The log-convexity of some sequences related to Cauchy numbers of two kinds

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Abstract

For Cauchy numbers of the first kind $\{a_n\}_{n\geq 0}$ and Cauchy numbers of the second kind $\{b_n\}_{n\geq 0}$, this paper focuses on the log-convexity of some sequences related to $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$. For example, we discuss log-convexity of $\{n|a_n| - |a_{n+1}|\}_{n\geq 1}$, $\{b_{n+1} - nb_n\}_{n\geq 1}$, $\{n|a_n|\}_{n\geq 1}$ and $\{(n+1)b_n\}_{n\geq 0}$. In addition, we investigate log-balancedness of some sequences involving a_n (or b_n).

Key words. Cauchy numbers, log-convexity, log-concavity, log-balancedness.

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1 Introduction

Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ denote the Cauchy numbers of the first and the second kind, respectively. The definitions of a_n and b_n are (see Comtet [2])

$$a_n = \int_0^1 (x)_n dx, \quad b_n = \int_0^1 \langle x\rangle_n dx,$$

where

$$(x)_n = \begin{cases} x(x-1)\cdots(x-n+1), & n \geq 1, \\ 1, & n = 0, \end{cases}$$

$$\langle x\rangle_n = \begin{cases} x(x+1)\cdots(x+n-1), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

The exponential generating functions of $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are

$$\sum_{n=0}^{\infty} \frac{a_n z^n}{n!} = \frac{z}{\ln(1+z)}, \quad \sum_{n=0}^{\infty} \frac{b_n z^n}{n!} = -\frac{z}{(1-z)\ln(1-z)}.$$

Some values of $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6	7	8	9
a_n	1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{4}$	$-\frac{19}{30}$	$\frac{9}{4}$	$-\frac{863}{84}$	$\frac{1375}{24}$	$-\frac{33953}{90}$	$\frac{57281}{20}$
b_n	1	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{9}{4}$	$\frac{251}{30}$	$\frac{475}{12}$	$\frac{19087}{84}$	$\frac{36799}{24}$	$\frac{1070017}{90}$	$\frac{2082753}{20}$

Cauchy numbers play important roles in many subjects such as combinatorics, approximate integrals and difference-differential equations. See for instance [1,4,5,7]. Cauchy numbers of two kinds are related to some known numbers such as Stirling numbers, Bernoulli numbers and harmonic numbers. Hence their properties deserve to be studied. For the properties of Cauchy numbers, see [2,6,8–12]. In this paper, we consider log-convexity of some sequences related to $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$.

Now we recall some definitions in combinatorics. For a positive sequence $\{z_n\}_{n\geq 0}$, it is said to be *log-convex* (or *log-concave*) if $z_n^2 \leq z_{n-1}z_{n+1}$ (or $z_n^2 \geq z_{n-1}z_{n+1}$) for all $n \geq 1$. For an alternating sequence, its log-convexity (or log-concavity) can be defined in the same way. An alternative (and equivalent) definition is that an alternating sequence $\{w_n\}_{n\geq 0}$ is log-convex (log-concave) if the sequence of its absolute values $\{|w_n|\}_{n\geq 0}$ is log-convex (log-concave). For a log-convex sequence, Došlić [3] gave the following definition:

Definition 1.1 *A log-convex sequence $\{z_n\}_{n\geq 0}$ is said to be log-balanced if $\{\frac{z_n}{n!}\}_{n\geq 0}$ is log-concave.*

It is well known that for a positive sequence $\{z_n\}_{n\geq 0}$ is log-convex (log-concave) if and only if its quotient sequence $\{\frac{z_{n+1}}{z_n}\}_{n\geq 0}$ is nondecreasing (nonincreasing). For an alternating sequence $\{w_n\}_{n\geq 0}$, it is log-convex (log-concave) if and only if the quotient sequence of $\{|w_n|\}_{n\geq 0}$ is nondecreasing (nonincreasing). Clearly, a log-balanced sequence is a kind of log-convex sequences whose quotient sequence does not grow too fast.

Log-concavity and log-convexity are not only the important parts of unimodality problems, but also fertile sources of combinatorial inequalities. In combinatorics and number theory, there are many log-convex sequences such as the Catalan numbers, the Bell numbers, the Motzkin numbers, the Fine numbers, the Franel numbers of order 3 and 4, the Apéry numbers, the large and little Schröder numbers, the derangements numbers, the central Delannoy numbers and the Catalan-Larcombe-French numbers. Zhao [12] investigated log-convexity of Cauchy numbers and she proved that $\{|a_n|\}_{n\geq 1}$ and $\{b_n\}_{n\geq 0}$ are log-convex. In this paper, we mainly discuss log-convexity of some sequences related to $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$. For example, we discuss log-convexity of $\{n|a_n| - |a_{n+1}|\}_{n\geq 1}$, $\{b_{n+1} - nb_n\}_{n\geq 1}$, $\{n|a_n|\}_{n\geq 1}$ and $\{(n+1)b_n\}_{n\geq 0}$. In addition, we investigate log-balancedness of some sequences involving a_n (or b_n).

2 Log-convexity of some sequences involving Cauchy numbers of two kinds

In this section, we discuss log-convexity of some sequences related to Cauchy numbers of two kinds.

Theorem 2.1 *The sequences* $\{n|a_n| - |a_{n+1}|\}_{n\geq 1}$, $\{b_{n+1} - nb_n\}_{n\geq 1}$, $\{n|a_n|\}_{n\geq 1}$ *and* $\{(n+1)b_n\}_{n\geq 0}$ *are log-convex.*

Proof. For $n \geq 1$, let

$$\delta_n = \int_0^1 x^2(1-x)\cdots(n-1-x)dx, \quad \eta_n = \int_0^1 x^2(x+1)\cdots(x+n-1)dx.$$

From the definition of $\{a_n\}$ and $\{b_n\}$, we derive

$$\begin{aligned} n|a_n| - |a_{n+1}| &= \delta_n, \\ b_{n+1} - nb_n &= \eta_n. \end{aligned} \tag{2.1}$$

Now we show that $\{\delta_n\}_{n\geq 1}$ and $\{\eta_n\}_{n\geq 1}$ are both log-convex. For $n \geq 1$, put

$$s_n = \frac{\delta_{n+1}}{\delta_n}, \quad t_n = \frac{\eta_{n+1}}{\eta_n}.$$

We note that

$$\begin{aligned} s_n &= n - \frac{\int_0^1 x^3(1-x)\cdots(n-1-x)dx}{\delta_n}, \\ t_n &= n + \frac{\int_0^1 x^3(x+1)\cdots(x+n-1)dx}{\eta_n}. \end{aligned}$$

Since

$$\begin{aligned} 0 &< \frac{\int_0^1 x^3(1-x)\cdots(n-1-x)dx}{\delta_n} < 1, \quad n \geq 2, \\ 0 &< \frac{\int_0^1 x^3(x+1)\cdots(x+n-1)dx}{\eta_n} < 1, \quad n \geq 1, \end{aligned}$$

we derive

$$n-1<s_n<n, \quad n\geq 2,$$

$$n<t_n<n+1, \quad n\geq 1.$$

This means that $\{s_n\}_{n\geq 2}$ and $\{t_n\}_{n\geq 1}$ are both increasing. Then the sequences $\{\delta_n\}_{n\geq 1}$ and $\{\eta_n\}_{n\geq 1}$ are both log-convex. Hence the sequences $\{n|a_n| - |a_{n+1}|\}_{n\geq 1}$, $\{b_{n+1}-nb_n\}_{n\geq 1}$ are log-convex.

From (2.1), we obtain

$$n|a_n| = |a_{n+1}| + \delta_n.$$

On the other hand, we note that the sequence $\{|a_n|\}_{n\geq 1}$ is log-convex (see Zhao [12]). As a result, the sequence $\{n|a_n|\}_{n\geq 1}$ is log-convex.

It follows from the exponential generating functions of $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ that

$$(1-z)\sum_{n=0}^{\infty}\frac{b_n z^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n a_n z^n}{n!}. \tag{2.2}$$

Comparing the coefficients of z^n on both sides of (2.2), we get

$$b_n-(-1)^n a_n = nb_{n-1}, \quad n\geq 1.$$

It is clear that $|a_n|=(-1)^{n-1}a_n$ for $n\geq 1$. Then we obtain

$$(n+1)b_n = b_{n+1}+|a_{n+1}|, \quad (n\geq 0).$$

It follows from the log-convexity of $\{|a_n|\}_{n\geq 1}$ and $\{b_n\}_{n\geq 0}$ that the sequence $\{(n+1)b_n\}_{n\geq 0}$ is log-convex. ■

At the end of this section, we discuss log-balancedness of some sequences involving a_n (or b_n).

Theorem 2.2 *Suppose that $r\geq 2$ is a real number.*

(i) *For* $r > 2$, *the sequence* $\{\sqrt[r]{|a_n|}\}_{n\geq N_1}$ *is log-balanced, where* N_1 *is a positive integer such that* $N_1 \geq \frac{r+3}{2r-3}$, *and the sequence* $\{\sqrt[r]{b_n}\}_{n\geq 0}$ *is log-balanced.*

(ii) *The sequences* $\{\sqrt{|a_n|}\}_{n\geq 2}$ *and* $\{\sqrt{b_n}\}_{n\geq 0}$ *are log-balanced.*

Proof. Because the sequences $\{|a_n|\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are log-convex, $\{\sqrt[r]{|a_n|}\}_{n\geq 1}$ and $\{\sqrt[r]{b_n}\}_{n\geq 1}$ are log-convex. Next we investigate log-concavity of $\{\frac{\sqrt[r]{|a_n|}}{n!}\}_{n\geq 1}$ and $\{\frac{\sqrt[r]{b_n}}{n!}\}_{n\geq 0}$.

For $n \geq 1$, let $x_n = \frac{|a_{n+1}|}{|a_n|}$, $y_n = \frac{b_{n+1}}{b_n}$. We note that

$$\begin{aligned} x_n &= n - \frac{\int_0^1 x^2(1-x)\cdots(n-1-x)dx}{|a_n|}, \\ y_n &= n + \frac{\int_0^1 x^2(x+1)\cdots(x+n-1)dx}{b_n}. \end{aligned}$$

We can verify that

$$\begin{aligned} &\int_0^1 2x^2(1-x)\cdots(n-1-x)dx - \int_0^1 x(1-x)\cdots(n-1-x)dx \\ &= \int_0^1 2(1-x)^2x(x+1)\cdots(n-2+x)dx \\ &\quad - \int_0^1 (1-x)x(x+1)\cdots(n-2+x)dx \\ &= \int_0^{\frac{1}{2}} (1-2x)(1-x)x(x+1)\cdots(n-2+x)dx \\ &\quad + \int_{\frac{1}{2}}^1 (1-2x)(1-x)x(x+1)\cdots(n-2+x)dx \\ &\leq \left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+n-2\right)\int_0^{\frac{1}{2}}(1-2x)(1-x)xdx \\ &\quad + \left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+n-2\right)\int_{\frac{1}{2}}^1(1-2x)(1-x)xdx \\ &= \left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+n-2\right)\int_0^1(1-2x)(1-x)xdx \\ &= 0 \quad (n \geq 2), \end{aligned}$$

$$\int_0^1 2x^2(x+1)\cdots(x+n-1)dx - \int_0^1 x(x+1)\cdots(x+n-1)dx$$
$$= \int_0^1 (2x-1)x(x+1)\cdots(x+n-1)dx$$
$$= \int_0^{\frac{1}{2}} (2x-1)x(x+1)\cdots(x+n-1)dx$$
$$+ \int_{\frac{1}{2}}^1 (2x-1)x(x+1)\cdots(x+n-1)dx$$
$$\geq \frac{1}{2}\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+n-1\right)\left[\int_0^{\frac{1}{2}}(2x-1)dx + \int_{\frac{1}{2}}^1 (2x-1)dx\right]$$
$$= 0 \quad (n \geq 1).$$

Then we have

$$n - \frac{1}{2} < x_n < n, \quad n \geq 2, \tag{2.3}$$
$$n + \frac{1}{2} < y_n < n+1, \quad n \geq 1. \tag{2.4}$$

(i) For $r > 2$, set

$$p_n = \frac{\sqrt[r]{|a_n|}}{n!}, \quad q_n = \frac{\sqrt[r]{b_n}}{n!},$$
$$u_n = \frac{p_{n+1}}{p_n}, \quad v_n = \frac{q_{n+1}}{q_n}.$$

It is clear that

$$u_n = \frac{1}{n+1}\sqrt[r]{x_n}, \quad v_n = \frac{1}{n+1}\sqrt[r]{y_n}.$$

It is well known that $u_n - u_{n+1} \geq 0$ if and only if $(n+2)^r x_n - (n+1)^r x_{n+1} \geq 0$ and $v_n - v_{n+1} \geq 0$ if and only if $(n+2)^r x_n - (n+1)^r x_{n+1} \geq 0$. In order to prove that $\{u_n\}$ and $\{v_n\}$ are decreasing, we only need to show that $(n+2)^r x_n - (n+1)^r x_{n+1} \geq 0$ and $(n+2)^r y_n - (n+1)^r y_{n+1} \geq 0$. Applying

(2.3) and (2.4), we derive

$$\begin{aligned}
&(n+2)^r x_n - (n+1)^r x_{n+1} \\
&> (n+2)^r\left(n-\frac{1}{2}\right) - (n+1)^{r+1} \\
&= (n+1)^r\left[\left(n-\frac{1}{2}\right)\left(1+\frac{1}{n+1}\right)^r - n - 1\right] \\
&> (n+1)^r\left[\left(n-\frac{1}{2}\right)\left(1+\frac{r}{n+1}\right) - n - 1\right] \\
&= (n+1)^{r-1}\left[n\left(r-\frac{3}{2}\right) - \frac{r+3}{2}\right],
\end{aligned}$$

and

$$\begin{aligned}
&(n+2)^r y_n - (n+1)^r y_{n+1} \\
&> (n+2)^r\left(n+\frac{1}{2}\right) - (n+1)^r(n+2) \\
&= (n+1)^r\left[\left(n+\frac{1}{2}\right)\left(1+\frac{1}{n+1}\right)^r - n - 2\right] \\
&> (n+1)^r\left[\left(n+\frac{1}{2}\right)\left(1+\frac{r}{n+1}\right) - n - 2\right] \\
&= (n+1)^{r-1}\left[n\left(r-\frac{3}{2}\right) + \frac{r-3}{2}\right].
\end{aligned}$$

Since $(n+2)^r x_n - (n+1)^r x_{n+1} \geq 0$ when $n \geq \frac{r+3}{2r-3}$, there exists a positive integer $N_1 \geq \frac{r+3}{2r-3}$ such that $(n+2)^r x_n - (n+1)^r x_{n+1} \geq 0$ when $n \geq N_1$. It is clear that $(n+2)^r y_n - (n+1)^r y_{n+1} \geq 0$ when $n \geq 1$. On the other hand, we find that $(n+2)^r y_n - (n+1)^r y_{n+1} > 0$ for $n = 0$. Since the sequences $\{\sqrt[r]{|a_n|}\}_{n\geq N_1}$ and $\{\sqrt[r]{b_n}\}_{n\geq 0}$ are both log-convex and $\{\frac{\sqrt[r]{|a_n|}}{n!}\}_{n\geq N_1}$ and $\{\frac{\sqrt[r]{b_n}}{n!}\}_{n\geq 0}$ are both log-concave, $\{\sqrt[r]{|a_n|}\}_{n\geq N_1}$ and $\{\sqrt[r]{b_n}\}_{n\geq 0}$ are log-balanced.

(ii) Now we prove that $\{\sqrt{|a_n|}\}_{n\geq 2}$ and $\{\sqrt{b_n}\}_{n\geq 0}$ are log-balanced. We only need to show that

$$\begin{aligned}
(k+2)^2 x_k - (k+1)^2 x_{k+1} &\geq 0 \quad (k \geq 2), \\
(k+2)^2 y_k - (k+1)^2 y_{k+1} &\geq 0 \quad (k \geq 0).
\end{aligned}$$

By means of (2.3) and (2.4), we have

$$
\begin{aligned}
(n+2)^2 x_n - (n+1)^2 x_{n+1} &\geq (n+2)^2\left(n-\frac{1}{2}\right) - (n+1)^3 \\
&= \frac{n^2-2n-6}{2} \\
&> 0 \quad (n \geq 4)
\end{aligned}
$$

and

$$
\begin{aligned}
(n+2)^2 y_n - (n+1)^2 y_{n+1} &\geq (n+2)^2\left(n+\frac{1}{2}\right) - (n+1)^2(n+2) \\
&= \frac{n(n+2)}{2} \\
&> 0 \quad (n \geq 1).
\end{aligned}
$$

On the other hand, we observe that $(k+2)^2 x_k - (k+1)^2 x_{k+1}$ for $k = 2, 3$ and $4y_0 - y_1 > 0$. Then we obtain $(k+2)^2 x_k - (k+1)^2 x_{k+1} \geq 0$ for $k \geq 2$ and $(k+2)^2 y_k - (k+1)^2 y_{k+1} \geq 0$ for $k \geq 0$. Hence, the sequences $\{\sqrt{|a_n|}\}_{n\geq 2}$ and $\{\sqrt{b_n}\}_{n\geq 0}$ are log-balanced. ■

3 Conclusions

For the Cauchy numbers of the first kind $\{a_n\}_{n\geq 0}$ and the Cauchy numbers of the second kind $\{b_n\}_{n\geq 0}$, we mainly discussed the log-convexity of some sequences involving a_n (or b_n). For example, we prove that sequences $\{n|a_n| - |a_{n+1}|\}_{n\geq 1}$, $\{b_{n+1} - nb_n\}_{n\geq 1}$, $\{n|a_n|\}_{n\geq 1}$ and $\{(n+1)b_n\}_{n\geq 0}$ are all log-convex. In particular, we show that $\{\sqrt{|a_n|}\}_{n\geq 2}$ and $\{\sqrt{b_n}\}_{n\geq 0}$ are log-balanced. Our future work is to study the log-behavior of sequences $\{\sqrt[n]{|a_n|}\}_{n\geq 1}$ and $\{\sqrt[n]{b_n}\}_{n\geq 1}$.

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