The Matching Number and Hamiltonicity of Quasi-Claw-Free Graphs

Rao Li
Dept. of mathematical sciences
University of South Carolina Aiken
Aiken, SC 29801
Email: raol@usca.edu

Submitted on Mar. 6, 2017; accepted on Jan. 27, 2018

Abstract

A graph G is quasi-claw-free if it satisfies the property: $d(x, y) = 2 \Longrightarrow$ there exists $u \in N(x) \cap N(y)$ such that $N[u] \subseteq N[x] \cup N[y]$. The matching number of a graph G is the size of a maximum matching in the graph. In this note, we present a sufficient condition involving the matching number for the Hamiltonicity of quasi-claw-free graphs.

2010 *Mathematics Subject Classification*: $05C70$, $05C45$.

Keywords: Matching Number, Hamiltonicity, Quasi-Claw-Free Graph

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. Let $G = (V(G), E(G))$ be a graph. A matching M in G is a set of pair-wise non-adjacent edges. A maximum matching is a matching that contains the largest possible number of edges. The matching number, denoted $m(G)$, of a graph G is the size of a maximum matching. For a vertex u and a vertex subset U in G, we use $N[u]$ to denote the union between u and all the neighbors of u; we use $N_U(u)$ to denote all the neighbors of u in U. For two disjoint vertex subsets S and T in G, we define $E(S,T)$ as $\{st : \text{where } s \in S, t \in T, \text{and } st \in E\}$. For two distinct vertices x and y in a graph G, we use $d(x, y)$ to denote the distance between x and y in G. A graph is called claw-free graph if it does not contain a $K_{1,3}$ as an induced subgraph. The concept of quasi-claw-free graphs was introduced by Ainouche [1]. For two vertices x and y, we define

$J(x,y) = \{ u \in N(x) \cap N(y) : N[u] \subseteq N[x] \cup N[y] \}$. A graph G is quasi-claw-free if it satisfies the property: $d(x,y) = 2 \Longrightarrow J(x,y) \neq \emptyset$. Clearly, every claw-free graph is quasi-claw-free. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle.

The purpose of this note is to present a sufficient condition based on the matching number for the Hamiltonicity of quasi-claw-free graphs. The main result and its proofs are as follows. Some ideas in [3] are used in our proofs.

Theorem 1. Let G be a quasi-claw-free graph of order $n \geq 10$ with matching number m and connectivity κ ($\kappa \geq 2$). If $m \leq 2\kappa$, then G is Hamiltonian.

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1. Suppose G is not Hamiltonian. Since $\kappa \geq 2$, G contains a cycle. Choose a longest cycle C in G and give an orientation on C. For a vertex u on C, we use u^+ (resp. u^-) to denote the successor (resp. predecessor) of u along the direction of C. u^{+2} (resp. u^{-2}) is defined as the successor of u^+ (resp. predecessor of u^-) along the direction of C. For two vertices x and y on C, we use $\overrightarrow{C}[x,y]$ to denote the segment (and set of vertices) of C which is along the direction of C from x to y. Since G is not Hamiltonian, there exists a vertex $x_0 \in V(G)\backslash V(C)$. Let H be a component in $V(G)\backslash V(C)$ such that $x_0 \in V(H)$. Define $N_C(V(H)) := \{u_1, u_2, ..., u_s\}$. We also assume that the order of the appearance of $u_1, u_2, ..., u_s$ agrees with the direction of C. From the proofs of Lemma 2.1 in [4], we have the following true claim.

Claim 1. For each i with $1 \leq i \leq s$, we have $J(x_i, u_i^-) = J(x_i, u_i^+) = \{ u_i \}$ and $u_i^- u_i^+ \in E$, where $x_i \in N_H(u_i)$.

Since C is a longest cycle in G, we have that $|\overrightarrow{C}[u_i, u_{i+1}]| \geq 5$, for each i with $1 \leq i \leq s$, where u_{s+1} is regarded as u_1. Again since C is a longest cycle in G, We further have the following true claim.

Claim 2. For each pair of i and j with $1 \leq i \neq j \leq s$, we have $E(\{u_i, u_i^+, u_i^{+2}\}, \{u_j^+, u_j^{+2}\}) = E(\{ u_i, u_i^-, u_i^{-2} \}, \{ u_j^-, u_j^{-2} \}) = \emptyset$.

Obviously, the edges of $u_1u_1^+$, $u_2^{-2}u_2^-$, $u_2u_2^+$, $u_3^{-2}u_3^-$, ..., $u_su_s^+$, and $u_1^{-2}u_1^-$ form a matching in G. Thus $2\kappa \leq 2s \leq m \leq 2\kappa$. Therefore $2\kappa = 2s = m$.

Claim 3. H consists of the singleton x_0.

Proof of Claim 3. Suppose, to the contrary, that Claim 3 is not true. Then we can find an edge, say e, in H. Then the edges of e, $u_1u_1^+$, $u_2^{-2}u_2^-$, $u_2u_2^+$, $u_3^{-2}u_3^-$, ..., $u_su_s^+$, and $u_1^{-2}u_1^-$ form a matching in G, giving a contradiction of $2\kappa + 1 = 2s + 1 \leq m = 2\kappa$.

Claim 4. $u_i^{+2} = u_{i+1}^{-2}$ for each i with $1 \leq i \leq s$, where u_{s+1} is regarded as u_1. Namely, $C = u_1u_1^+u_1^{+2}u_2^-u_2u_2^+u_2^{+2}u_3^{-1}u_3 \dots u_su_s^+u_s^{+2}u_1^-u_1$.

Proof of Claim 4. Suppose, to the contrary, that there exists one i with $1 \leq i \leq s$ such that $u_i^{+2} \neq u_{i+1}^{-2}$. Without loss of generality, we assume that $u_1^{+2} \neq u_2^{-2}$. Then the edges of x_0u_1, $u_1^+u_1^{+2}$, $u_2^{-2}u_2^-$, $u_2u_2^+$, $u_3^{-2}u_3^-$, ..., $u_su_s^+$, and $u_1^{-2}u_1^-$ form a matching in G, giving a contradiction of $2\kappa + 1 = 2s + 1 \leq m = 2\kappa$.

Claim 5. If $V(G)\backslash(V(C) \cup \{x_0\})$ is not empty, then $V(G)\backslash(V(C) \cup \{x_0\})$ is an independent set.

Proof of Claim 5. Using the similar arguments as the ones in the proofs of Claims 1, 2, and 3, we can prove that Claim 5 is true.

Claim 6. If the independent set $V(G)\backslash(V(C) \cup \{x_0\}) := \{w_1, w_2, ..., w_r\}$ is nonempty, then $N_C(w_i) = \{u_1, u_2, ..., u_s\}$ for each i with $1 \leq i \leq r$.

Proof of Claim 6. Suppose, to the contrary, that there exists one i with $1 \leq i \leq s$ such that $N_C(w_i) \neq \{u_1, u_2, ..., u_s\}$. Without loss of generality, we assume that $N_C(w_1) \neq \{u_1, u_2, ..., u_s\}$. Using the similar arguments as the ones in the proofs of Claim 4, we can prove that

$$C = z_1z_1^+z_1^{+2}z_2^-z_2z_2^+z_2^{+2}z_3^{-1}z_3 \dots z_sz_s^+z_s^{+2}z_1^-z_1,$$

where $N_C(w_1) = \{z_1, z_2, ..., z_s\}$. Since $N_C(w_1) \neq \{u_1, u_2, ..., u_s\}$, we must have that $N_C(x_1) = \{z_1, z_2, ..., z_s\} = \{u_1^+, u_2^+, ..., u_s^+\}$ or $\{u_1^{+2}, u_2^{+2}, ..., u_s^{+2}\}$ or $\{u_1^-, u_2^-, ..., u_s^-\}$. In each of those cases, we can easily find a cycle in G which is longer than C, giving a contradiction.

Claim 7. $V(G) = (V(C) \cup \{x_0\})$.

Proof of Claim 7. Suppose, to the contrary, that $V(G) \neq (V(C) \cup \{x_0\})$. Choose one vertex, say w_1, in $V(G)\backslash(V(C) \cup \{x_0\})$. Then $d(x_0, w_1) = 2$. Since G is quasi-claw-free, $J(x_0, w_1) \neq \emptyset$. Let a be an element in $J(x_0, w_1) \neq \emptyset$. Then a must be in $\{u_1, u_2, ..., u_s\}$. Without loss of generality, we assume that $a = u_1$. Since $u_1^+ \in N[u_1] \subseteq N[x_0] \cup N[w_1]$, we have

$u_1^+x_0 \in E$ or $u_1^+w_1 \in E$. In either of the two cases, we can easily find a cycle in G which is longer than C, giving a contradiction.

Notice that u_1^{+2} just can be adjacent to u_1^+ and u_1^-. Thus $d(u_1^{+2}) = 2$. Therefore $2 \leq \kappa \leq \delta \leq d(u_1^{+2}) = 2$. Hence $\kappa = 2$ and $n = 9$, a contradiction.

So we complete the proof of Theorem 1.

Obviously, Theorem 1 has the following corollary.

Corollary 1. Let G be a claw-free graph of order $n \geq 10$ with matching number m and connectivity κ ($\kappa \geq 2$). If $m \leq 2\kappa$, then G is Hamiltonian.

References

[1] A. Ainouche, Quasi-claw-free graphs, *Discrete Mathematics* **179** (1998), 13 - 26.

[2] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York (1976).

[3] R. Li, The matching number and Hamiltonicity of graphs, manuscript, Feb. 2017.

[4] M. Zhan, Vertex pancyclicity in quasi-claw-free graphs, *Discrete Mathematics* **307** (2007), 1679 - 1683.