The log-convexity of a class of linear recurrence sequences

Rui-Li Liu
Department of Mathematics,
Shanghai University,
Shanghai 200444, China.
{E-mail: liuruili@i.shu.edu.cn}

Feng-Zhen Zhao[1]
Department of Mathematics,
Shanghai University,
Shanghai 200444, China.
{E-mail: fengzhenzhao@shu.edu.cn}

Abstract

In this paper, we consider the sequences $\{F(n,k)\}_{n\geq k}$ $(k \geq 1)$ defined by $F(n,k) = (n-2)F(n-1,k) + F(n-1,k-1), F(n,1) = \frac{n!}{2}, F(n,n) = 1$. We mainly study the log-convexity of $\{F(n,k)\}_{n\geq k}$ $(k \geq 1)$ when k is fixed. We prove that $\{F(n,3)\}_{n\geq 3}$, $\{F(n,4)\}_{n\geq 5}$, and $\{F(n,5)\}_{n\geq 6}$ are log-convex. In addition, we discuss the log-behavior of some sequences related to $F(n,k)$.

Key words. log-convexity, log-concavity, log-balancedness, ratio log-concavity.

[1]Corresponding author

1 Introduction

We first recall some definitions involved in this paper. For a sequence of positive real numbers $\{z_n\}_{n\geq 0}$, it is said to be *log-convex* (or *log-concave*) if $z_n^2 \leq z_{n-1}z_{n+1}$ (or $z_n^2 \geq z_{n-1}z_{n+1}$) for each $n \geq 1$. For a log-convex sequence $\{z_n\}_{n\geq 0}$, it is said to be *log-balanced* if $\{\frac{z_n}{n!}\}_{n\geq 0}$ is log-concave (Došlić [5] gave this definition). It is well known that $\{z_n\}_{n\geq 0}$ is log-convex (or log-concave) if and only if its quotient sequence $\{\frac{z_{n+1}}{z_n}\}_{n\geq 0}$ is nondecreasing (or nonincreasing) and a log-convex sequence $\{z_n\}_{n\geq 0}$ is log-balanced if and only if $\frac{(n+1)z_n}{z_{n-1}} \geq \frac{nz_{n+1}}{z_n}$ for each $n \geq 1$. It is evident that the quotient sequence of a log-balanced sequence does not grow too fast. A sequence $\{z_n\}_{n\geq 0}$ is said to be *ratio log-concave* (or *ratio log-convex*) if $\{z_{n+1}/z_n\}_{n\geq 0}$ is log-concave (or log-convex). See Chen, Guo, and Wang [3] for more details about ratio log-behavior of sequences.

Log-concavity and log-convexity play important roles in combinatorics, they are not only instrumental in obtaining the growth rate of a combinatorial sequence, but also fertile sources of inequalities. For applications of log-convexity and log-concavity in other subjects, see [1, 2, 6, 7, 9–11]. Hence the log-convexity (log-concavity) of sequences deserves to be studied. There exist many log-concave (log-convex) sequences in combinatorics. The binomial coefficients $\binom{n}{k}$, the Eulerian numbers $A(n,k)$, the Stirling numbers $c(n,k)$ and $S(n,k)$ of two kinds are log-concave for k when n is fixed. Some famous combinatorial sequences, including the Bell numbers, the Catalan numbers, the central binomial coefficients, the central Delannoy numbers, the Motzkin numbers, the Fine numbers, the little and large Schröder numbers, are log-convex. In this paper, we are interested in the log-convexity of a class of linear recurrence sequences $\{F(n,k)\}_{n\geq k}$ $(k \geq 1)$ defined by

$$F(n,k) = (n-2)F(n-1,k) + F(n-1,k-1), \tag{1.1}$$

where $F(n,1) = \frac{n!}{2}$, $F(n,n) = 1$; see Table 1 for some information about $F(n,k)$. The sequence $\{F(n,k)\}$ is related to permutations with ordered

Table 1: Some initial values of $\{F(n,k)\}$

n \ k	1	2	3	4	5	6	7	8	9
1	1								
2	1	1							
3	3	2	1						
4	12	7	4	1					
5	60	33	19	7	1				
6	360	192	109	47	11	1			
7	2520	1320	737	344	102	16	1		
8	20160	10440	5742	2801	956	198	22	1	
9	181440	93240	50634	25349	9493	2342	352	29	1

orbits. Let $\mathfrak{S}[n]$ denote the group of permutations of $[n]$, where $[n] = \{1, 2, \cdots, n\}$. $F(n,k)$ is the number of a kind of permutation σ of $[n]$ ($\sigma \in \mathfrak{S}[n]$). For more properties of $\{F(n,k)\}$, see Comtet [4].

This paper is devoted to the study of the log-convexity of $\{F(n,k)\}_{n\geq k}$ for fixed k and is organized as follows. In Section 2, we mainly study the log-convexity of $\{F(n,k)\}_{n\geq k}$ $(k \geq 3)$ when k is fixed. We prove that $\{F(n,3)\}_{n\geq 3}$, $\{F(n,4)\}_{n\geq 5}$ and $\{F(n,5)\}_{n\geq 6}$ are log-convex. In Section 3, we show that $\{F(n,2)\}_{n\geq 3}$ is ratio log-concave. In addition, we discuss the log-behavior of some sequences related to $F(n,k)$.

2 Log-convexity of $\{F(n,k)\}_{n\geq k}$ for fixed k

We first give some lemmas.

Lemma 2.1 *For $n \geq 2$, let $g_n = \frac{F(n+1,2)}{F(n,2)}$. Then we have*

$$n + \frac{1}{2} \leq g_n \leq n+1, \quad n \geq 3. \tag{2.1}$$

Proof. By applying (1.1), we derive

$$F(n+1,2) = (2n-1)F(n,2) - n(n-2)F(n-1,2), \quad n \geq 3. \tag{2.2}$$

It follows from (2.2) that

$$g_n = 2n - 1 - \frac{n(n-2)}{g_{n-1}}, \quad n \geq 3. \tag{2.3}$$

Noting that $g_3 = \frac{7}{2}$, $g_4 = \frac{33}{7}$, and $g_5 = \frac{64}{11}$, it is clear that $j + \frac{1}{2} \leq g_j \leq j+1$ for $j = 3, 4, 5$. Assume that $j + \frac{1}{2} \leq g_j \leq j + 1$ for $j \geq 5$. By means of (2.3), we get

$$g_{j+1} - j - \frac{3}{2} = j - \frac{1}{2} - \frac{j^2-1}{g_j} \quad \text{and} \quad g_{j+1} - j - 2 = j - 1 - \frac{j^2-1}{g_j}.$$

Since $j + \frac{1}{2} \leq g_j \leq j+1$, we have

$$\begin{aligned} g_{j+1} - j - \frac{3}{2} &= \frac{(j-\frac{1}{2})g_j - j^2 + 1}{g_j} \\ &\geq \frac{3}{4g_j} \\ &> 0 \end{aligned}$$

and

$$g_{j+1} - j - 2 = \frac{(j-1)g_j - j^2 + 1}{g_j} \leq 0.$$

Hence, we have $n + \frac{1}{2} \leq g_n \leq n+1$ when $n \geq 3$. ■

Lemma 2.2 *For $n \geq 3$, put $h_n = \frac{F(n+1,3)}{F(n,3)}$. Then we have*

$$n + \frac{1}{2} \leq h_n \leq n+1, \quad n \geq 3. \tag{2.4}$$

Proof. For $n \geq 2$, set $g_n = \frac{F(n+1,2)}{F(n,2)}$. By using (1.1), we obtain

$$F(n+1,3) = (n-1+g_{n-1})F(n,3) - (n-2)g_{n-1}F(n-1,3), \quad n \geq 4. \tag{2.5}$$

It follows from (2.5) that

$$h_n = n - 1 + g_{n-1} - \frac{(n-2)g_{n-1}}{h_{n-1}}, \quad n \geq 4. \tag{2.6}$$

For $j = 3, 4, 5$, we find that $j + \frac{1}{2} \leq h_j \leq j+1$. For $j \geq 5$, assume that $j + \frac{1}{2} \leq h_j \leq j+1$. By applying (2.6), we derive

$$h_{j+1} - j - \frac{3}{2} = \frac{(g_j - \frac{3}{2})h_j - (j-1)g_j}{h_j}$$

and

$$h_{j+1} - j - 2 = \frac{(g_j - 2)h_j - (j-1)g_j}{h_j}.$$

It follows from (2.1) that $g_j - \frac{3}{2} \geq 0$ and $g_j - 2 > 0$ for $j \geq 3$. Since $j + \frac{1}{2} \leq h_j \leq j+1$, we get

$$\begin{aligned} h_{j+1} - j - \frac{3}{2} &\geq \frac{(g_j - \frac{3}{2})(j + \frac{1}{2}) - (j-1)g_j}{h_j} \\ &= \frac{3(g_j - j - \frac{1}{2})}{2h_j} \end{aligned}$$

and

$$h_{j+1} - j - 2 \leq \frac{2(g_j - j - 1)}{h_j}.$$

It follows from (2.1) that $h_{j+1} - j - \frac{3}{2} \geq 0$ and $h_{j+1} - j - 2 \leq 0$. Thus, we have $n + \frac{1}{2} \leq h_n \leq n+1$ when $n \geq 3$. ■

Lemma 2.3 *Let $\{z_n\}_{n\geq 0}$ be a sequence of positive real numbers and satisfy the three-term recurrence*

$$z_{n+1} = (n-1+w_{n-1})z_n - (n-2)w_{n-1}z_{n-1}, \quad n \geq 2, \tag{2.7}$$

where the sequence $\{w_n\}_{n\geq 0}$ is monotonic increasing and $w_0 \geq 0$. For $n \geq 0$, let $x_n = \frac{z_{n+1}}{z_n}$. Suppose that there exists a positive integer $n_0 \geq 1$ such that $\{z_{n_0}, z_{n_0+1}, z_{n_0+2}\}$ is log-convex. If $x_{n+1} \geq n$ and $x_{n+1} \geq w_n$ for $n \geq n_0$, then the sequence $\{z_n\}_{n\geq n_0}$ is log-convex.

Proof. It follows from (2.7) that

$$x_n = n - 1 + w_{n-1} - \frac{(n-2)w_{n-1}}{x_{n-1}}, \quad n \geq 2. \tag{2.8}$$

Now we prove by induction that the sequence $\{x_n\}_{n\geq n_0}$ is monotonic increasing. By the assumption that $\{z_{n_0}, z_{n_0+1}, z_{n_0+2}\}$ is log-convex, we have $x_{n_0} \leq x_{n_0+1}$. For $j \geq n_0$, assume that $x_j \leq x_{j+1}$. It follows from (2.8) that

$$x_{j+2} - x_{j+1} = 1 + w_{j+1} - w_j + \frac{(j-1)w_j}{x_j} - \frac{jw_{j+1}}{x_{j+1}}.$$

Since $x_j \leq x_{j+1}$, we get

$$\begin{aligned} x_{j+2} - x_{j+1} &\geq 1 + w_{j+1} - w_j + \frac{(j-1)w_j - jw_{j+1}}{x_{j+1}} \\ &= \frac{x_{j+1} + (w_{j+1} - w_j)(x_{j+1} - j) - w_j}{x_{j+1}}. \end{aligned}$$

Noting that $x_{j+1} \geq j$ and $x_{j+1} \geq w_j$ for $j \geq n_0$, we have $x_{j+2} - x_{j+1} \geq 0$. Hence the sequence $\{x_n\}_{n\geq n_0}$ is monotonic increasing. Thus $\{z_n\}_{n\geq n_0}$ is log-convex. ■

Zhao [12] proved that the sequence $\{F(n,2)\}_{n\geq 2}$ is log-convex. Now we discuss the log-convexity of $\{F(n,3)\}_{n\geq 3}$, $\{F(n,4)\}_{n\geq 5}$, and $\{F(n,5)\}_{n\geq 6}$.

Theorem 2.1 *The sequences $\{F(n,3)\}_{n\geq 3}$, $\{F(n,4)\}_{n\geq 5}$, and $\{F(n,5)\}_{n\geq 6}$ are log-convex.*

Proof. For $n \geq 3$, let

$$h_n = \frac{F(n+1,3)}{F(n,3)},$$

$$u_n = F(n,4), \quad x_n = \frac{u_{n+1}}{u_n}, \quad n \geq 4,$$

$$z_n = F(n,5), \quad y_n = \frac{z_{n+1}}{z_n}, \quad n \geq 5.$$

(i) It follows from (2.4) that $\{h_n\}_{n\geq 3}$ is monotonic increasing. Then $\{F(n,3)\}_{n\geq 3}$ is log-convex.

(ii) In order to prove that the sequence $\{F(n,4)\}_{n\geq 5}$ is log-convex, we show that $\{x_n\}_{n\geq 5}$ is monotonic increasing. By using (1.1), we have

$$u_{n+1} = (n-1+h_{n-1})u_n - (n-2)h_{n-1}u_{n-1}, \quad n \geq 5. \tag{2.9}$$

It follows from (2.9) that

$$x_n = n-1+h_{n-1} - \frac{(n-2)h_{n-1}}{x_{n-1}}, \quad n \geq 5. \tag{2.10}$$

Now we prove by induction that

$$n + \frac{1}{2} \leq x_n \leq n + \frac{3}{2}, \quad n \geq 6. \tag{2.11}$$

We observe that $\frac{13}{2} < x_6 < \frac{15}{2}$. For $j \geq 6$, assume that $j + \frac{1}{2} \leq x_j \leq j + \frac{3}{2}$. By applying (2.10), we get

$$x_{j+1} - j - \frac{3}{2} = \frac{(h_j - \frac{3}{2})x_j - (j-1)h_j}{x_j},$$

$$x_{j+1} - j - \frac{5}{2} = \frac{(h_j - \frac{5}{2})x_j - (j-1)h_j}{x_j}.$$

It follows from $j + \frac{1}{2} \leq x_j \leq j + \frac{3}{2}$ and (2.4) that

$$\begin{aligned} x_{j+1} - j - \frac{3}{2} &\geq \frac{3(h_j - j - \frac{1}{2})}{2x_j} \\ &\geq 0, \\ x_{j+1} - j - \frac{5}{2} &\leq \frac{5(h_j - j - \frac{3}{2})}{2x_j} \\ &\leq 0. \end{aligned}$$

Thus, $n + \frac{1}{2} \le x_n \le n + \frac{3}{2}$ when $n \ge 6$. On the other hand, we find that $x_5 < x_6$. Then the sequence $\{x_n\}_{n\ge5}$ is monotonic increasing.

(iii) The above two parts are based on (2.4) or (2.11), which may be obtained by observing the first several values of the sequences involved. However, we can not observe that the sequence $\{y_n\}_{n\ge5}$ has a similar property. In what follows, we make use of Lemma 2.3 to prove that $\{z_n\}_{n\ge6}$ is log-convex.

By using (1.1), we have

$$z_{n+1} = (n-1+x_{n-1})z_n - (n-2)x_{n-1}z_{n-1}, \quad n \ge 6. \tag{2.12}$$

It follows from (2.12) that

$$y_n = n - 1 + x_{n-1} - \frac{(n-2)x_{n-1}}{y_{n-1}}, \quad n \ge 6. \tag{2.13}$$

Now we prove by induction that $y_n \ge n + \frac{1}{2}$ for $n \ge 5$. It is clear that $y_5 > \frac{11}{2}$ and $y_6 > \frac{13}{2}$. For $j \ge 6$, assume that $y_j \ge j + \frac{1}{2}$. It follows from (2.13) that

$$y_{j+1} - j - \frac{3}{2} = \frac{y_j(x_j - \frac{3}{2}) - (j-1)x_j}{y_j}.$$

Since $y_j \ge j + \frac{1}{2}$,

$$y_{j+1} - j - \frac{3}{2} \ge \frac{3(x_j - j - \frac{1}{2})}{2y_j}.$$

By means of (2.11), we have $y_{j+1} - j - \frac{3}{2} \ge 0$. Then $y_n \ge n + \frac{1}{2}$ holds for $n \ge 5$. We note that $\{z_6, z_7, z_8\}$ is log-convex. It is evident that $y_{j+1} > j$ and $y_{j+1} \ge x_j$ for $j \ge 6$. We have from Lemma 2.3 that the sequence $\{z_n\}_{n\ge6}$ is log-convex. ■

We can verify that $\{F(n,k)\}$ satisfies the following recurrence

$$F(n+1,k) = \left[n - 1 + \frac{F(n,k-1)}{F(n-1,k-1)}\right]F(n,k) - \frac{(n-2)F(n,k-1)}{F(n-1,k-1)}F(n-1,k), \quad n \ge k+1.$$

When $k \geq 6$ is fixed, we can discuss the log-behavior of $\{F(n,k)\}_{n\geq k}$ by Lemma 2.3.

3 Log-behavior of some sequences involving $F(n,k)$

In this section, we discuss the log-behavior of some sequences involving $F(n,k)$. Chen, Guo, and Wang [3] showed that sequences of the derangement numbers, the Motzkin numbers, the Fine numbers, the central Delannoy numbers, the numbers of tree-like polyhexes and the Domb numbers are ratio log-concave. Now we prove that $\{F(n,2)\}_{n\geq 3}$ is also ratio log-concave.

Theorem 3.1 *The sequence $\{F(n,2)\}_{n\geq 3}$ is ratio log-concave.*

Proof. For $n \geq 2$, let $g_n = \frac{F(n+1,2)}{F(n,2)}$, $\lambda_n = n + \frac{1}{2}$, and $\mu_n = n + 1$. It is obvious that $\lambda_n \leq g_n \leq \mu_n$ for $n \geq 3$. In order to prove that the sequence $\{F(n,2)\}_{n\geq 3}$ is ratio log-concave, it suffices to show that $g_n^2 - g_{n-1}g_{n+1} \geq 0$ for $n \geq 4$. For $n \geq 4$, by applying (2.3), we get

$$\begin{aligned} & g_n^2 - g_{n-1}g_{n+1} \\ = & \frac{-g_n^4 + (2n-1)g_n^3 - n(n-2)(2n+1)g_n + n(n-2)(n^2-1)}{(2n-1-g_n)g_n}. \end{aligned}$$

For any $t \in (-\infty, +\infty)$, define a function

$$f(t) = -t^4 + (2n-1)t^3 - n(n-2)(2n+1)t + n(n-2)(n^2-1).$$

Then we have

$$\begin{aligned} f'(t) &= -4t^3 + 3(2n-1)t^2 - n(n-2)(2n+1), \\ f''(t) &= -12t^2 + 6(2n-1)t, \\ f'''(t) &= -24t + 6(2n-1). \end{aligned}$$

Since $f'''(t) < 0$ for $t \geq \lambda_n$, f'' is decreasing on $[\lambda_n, +\infty)$. We note that $f''(\lambda_n) = -12n - 6$. Then f' is decreasing on $[\lambda_n, +\infty)$. By calculation, we have

$$f'(\lambda_n) = -\frac{5(2n+1)}{4}.$$

This implies that $f'(t) < 0$ for $t > \lambda_n$. Thus, f is decreasing on $[\lambda_n, +\infty)$. Noting that

$$f(\mu_n) = (n+1)(n-2) > 0,$$

we obtain $f(g_n) > 0$. Hence $g_n^2 - g_{n-1}g_{n+1} > 0$ for $n \geq 4$. ■

Liu and Zhao [8] have proved that $\{\sqrt{F(n,2)}\}_{n\geq 3}$ is log-balanced. Now we discuss the log-balancedness of $\{\sqrt{F(n,3)}\}_{n\geq 3}$ and $\{\sqrt[3]{F(n,4)}\}_{n\geq 4}$.

Theorem 3.2 *The sequences $\{\sqrt{F(n,3)}\}_{n\geq 3}$ and $\{\sqrt[3]{F(n,4)}\}_{n\geq 5}$ are log-balanced.*

Proof. For $n \geq 3$, let $h_n = \frac{F(n+1,3)}{F(n,3)}$ and $x_n = \frac{F(n+1,4)}{F(n,4)}$ $(n \geq 4)$. Since the sequences $\{F(n,3)\}_{n\geq 3}$ and $\{F(n,4)\}_{n\geq 5}$ are log-convex, $\{\sqrt{F(n,3)}\}_{n\geq 3}$ and $\{\sqrt[3]{F(n,4)}\}_{n\geq 5}$ are also log-convex. We need to show that $\{\frac{\sqrt{F(n,3)}}{n!}\}_{n\geq 3}$ and $\{\frac{\sqrt[3]{F(n,4)}}{n!}\}_{n\geq 5}$ are log-concave. It is evident that the sequence $\{\frac{\sqrt{F(n,3)}}{n!}\}_{n\geq 3}$ (or $\{\frac{\sqrt[3]{F(n,4)}}{n!}\}_{n\geq 5}$) is log-concave if and only if $\frac{\sqrt{h_n}}{n+1} \geq \frac{\sqrt{h_{n+1}}}{n+2}$ (or $\frac{\sqrt{x_n}}{n+1} \geq \frac{\sqrt{x_{n+1}}}{n+2}$). It follows from $(n+2)^2 h_n - (n+1)^2 h_{n+1} \geq 0$ (or $(n+2)^3 x_n - (n+1)^3 x_{n+1} \geq 0$) that $\frac{\sqrt{h_n}}{n+1} \geq \frac{\sqrt{h_{n+1}}}{n+2}$ (or $\frac{\sqrt{x_n}}{n+1} \geq \frac{\sqrt{x_{n+1}}}{n+2}$).

Now we prove that $(n+2)^2 h_n - (n+1)^2 h_{n+1} \geq 0$ for $n \geq 3$ and $(n+2)^3 x_n - (n+1)^3 x_{n+1} \geq 0$ for $n \geq 5$. It follows from (2.4) that

$$\begin{aligned}(n+2)^2 h_n - (n+1)^2 h_{n+1} &\geq \frac{(n+2)^2(2n+1)}{2} - (n+1)^2(n+2) \\ &= \frac{n(n+2)}{2} \\ &> 0 \quad (n \geq 3).\end{aligned}$$

For $n \geq 6$, it follows from (2.11) that

$$(n+2)^3 x_n - (n+1)^3 x_{n+1} \geq n^3 + \frac{9}{2}n^2 + \frac{11}{2}n + \frac{3}{2} > 0.$$

On the other hand, we observe that $7^3 x_5 - 6^3 x_6 > 0$. Hence, $(n+2)^3 x_n - (n+1)^3 x_{n+1} \geq 0$ holds for $n \geq 5$. Therefore, the sequences $\{\sqrt{F(n,3)}\}_{n\geq 3}$ and $\{\sqrt[3]{F(n,4)}\}_{n\geq 5}$ are log-balanced. ■

Theorem 3.3 *The sequence* $\{\sqrt{\frac{F(n,2)}{n}}\}_{n\geq 3}$ *is log-balanced.*

Proof. For $n \geq 2$, let $t_n = \frac{nF(n+1,2)}{(n+1)F(n,2)}$ and $g_n = \frac{F(n+1,2)}{F(n,2)}$. Then we have $t_n = \frac{n}{n+1} g_n$. Since the sequences $\{F(n,2)\}_{n\geq 2}$ and $\{\frac{1}{n}\}_{n\geq 1}$ are both log-convex, $\{\frac{F(n,2)}{n}\}_{n\geq 2}$ is also log-convex. Then $\{\sqrt{\frac{F(n,2)}{n}}\}_{n\geq 2}$ is log-convex. In order to prove the log-balancedness of $\{\sqrt{\frac{F(n,2)}{n}}\}_{n\geq 3}$, it is sufficient to show that the sequence $\{\frac{1}{n!}\sqrt{\frac{F(n,2)}{n}}\}_{n\geq 3}$ is log-concave. We note that $\{\frac{1}{n!}\sqrt{\frac{F(n,2)}{n}}\}_{n\geq 3}$ is log-concave if and only if $\frac{1}{n+1}\sqrt{t_n} \geq \frac{1}{n+2}\sqrt{t_{n+1}}$ for $n \geq 3$. On the other hand, it follows from $n(n+2)^3 g_n - (n+1)^4 g_{n+1} \geq 0$ that $\frac{1}{n+1}\sqrt{t_n} \geq \frac{1}{n+2}\sqrt{t_{n+1}}$. Now we prove that $n(n+2)^3 g_n - (n+1)^4 g_{n+1} \geq 0$ holds for $n \geq 3$. Applying (2.1), we have

$$\begin{aligned} n(n+2)^3 g_n - (n+1)^4 g_{n+1} &\geq n(n+2)^3\left(n+\frac{1}{2}\right) - (n+1)^4(n+2) \\ &= \frac{n^3 - 4n - 2}{2} \\ &> 0 \quad (n \geq 3). \end{aligned}$$

Hence, the sequence $\{\sqrt{\frac{F(n,2)}{n}}\}_{n\geq 3}$ is log-balanced. ■

For $n \geq 2$, let $S_n = \frac{1}{n-1}\sum_{j=2}^{n} F(j,2)$ and $T_n = \frac{1}{n-2}\sum_{j=3}^{n} F(j,3)$. It is evident that $\{S_n\}_{n\geq 2}$ (or $\{T_n\}_{n\geq 3}$) is the mean value sequence of $\{F(n,2)\}_{n\geq 2}$ (or $\{F(n,3)\}_{n\geq 3}$). For some information of $\{S_n\}_{n\geq 2}$ and $\{T_n\}_{n\geq 3}$, see tables 2–3. At the end of this section, we discuss the log-behavior of $\{S_n\}_{n\geq 2}$ and $\{T_n\}_{n\geq 3}$.

Table 2: Some initial values of $\{S_n\}$

n	2	3	4	5	6	7
S_n	1	$\frac{3}{2}$	$\frac{10}{3}$	$\frac{43}{4}$	47	$\frac{1555}{6}$

Table 3: Some initial values of $\{T_n\}$

n	3	4	5	6	7	8
T_n	1	$\frac{5}{2}$	8	$\frac{133}{4}$	174	1102

Theorem 3.4 *The sequences* $\{S_n\}_{n\geq 2}$ *and* $\{T_n\}_{n\geq 3}$ *are log-convex.*

Proof. We only prove that the sequence $\{S_n\}_{n\geq 2}$ is log-convex. The proof for the log-convexity of $\{T_n\}_{n\geq 3}$ follows the same pattern and is omitted here.

For $n \geq 2$, let $g_n = \frac{F(n+1,2)}{F(n,2)}$, $W_n = \sum_{j=2}^{n} F(j,2)$, and $p_n = \frac{S_{n+1}}{S_n}$. In order to prove that $\{S_n\}_{n\geq 2}$ is log-convex, we need to show that $\{p_n\}_{n\geq 2}$ is increasing. For $n \geq 3$, noting that

$$\frac{W_{n+1} - W_n}{F(n+1,2)} = \frac{W_n - W_{n-1}}{F(n,2)},$$

we have

$$W_{n+1} = (1 + g_n)W_n - g_n W_{n-1}.$$

Then $\{S_n\}_{n\geq 2}$ satisfies the recurrence

$$S_{n+1} = \frac{(n-1)(1+g_n)}{n} S_n - \frac{(n-2)g_n}{n} S_{n-1}, \quad n \geq 3. \tag{3.1}$$

It follows from (3.1) that

$$p_n = \frac{(n-1)(1+g_n)}{n} - \frac{(n-2)g_n}{np_{n-1}}, \quad n \geq 3. \tag{3.2}$$

Now we prove by induction that $\{p_n\}_{n\geq 2}$ is increasing. We find that $j-1 < p_j < j$ for $j = 2, 3, 4$. For $j \geq 4$, assume that $j-1 \leq p_j \leq j$. It follows from (3.2) that

$$\begin{aligned} p_{j+1} - j &= \frac{j(1+g_{j+1})}{j+1} - \frac{(j-1)g_{j+1}}{(j+1)p_j} - j, \\ p_{j+1} - j - 1 &= \frac{j(1+g_{j+1})}{j+1} - \frac{(j-1)g_{j+1}}{(j+1)p_j} - j - 1. \end{aligned}$$

Since $j-1 \leq p_j \leq j$, we get

$$\begin{aligned} p_{j+1} - j &\geq \frac{j(1+g_{j+1})}{j+1} - \frac{g_{j+1}}{j+1} - j \\ &= \frac{(j-1)g_{j+1} - j^2}{j+1}, \\ p_{j+1} - j - 1 &\leq \frac{j(1+g_{j+1})}{j+1} - \frac{(j-1)g_{j+1}}{(j+1)j} - j - 1 \\ &= \frac{(j^2-j+1)g_{j+1} - j^3 - j^2 - j}{j(j+1)}. \end{aligned}$$

Using (2.1), we have

$$p_{j+1} - j \geq 0 \quad \text{and} \quad p_{j+1} - j - 1 \leq 0.$$

Hence, the sequence $\{p_n\}_{n\geq 2}$ is increasing. ■

4 Conclusions

We have discussed the log-convexity of the sequence $\{F(n,k)\}_{n\geq k}$ when k is fixed. We have also discussed the log-behavior of some sequences related to $F(n,k)$. We have shown that $\{F(n,3)\}_{n\geq 3}$, $\{F(n,4)\}_{n\geq 5}$, and $\{F(n,5)\}_{n\geq 6}$ are log-convex. When $k \geq 6$ is fixed, we give the following conjecture for $\{F(n,k)\}_{n\geq k}$.

Conjecture 4.1 *Suppose that $k \geq 6$ is fixed. For the sequence $\{F(n,k)\}_{n\geq k}$, there exists a positive integer $N_k \geq k$ such that $\{F(n,k)\}_{n\geq N_k}$ is log-convex.*

Acknowledgements. The authors would like to thank the anonymous referee for his/her useful comments and suggestions.

References

[1] N. Asai, I. Kubo and H. H. Kuo, Roles of log-concavity, log-convexity and growth order in white noise analysis, Infinite Dimensional Analysis, Quantum Probability and Related Topics, **4** (2001), 59–84.

[2] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: An update, Contemporary Mathematics, **178** (1994), 71–89.

[3] W.Y.C. Chen, J.J.F. Guo, and L.X.W. Wang, Infinitely log-monotonic combinatorial sequences, Advances in Applied Mathematics, **52** (2014), 99–120.

[4] L. Comtet, Advanced Combinatorics, Reidel Dordrecht, 1974.

[5] T. Došlić, Log-balanced combinatorial sequences, International Journal of Mathematics and Mathematical Sciences, **4** (2005), 507–522.

[6] T. Došlić, Seven (lattice) paths to log-convexity, Acta Applicandae Mathematicae, **110** (2010), 1373–1392.

[7] T. Došlić, D. Svrtan and D. Veljan, Enumerative aspects of secondary structures, Discrete Mathematics, **285** (2004), 67–82.

[8] R.L. Liu and F.Z. Zhao, New sufficient conditions for log-balancedness and their applications in combinatorial sequences, submitted.

[9] O. Milenkovic and K.J. Compton, Probabilistic transforms for combinatorial urn models, Combinatorics, Probability and Computing, **13** (2004), 645–675.

[10] D. Prelec, Decreasing impatience: A criterion for non-stationary time preference and "hyperbolic" discounting, The Scandinavian Journal of Economics, **106** (2004), 511–532.

[11] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, Annals of the New York Academy of Sciences, **576** (1989), 500–535.

[12] F.Z. Zhao, Log-convexity of some recurrence sequences, Journal of the Indian Mathematical Society. New Series, **82** (2015), 207–224.

2010 Mathematics Subject Classification. 05A20, 11B73, 11B83.