Power Domination in Certain Nanotori

J. Anitha
Department of Mathematics,
Easwari Engineering College, Chennai, India-600 089.
anithaharish78@gmail.com

Abstract

A set S of vertices in a graph G is called a dominating set of G if every vertex in $V(G)\backslash S$ is adjacent to some vertex in S. A set S is said to be a power dominating set of G if every vertex in the system is monitored by the set S following a set of rules for power system monitoring. The power domination number of G is the minimum cardinality of a power dominating set of G. In this paper, we solve the power domination number for certain nanotori such as H-Naphtelanic, $C_5C_6C_7[m,n]$ nanotori and $C_4C_6C_8[m,n]$ nanotori.

Keywords: Power domination; H-Naphtelanic$[m,n]$ nanotori; $C_5C_6C_7[m,n]$; $C_4C_6C_8[m,n]$ nanotori.

1 Introduction

The concept of power domination in graphs arises from the problem of monitoring electrical systems. Electric systems wish to constantly investigate the nature of the system by placing Phase Measurement Units, called PMUs, at selected regions in the system. The cost of such a synchronized devise is very high, and hence it is required to fetch a smallest set of devises while maintaining the ability to supervising the entire system. In 2002, Hayens et al. [1] considered this problems as the *power domination problem* in graphs which is a variation of the *domination problem.* Indeed, an electric power network can be modeled by a graph where the vertices represent the electric nodes and the edges are associated with the transmission lines joining two electrical nodes. In this model, the power domination problem in graphs consists of finding a minimum set of vertices from where the entire graph can be observed according to certain rules. In terms of the physical network, such a minimum set of vertices will provide the locations where the PMUs should be placed in order to monitor the entire graph at minimum cost [3]. In 2012, Paul Dorbec et al., presented the idea of k-power domination problem which is a generalization of power domination problem in graphs [2].

A graph $G = (V, E)$ is an ordered pair formed by a finite nonempty set of vertices $V = V(G)$ and a set of edges $E = E(G)$ containing unordered pairs of distinct vertices. The order of G is denoted by $|G| = |V(G)|$. We say that the vertices u and v are adjacent or are neighbours, if $\{u, v\} \in E$. For $v \in V(G)$, the open neighbourhood of v, denoted as $N(v)$, is the set of vertices adjacent with v; and the closed neighbourhood of v, denoted by $N[v]$, is $N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighbourhood of S is defined as $N(S) = \bigcup\limits_{v \in S} N(v)$ and the closed neighbourhood of S is defined as $N[S] = N(S) \cup S$ [3].

A vertex v in a graph G is said to dominate itself and all of its neighbors in G. A set of vertices S is a dominating set of G if every vertex of G is dominated by a vertex in S. The minimum cardinality of a dominating set is the domination number of G and is denoted by $\gamma(G)$ [3].
In 2002 Haynes et al. introduced the related concept of power domination by presenting propagation rules in terms of vertices and edges in a graph. Let $G(V, E)$ be a graph and let $S \subseteq V(G)$. We define the sets $M^i(S)$ of vertices monitored by S at level i, $i \geq 0$, inductively as follows:
1. $M^0(S) = N[S]$.
2. $M^{i+1}(S) = M^i(S) \cup \{w : \exists v \in M^i(S), N(v) \cap (V(G) \backslash M^i(S)) = w\}$.
If $M^\infty(S) = V(G)$, then the set S is called a power dominating set of G. The minimum cardinality of a power dominating set in G is called the power domination number of G written $\gamma_p(G)$ [3].

The power domination problem is NP-complete [1]. In fact, the problem has been shown to be NP-complete even when restricted to bipartite graphs and chordal graphs [1]. Lower bound and upper bound on the power domination number for any graph G, $1 \leq \gamma_p(G) \leq \gamma(G)$, was obtained in [1]. In the same paper, power domination problem is well studied for trees [1]. In 2005 Liao et al. have shown that the problem of finding the power domination number for split graphs, a subclass of chordal graphs, is NP-complete. In addition, they present a linear time algorithm for finding power domination number of an interval graph G, if the interval ordering of the graph is provided, and also provided an algorithm with $O(n\ logn)$ time complexity. They also discussed that the same results hold for the class of proper circular-arc graphs[4]. The power domination number is completely determined for grids [5]. In 2006 Zhao et al. investigated claw-free cubic graphs. Moreover, the power domination number satisfies $\gamma_p(G) \leq \frac{n}{3}$ for any graph G of order $n \geq 3$, and extremal graphs that attained these upper bounds were characterized[6]. The power domination problem is also solved for block graphs [7], product graphs [8], cylinder, torus and generalized Petersen graphs [9], certain chemical structures [10], honeycomb network [3], hexagonal grid [11] and so on.

In 2012, Chang et al. [2] extended the power domination problem to k-power domination problem and they obtained the following results for a connected graph G. If G is connected and $\Delta(G) \leq k+1$, then $\gamma_{p,k}(G) = 1$. In the same paper, an upper bound for the k-power domination number of any connected graph G of order n $\gamma_{p,k}(G) \leq \frac{n}{k+2}$ [2]. Also any claw-free $(k+2)$-regular graph of order n satisfies $\gamma_{p,k}(G) \leq \frac{n}{k+3}$ [12]. The k-power domination has been well studied for trees [2], regular graphs [12], Sierpinski graphs [15], hyper graphs [14] and block graphs [13]. In general, the k-power domination problem is NP-complete [2].

2 Main Results

In this section, we compute the power domination number for H-Naphtalenic $[m,n]$ nanotori, $C_4C_6C_8[m,n]$ nanotori and $C_5C_6C_7[m,n]$.

Stephen et al. [10] have given a lower bound for the power domination number for any graph G and is quoted below.

Theorem 2.1. *Let $H_1, H_2, \ldots, H_k$ be pairwise disjoint subgraphs of G satisfying the following conditions.*
1. *$V(H_i) = V_1(H_i) \cup V_2(H_i)$ where $V_1(H_i) = \{x \in V(H_i) \backslash x \sim y \; for \; some \; y \in V(G) - V(H_i)\}$ and $V_2(H_i) = \{x \in V(H_i) \backslash x \nsim y \; for \; all \; y \in V(G) - V(H_i)\}$.*
2. *$V_2(H_i) \neq \emptyset$ and for each $x \in V_1(H_i)$, there exists at least 2 vertices in $V_2(H_i)$ which are adjacent to x.*

If $V_1(H_i)$ is monitored and if l_i is the minimum number of vertices required to monitoring $V(H_i)$, then $\gamma_p \geq \sum_{i=1}^{k} l_i$.

2.1 H-Naphtalenic $[m,n]$ nanotori

A H-Naphtalenic $[m,n]$ nanotori is a trivalent decoration made by alternating squares C_4, pair of hexagons C_6 and octagons C_8. It is a bi-regular graph with m number of rows and n number of columns, each column comprising of the pair of hexagons C_6 viewed vetically and each row comprising of the pair of hexagons C_6 viewed horizontally. Each column of $G \cong H$-Naphtalenic $[m,n]$ nanotori comprises of 5 levels of disjoint sets of vertices, viewed vertically and the $5n$ levels of vertices labeled as level 1, level 2, ...,level $5n$ from left to right [16]. See Figure 1(b). In this section, for convenience, we write H-Naphtalenic $[m,n]$ nanotori simply as Naphtalenic $[m,n]$ nanotori.

The following lemma establishes a critical subgraph H of G in the sense that H contains at least one vertex of any power dominating set of G.

Lemma 2.2. *Let G be a graph and H be a subgraph G as shown in Figure 1(a). Then H is a critical subgraph of G.*

Proof. Suppose Row i, $1 \leq i \leq m$ does not contain any member of a power dominating set, then each vertices u_{ij}, $1 \leq j \leq 2$ is adjacent to two unmonitored vertices in Row i. □

Lemma 2.3. *Let G be a Naphtalenic $[m,n]$ nanotori, $m \leq n$, $m,n \geq 2$. Then $\gamma_p(G) \geq m$.*

Proof. In Naphtalenic $[m,n]$ nanotori, there are m vertex disjoint copies of H as described in Lemma 2.2. By taking $H_i^{'}s$ of Theorem 2.1 as the subgraph H in G, we conclude $l_i = 1$. Hence $\gamma_p(G) \geq m$. □

The Algorithm given below computes the power domination number in Naphtalenic $[m,n]$ nanotori.

Input: Let G be a Naphtalenic $[m,n]$ nanotori, $m \leq n$, $m,n \geq 2$.

Algorithm: Label the vertices of Naphtalenic $[m,n]$ nanotori, $m,n \geq 2$ as 1 to $30mn$ sequentially from top to bottom, level wise beginning with the top most vertex of level 1. Select the vertices $\{2m+2, 2m+4, 2m+6\ldots, 4m\}$ of column 1 in S. See Figure 1(b).

Output: $\gamma_p(G) = m$.

Proof of Correctness: Let S be the set of vertices labeled $\{2m+2, 2m+4, 2m+6\ldots, 4m\}$. See Figure 1($b$). Now $M^0[S] = \{2, 4, \ldots, 2m, 2m+2, \ldots, 4m, 4m+2, 4m+4, \ldots, 6m\}$. Then the vertices labeled as $\{2m, 2m-2, \ldots, 2, 6m, 6m-2, \ldots, 4m+2\}$ colored blue in $M^0(S)$ is adjacent to exactly one unmonitored vertices say, $\{2m-1, 2m-3, \ldots, 1, 6m-1, 6m-3, \ldots, 4m+1\}$ these vertices monitored in $M^1(S)$. In the next step vertices labeled as $\{1, 2m+3, 2m+5, \ldots, 4m-1\}$ is adjacent to exactly one unmonitored vertices say, $\{2m+1, 4m+3, 4m+4, \ldots, 6m-1\}$ and these vertices monitred in $M^2(S)$. Then the vertex labeled as $2m+1$ in $M^2(S)$ is adjacent to exactly one unmonitored vertex say, $4m+1$ is monitored in $M^3(S)$. Now every vertex in level 3 of column 1 is adjacent to exactly one unmonitored vertex. Proceeding inductively, for every vertex $v \in M^i[S]$, $|N[v] \in M^i[S]| = 1$, at every inductive step $i, i \geq 4$. Now $S = \{2m+2, 2m+4, 2m+6\ldots, 4m\}$ is a power dominating set of Naphtalenic $[m,n]$ nanotori. This implies that, $\gamma_p(Naphtalenic[\text{m, n}]nanotori) = m$. Hence the proof.

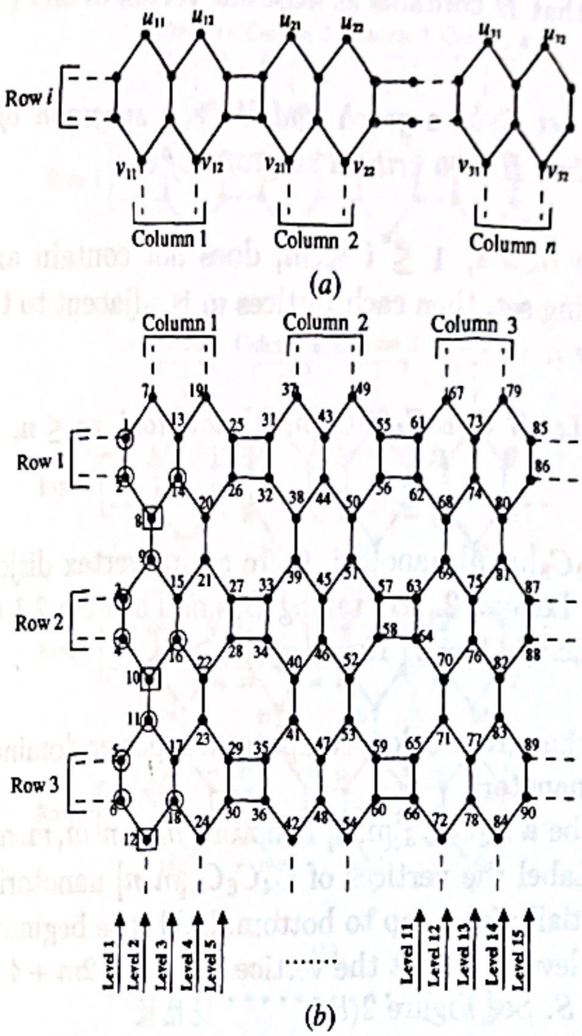

Figure 1: (a) Subgraph induced by Naphtalenic $[m, n]$ nanotori (b) Red colored squared vertices constitutes a Power dominating set of Naphtalenic $[3, 3]$ nanotori

Theorem 2.4. *Let G be a Naphtalenic $[m, n]$, $m \leq n$, $m, n \geq 2$ nanotori. Then $\gamma_p(G) = m$.*

2.2 $C_4C_6C_8[m, n]$ NANOTORI

A $C_4C_6C_8[m, n]$ nanotori is a trivalent decoration made by alternating squares C_4, hexagons C_6 and octagons C_8. It is a bi-regular graph with m number of rows and n number of columns, each column comprising of hexagons C_6 viewed vertically and each row comprising of hexagons C_6 viewed horizontally. Each column of $G \cong C_4C_6C_8[m, n]$ nanotori comprises of 3 levels of disjoint set of vertices, viewed vertically and the $3n$ levels of vertices are labeled as level 1, level 2, ..., level $3n$ from left to right [16]. See Figure 2(b). The following lemma establishes a critical subgraph H of

G in the sense that H contains at least one vertex of any power dominating set of G.

Lemma 2.5. *Let G be a graph and H be a subgraph of G as shown in Figure 2(a). Then H is a critical subgraph of G.*

Proof. Suppose Row i, $1 \leq i \leq m$, does not contain any member of a power dominating set, then each vertices u_i is adjacent to two unmonitored vertices in Row i. □

Lemma 2.6. *Let G be a $C_4C_6C_8[m,n]$ nanotori, $m \leq n$, $m,n \geq 2$. Then $\gamma_p(G) \geq m$.*

Proof. In $C_4C_6C_8[m,n]$ nanotori, there are m vertex disjoint copies of H as described in Lemma 2.. By taking $H_i^{'}s$ of Theorem 2.1 as the subgraph H in G, we conclude $l_i = 1$. Hence $\gamma_p(G) \geq m$. □

The Algorithm given below computes the power domination number in $C_4C_6C_8[m,n]$ nanotori.

Input: Let G be a $C_4C_6C_8[m,n]$ nanotori, $m \leq n$ $m, m, n \geq 2$.

Algorithm: Label the vertices of $C_4C_6C_8[m,n]$ nanotori, $m,n \geq 2$ as 1 to $6mn$ sequentially from top to bottom, level wise beginning with the top most vertex of level 1. Select the vertices $\{2m+2, 2m+4, 2m+6\ldots, 4m\}$ of column 1 in S. See Figure 2(b).

Output: $\gamma_p(G) = m$.

Proof of Correctness: Let S be the set of vertices labeled $\{2m+2, 2m+4, 2m+6\ldots, 4m\}$. See Figure 2(b). Now $M^0[S] = \{2, 4, \ldots, 2m, 2m+2, \ldots, 4m, 4m+2, 4m+4, \ldots, 6m\}$. Then the vertices labeled as $\{2m, 2m-2, \ldots, 2, 6m, 6m-2, \ldots, 4m+2\}$ colored blue in $M^0(S)$ is adjacent to exactly one unmonitored vertices say, $\{2m-1, 2m-3, \ldots, 1, 6m-1, 6m-3, \ldots, 4m+1\}$ and these vertices monitored in $M^1(S)$. In the next step vertices labeled as $\{1, 2m+3, 2m+5, \ldots, 4m-1\}$ is adjacent to exactly one unmonitored vertices say, $\{2m+1, 4m+3, 4m+4, \ldots, 6m-1\}$ and these vertices monitred in $M^2(S)$. Then the vertex labeled as $2m+1$ in $M^2(S)$ is adjacent to exactly one unmonitored vertex say, $4m+1$ is monitored in $M^3(S)$. Now every vertex in level 3 of column 1 is adjacent to exactly one unmonitored vertex. Proceeding inductively, for every vertex $v \in M^i[S]$, $|N[v] \in M^i[S]| = 1$, at every inductive step $i, i \geq 4$. Now $S = \{2m+2, 2m+4, 2m+6\ldots, 4m\}$ is a power dominating set of $C_4C_6C_8[m,n]$ nanotori. This implies that, $\gamma_p(C_4C_6C_8[m,n] nanotori) = m$. Hence the proof.

Theorem 2.7. *Let G be a $C_4C_6C_8[m,n]$, $m \leq n$, $m,n \geq 2$ nanotori. Then $\gamma_p(G) = m$.*

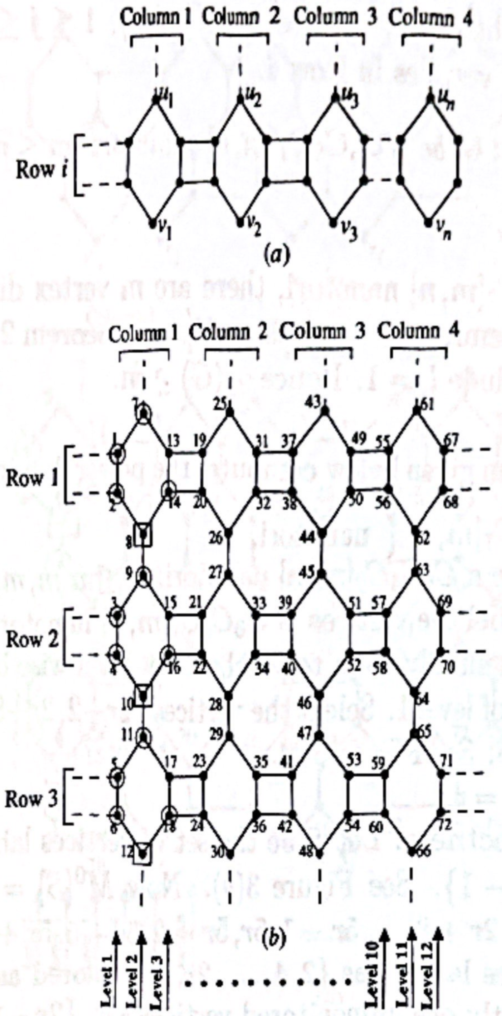

Figure 2: (*a*) Subgraph induced by $C_4C_6C_8[m,n]$ nanotori (*b*) Red colored squared vertices constitutes a Power dominating set of Naphtalenic $C_4C_6C_8[m,n]$ nanotori

2.3 $C_5C_6C_7[m,n]$ NANOTORI

A $C_5C_6C_7[m,n]$ nanotori is a trivalent decoration made by alternating pentagon C_5, hexagon C_6 and heptagons C_7 and is a bi-regular graph. We denote the number of pentagons in the first rows by m. In this nanotori three first rows of vertices and edges are repeated alternatively, and we denote this number by n. In each period there are $16mn$ vertices and $2m$ vertices which are joined to the end of the graph and hence the number of vertices in the nanotori is equal to $16mn + 2m$ [16] .

The following lemma establishes a critical subgraph H of G in the sense that H contains at least one vertex of any power dominating set of G.

Lemma 2.8. *Let G be a graph and H be a subgraph of G as shown in Figure 3(a). Then H is a critical subgraph of G.*

Proof. Suppose Row i, $1 \leq i \leq r$ does not contain any member of a power dominating set, then each vertices of u_j and v_j, $1 \leq j \leq 2m$ is adjacent to two unmonitored vertices in Row i. □

Lemma 2.9. *Let G be a $C_5C_6C_7[m,n]$ nanotori, $m \leq n$, $m,n \geq 2$. Then $\gamma_p(G) \geq r$.*

Proof. In $C_5C_6C_7[m,n]$ nanotori, there are m vertex disjoint copies of H as described in Lemma 2.8. By taking $H_i^{'}s$ of Theorem 2.1 as the subgraph H in G, we conclude $l_i = 1$. Hence $\gamma_p(G) \geq m$. □

The Algorithm given below computes the power domination in

$C_5C_6C_7[m,n]$ nanotori.

Input: Let G be a $C_5C_6C_7[m,n]$ nanotori, $m \leq n$ $m, m,n \geq 2$.

Algorithm: Label the vertices of $C_5C_6C_7[m,n]$ nanotori, $m,n \geq 2$ as 1 to $16mn + 2m$ sequentially from top to bottom, level wise beginning with the top most vertex of level 1. Select the vertices $\{2r+2, 2r+5, 2r+8\ldots, 5r-1\}$ of column 1 in S. See Figure 3(b).

Output: $\gamma_p(G) = r$.

Proof of Correctness: Let S be the set of vertices labeled $\{2r+2, 2r+5, 2r+8\ldots, 5r-1\}$. See Figure 3($b$). Now $M^0[S] = \{2,4,\ldots,2r, 2r+1, 2r+2, 2r+5, 2r+8\ldots, 5r-1, 5r, 5r+2, 5r+5, 5r+8, \ldots, 8r-1, 8r\}$. Then the vertices labeled as $\{2,4,\ldots,2r,5r\}$ colored as blue in $M^0(S)$ is adjacent to exactly one unmonitored vertices say, $\{2r-1, 2r-3, \ldots, 1, 8r\}$ and these vertices are monitored in $M^1(S)$. In the next step vertex labeled as $\{1,3,\ldots,2r,5r\}$ is adjacent to exactly one unmonitored vertices say, $\{2r+1, 2r+4, \ldots, 5r-2\}$ and these vertices monitred in $M^2(S)$. Then the vertex labeled as $2r+1$ in $M^2(S)$ is adjacent to exactly one unmonitored vertex say, $5r+1$ is monitored in $M^3(S)$. Now every vertex in level 3 of column 1 is adjacent to exactly one unmonitored vertex. Proceeding inductively, for every vertex $v \in M^i[S]$, $|N[v] \in M^i[S]| = 1$, at every inductive step $i, i \geq 4$. Now $S = \{2r+2, 2r+5, 2r+8\ldots, 5r-1\}$ is a power dominating set of $C_5C_6C_7[m,n]$ nanotori. This implies that, $\gamma_p(C_5C_6C_7[m,n] nanotori) = r$. Hence the proof.

Theorem 2.10. *Let G be a $C_5C_6C_7[m,n]$, $m,n \geq 2$ nanotori. Then $\gamma_p(G) = r$.*

3 Conclusion

In this paper, we have obtained the power domination number for H-Naphtelanic, $C_5C_7[m,n]$ nanotori, $C_5C_6C_7[m,n]$ nanotori and $C_4C_6C_8[m,n]$

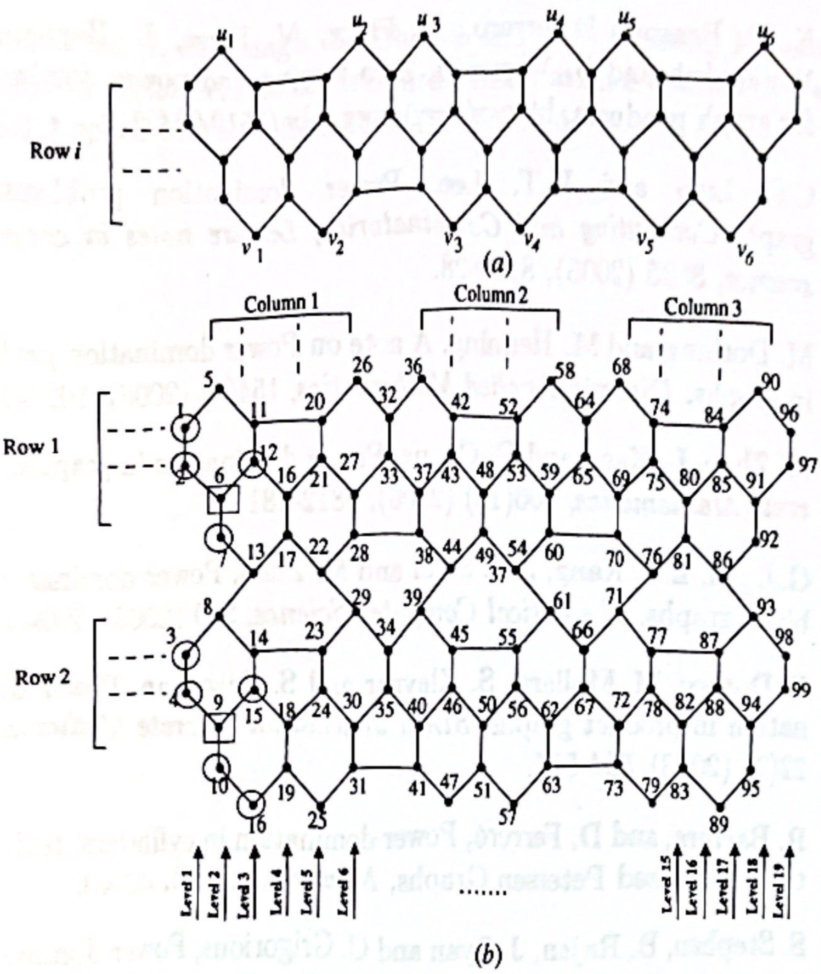

Figure 3: (a) Subgraph induced by $C_5C_6C_7[m,n]$ nanotori (b) Red colored squared vertices constitutes a Power dominating set of Naphtalenic $C_5C_6C_7[m,n]$ nanotori

nanotori.

References

[1] T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi and M.A. Henning, Power domination in graphs applied to electrical power networks, *SIAM Journal on Discrete Mathematics*, 15(4) (2002), 519-529.

[2] G.J. Chang, P. Dorbec, M. Montassier and A. Raspud, Generalized power domination of graphs, *Discrete Applied Mathematics*, 160(12) (2012), 1691-1698.

[3] K. F. Benson, D.Ferrero, M. Flagg, V. Furst, L. Hogben, V. Vasilevskak and B. Wissman, Zero forcing and power domination for graph products, https://arxiv.org/abs/1510.02421.

[4] C.S. Liao and D.T. Lee, Power domination problems in graphs. *Computing and Combinatorics, Lecture notes in computer science*, 3595 (2005), 818-828.

[5] M. Dorfling and M. Henning, A note on Power domination problem in graphs, *Discrete Applied Mathematics*, 154(6) (2006), 1023-1027.

[6] M. Zhao, L. Kang and G. Chang, Power domination in graphs, *Discrete Mathematics*, 306(15) (2006), 1812-1816.

[7] G.J. Xu, L.Y. Kang, E.F. Shan and M. Zhao, Power domination in block graphs, *Theoretical Computer Science*, 359 (2006), 299-305.

[8] P. Dorbec, M. Mollard, S. Klavzar and S. Spacapan, Power domination in product graphs, *SIAM Journal on Discrete Mathematics*, 22(2) (2008), 554-567.

[9] R. Barrera, and D. Ferrero, Power domination in cylinders, tori, and the generalized Petersen Graphs, *Networks*, (2009), 43-49.

[10] S. Stephen, B. Rajan, J. Ryan and C. Grigorious, Power domination in certain chemical structures, *Journal of Discrete Algorithms*, 33 (2015), 10-18.

[11] P. Bose, C.Pennarun and S. Verdonschot, Power domination on triangular grids, http://arxiv.org/abs/1707.02760.

[12] P. Dorbec, M.A. Henning, C. Lowenstien, M. Montassier and A. Raspaud, Generalized power domination in regular graphs, *SIAM Journal on Discrete Mathematics*, 27(3) (2013) 1559-1574.

[13] C. Wang, L. Chen and C. Lu, k-power domination in block graphs, *Journal of Combinatorial Optimization*, 31(2) (2016), 865-873.

[14] G.J. Chang and N. Roussel, On the k-power domination of hypergraphs, *Journal of Combinatorial Optimization*, 30(4) (2015), 1095-1106.

[15] P. Dorbec and S. Klazvar, Generalized power domination: Propagation radius and Sierpinski graphs, *Acta Applicandae Mathematicae*, 1 (2014), DOI-10.1007/s10440-014-9870-7.

[16] P. Manuel, I. Rajasingh, R. Bharati and A.S. Shanthi, Excessive index of certain chemical structures, International Journal of Pure and Applied Mathematics, 84(2) (2013), 39-48.