{E}_2=\{e'\cup\{u\}, u\in V_1 \mid e' \text{ is not an edge in } \bar{G}\}$
$E_x=\{\{x,u_l,v_s\}\mid \text{both } l \text{ and } s \text{ are either odd or even}\}$
$\bar{E}_x=\{\{x,u_l,v_s\}\mid \text{exactly one of } l \text{ and } s \text{ is odd}\}$
$E_y=\{\{y,u_l,v_s\}\mid \text{exactly one of } l \text{ and } s \text{ is odd}\}$

$\bar{E}_y = \{\{y, u_l, v_s\} |$ both l and s are either odd or even $\}$
$E_{(x,y)} = \{\{x, y, u_l\} | \ l$ is even $\} \cup \{\{x, y, v_s\} | \ s$ is odd $\}$
$\bar{E}_{(x,y)} = \{\{x, y, u_l\} | \ l$ is odd $\} \cup \{\{x, y, v_s\} | \ s$ is even $\}$.

Let H be a 3-uniform hypergraph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup E_x \cup E_y \cup E_{(x,y)}$ so that $\bar{H}$ will have the edge set $\bar{E} = \bar{E}_1 \cup \bar{E}_2 \cup \bar{E}_x \cup \bar{E}_y \cup \bar{E}_{(x,y)}$. Clearly, H is a bipartite 3-uniform hypergraph.

To prove that H is self-complementary, we define a bijection $\sigma : V(H) \to V(H)$ as $\sigma = (\prod_{i=1}^{4k+1} (u_i \ v_i))(x \ y)$. σ is clearly a complementing permutation.

Finally, we show that H is quasi-regular by counting the degree of each vertex of H.

For any $u_i \in V_1$ where $i = 1, 3, ..., 4k+1$, u_i is in $2k(4k+2)$ triples of E_1, $k(4k+1)$ triples of E_2, $2k+1$ triples of E_x and $2k$ triples of E_y. Hence, $d_H(u_i) = 2k(4k+2) + k(4k+1) + 2k + 2k + 1 = 12k^2 + 9k + 1$.

For any $u_i \in V_1$ where $i = 2, 4, ..., 4k+2$, u_i is in $2k(4k+2)$ triples of E_1, $k(4k+1)$ triples of E_2, $2k$ triples of E_x and $2k$ triples of E_y and 1 triple of $E_{(x,y)}$. Hence, $d_H(u_i) = 2k(4k+2)+k(4k+1)+2k+1+2k+1 = 12k^2+9k+2$.

For any $v_i \in V_2$ where $i = 1, 3, ..., 4k+1$, v_i is in $k(4k+1)$ triples of E_1, $2k(4k+2)$ triples of E_2, $2k+1$ triples of E_x and $2k+1$ triples of E_y and 1 triple of $E_{(x,y)}$. Hence,
$d_H(v_i) = 2k(4k+2) + k(4k+1) + 2k + 1 + 2k + 1 = 12k^2 + 9k + 2$.

For any $v_i \in V_2$ where $i = 2, 4, ..., 4k+2$, v_i is in $k(4k+1)$ triples of E_1, $2k(4k+2)$ triples of E_2, $2k$ triples of E_x and $2k+1$ triples of E_y. Hence, $d_H(v_i) = k(4k+1) + 2k(4k+2) + 2k + 2k + 1 = 12k^2 + 9k + 1$.
Lastly, $d_H(x) = k(4k+1) + (2k+1)(2k+1) + 2k(2k) + 2k + 2k + 1 = 12k^2 + 9k + 2$
and $d_H(y) = k(4k+1)+2k(2k+1)+2k(2k+1)+2k+2k+1 = 12k^2+9k+1$.

Case (iii) Suppose $m = n$ and n is congruent to 3 modulo 4, that is $n = 4k+3$ for some positive integer k.

Let $V_1 = \{u_1, u_2, ..., u_{4k+1}, x_1, x_2\}$. Let G be a regular self-complementary graph with vertex set $\{u_1, u_2, ..., u_{4k+1}\}$. Let $\bar{G}$ be its complement. Denote the vertices $u_1, u_2, ..., u_{4k+1}$ of $\bar{G}$ as $v_1, v_2, ..., v_{4k+1}$ respectively. Observe that both G and $\bar{G}$ have $k(4k+1)$ edges such that degree of each vertex is $2k$ and $\{u_i, u_j\}$ is an edge in G if and only if $\{v_i, v_j\}$ is not an edge in $\bar{G}$.
Let $V_2 = \{v_1, v_2, ..., v_{4k+1}, y_1, y_2\}$.
For $l, s = 1, 2, ...4k+1$, consider the following partition of the edge set of $K^3_{(m,n)}$.
$E_1 = \{e \cup \{v\}, v \in V_2 \mid e$ is an edge in $G\}$

$\bar{E_1} = \{e \cup \{v\}, v \in V_2 \mid e \text{ is not an edge in } G\}$
$E_2 = \{e' \cup \{u\}, u \in V_1 \mid e' \text{ is an edge in } \bar{G}\}$
$\bar{E_2} = \{e' \cup \{u\}, u \in V_1 \mid e' \text{ is not an edge in } \bar{G}\}$
$E_{x_1} = \{\{x_1, u_l, v_s\} \mid \text{ both } l \text{ and } s \text{ are either odd or even }\}$,
$\bar{E}_{x_1} = \{\{x_1, u_l, v_s\} \mid \text{ exactly one of } l \text{ and } s \text{ is odd }\}$
$E_{x_2} = \{\{x_2, u_l, v_s\} \mid \text{ both } l \text{ and } s \text{ are either odd or even }\}$,
$\bar{E}_{x_2} = \{\{x_2, u_l, v_s\} \mid \text{ exactly one of } l \text{ and } s \text{ is odd }\}$
$E_{y_1} = \{\{y_1, u_l, v_s\} \mid \text{ exactly one of } l \text{ and } s \text{ is odd }\}$,
$\bar{E}_{y_1} = \{\{y_1, u_l, v_s\} \mid \text{ both } l \text{ and } s \text{ are either odd or even }\}$
$E_{y_2} = \{\{y_2, u_l, v_s\} \mid \text{ exactly one of } l \text{ and } s \text{ is odd }\}$,
$\bar{E}_{y_2} = \{\{y_2, u_l, v_s\} \mid \text{ both } l \text{ and } s \text{ are either odd or even }\}$
$E_{(x_1,y_1)} = \{\{x_1, y_1, u_l\} \mid l \text{ is even }\} \cup \{\{x_1, y_1, v_s\} \mid s \text{ is odd }\}$
$\bar{E}_{(x_1,y_1)} = \{\{x_1, y_1, v_s\} \mid s \text{ is even }\} \cup \{\{x_1, y_1, u_l\} \mid l \text{ is odd }\}$
$E_{(x_1,y_2)} = \{\{x_1, y_2, u_l\} \mid l \text{ is odd }\} \cup \{\{x_1, y_2, v_s\} \mid s \text{ is odd }\}$
$\bar{E}_{(x_1,y_2)} = \{\{x_2, y_1, v_s\} \mid s \text{ is odd }\} \cup \{\{x_2, y_1, u_l\} \mid l \text{ is odd }\}$
$E_{(x_2,y_1)} = \{\{x_2, y_1, u_l\} \mid l \text{ is even }\} \cup \{\{x_2, y_1, v_s\} \mid s \text{ is even }\}$
$\bar{E}_{(x_2,y_1)} = \{\{x_1, y_2, v_s\} \mid s \text{ is even }\} \cup \{\{x_1, y_2, u_l\} \mid l \text{ is even }\}$
$E_{(x_2,y_2)} = \{\{x_2, y_2, u_l\} \mid l \text{ is odd }\} \cup \{\{x_2, y_2, v_s\} \mid s \text{ is even }\}$
$\bar{E}_{(x_2,y_2)} = \{\{x_2, y_2, v_s\} \mid s \text{ is odd }\} \cup \{\{x_2, y_2, u_l\} \mid l \text{ is odd }\}$
$E_{(x_1,x_2)} = \{\{\{x_1, x_2, v_s\} \mid s \text{ is even }\} \cup \{x_1, x_2, y_1\}$
$\bar{E}_{(x_1,x_2)} = \{\{y_1, y_2, u_l\} \mid l \text{ is even }\} \cup \{y_1, y_2, x_1\}$
$E_{(y_1,y_2)} = \{\{y_1, y_2, u_l\} \mid l \text{ is odd }\} \cup \{y_1, y_2, x_2\}$
$\bar{E}_{(y_1,y_2)} = \{\{x_1, x_2, v_s\} \mid s \text{ is even }\} \cup \{x_1, x_2, y_2\}$

Let H be a 3-uniform hypergraph whose vertex set is $V = V_1 \cup V_2$ and edge set is

$$E = E_1 \cup E_2 \cup E_{x_1} \cup E_{x_2} \cup E_{y_1} \cup E_{y_2} \cup E_{(x_1,y_1)} \cup E_{(x_1,y_2)} \cup E_{(x_2,y_1)} \cup E_{(x_2,y_2)} \cup E_{(x_1,x_2)} \cup E_{(y_1,y_2)}$$

so that $\bar{H}$ have the edge set

$$\bar{E} = \bar{E}_1 \cup \bar{E}_2 \cup \bar{E}_{x_1} \cup \bar{E}_{x_2} \cup \bar{E}_{y_1} \cup \bar{E}_{y_2} \cup \bar{E}_{(x_1,y_1)} \cup \bar{E}_{(x_1,y_2)} \cup \bar{E}_{(x_2,y_1)} \cup \bar{E}_{(x_2,y_2)} \cup \bar{E}_{(x_1,x_2)} \cup \bar{E}_{(y_1,y_2)}$$

Clearly, H is bipartite 3-uniform hypergraph. To prove that H is self-complementary, we define a bijection $\sigma : V(H) \to V(H)$ as

$$\sigma = (\prod_{i=1}^{4k+1} (u_i\ v_i))(x_1\ y_1)(x_2\ y_2).$$

Finally we show that H is quasi-regular by counting the degree of each vertex of H.

For any $u_i \in V_1$ where $i = 1, 3, ..., 4k+1$, u_i is in $2k(4k+3)$ triples of E_1, $k(4k+1)$ triples of E_2, $2k+1$ triples of E_{x_1}, $2k+1$ triples of E_{x_2}, $2k$ triples of E_{y_1} and $2k$ triples of E_{y_2}, 1 triple of $E_{(x_1,y_2)}$, 1 triple of $E_{(x_2,y_2)}$, and 1 triple of $E_{(y_1,y_2)}$. Hence,

$d_H(u_i) = 2k(4k+3) + k(4k+1) + 2(2k+1) + 4k + 3 = 12k^2 + 15k + 5.$

For any $u_i \in V_1$ where $i = 2, 4, ..., 4k$, u_i is in $2k(4k+3)$ triples of E_1, $k(4k+1)$ triples of E_2, $2k$ triples of E_{x_1}, $2k$ triples of E_{x_2}, $2k+1$ triples of E_{y_1} and $2k+1$ triples of E_{y_2}, 1 triple of $E_{(x_1,y_1)}$, 1 triple of $E_{(x_2,y_1)}$. Hence,
$d_H(u_i) = 2k(4k+3) + k(4k+1) + 2(2k) + 2(2k+1) + 2 = 12k^2 + 15k + 4.$
Similarly, it can be checked that,
$d_H(v_i) = 12k^2 + 15k + 4$ for $i = 1, 3, ..., 4k+1$.
$d_H(v_i) = 12k^2 + 15k + 5$ for $i = 2, 4, ..., 4k$.
$d_H(x_1) = 12k^2 + 15k + 5.$
$d_H(x_2) = 12k^2 + 15k + 4.$
$d_H(y_1) = 12k^2 + 15k + 4.$
$d_H(y_2) = 12k^2 + 15k + 5.$
Hence H is quasi-regular. □

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