Construction of Non-isomorphic Families of Halin Graphs with Same Split Domination Numbers

M.Priyadharshini[1], D.Anandhababu[2]
and A.Anuradha[3]
Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur, Tamil Nadu.
[1]priyadharshini.m@ktr.srmuniv.ac.in,
[2]anandhababu.d@ktr.srmuniv.ac.in,
[3]anuradha.a@ktr.srmuniv.ac.in

Abstract

Split domination number of a graph is the cardinality of a minimum dominating set whose removal disconnects the graph. In this paper, we define a special family of Halin graphs and determine the split domination number of those graphs. We show that the construction yield non-isomorphic families of Halin graphs but with same split domination numbers.

1 Introduction

All the graphs, which are dealt with this paper, are connected, finite, simple and undirected [1]. Domination theory in graphs is an important field which helps in optimization of the number of vertices and edges involved in a graph. This parameter is hence found to be applied in interconnection networks and communication theory. For a detailed survey on domination theory, the reader is referred to [2, 3, 8]. Many different domination parameters [2] have been defined and these are found to have various applications such as Locating Radio Stations Problem, Modeling Social Networks, Facility Location Problems and Modeling Biological Networks [7].

Kulli and Janakiram [5], in 1997, defined the parameter split domination number of a graph. Let $G = (V, E)$ be a graph with V as its vertex set and E

as its edge set. Let $|V| = n$ and $|E| = m$. Then $S \subset V$ is a split dominating set of G if each vertex in $V - S$ is adjacent to at least one vertex of S and the induced graph $< V - S >$ is a disconnected graph. The number of vertices in a minimum split dominating set is the split domination number, $\gamma_S(G)$, of G [5]. This number does not exist for all families of graphs. If $diam(G) = 1$ for a graph G, then there will exist no split dominating set for G. Therefore this parameter can be determined only for graphs with diameter at least two when the graphs are connected.

Split domination number for certain families of graphs have been determined by Tamizh Chelvam and R. Chellathurai [8], V. R. Kulli and Janakiram [5], Stephen T.Hedetnimi [7] and others [6]. Bounds for split domination number have been determined based on diameter, degree, chromatic number, independence number, etc [3]. For basic definitions and properties of the above mentioned parameters, the readers are referred to [1]. Throughout this paper, n, m denote respectively the order and size of a graph.

Halin graphs were introduced by the German Mathematician, Rudolf Halin in 1971. Prior to him, these graphs were also studied independently by Thomas Kirkman [4] in 1856 and by Hans Rademacher in 1965. Let T be a tree. Then clearly, T contains at least two vertices of degree one. Now construct a graph H from T by joining all degree 1 vertices into a cycle. This graph H is called the Halin graph. The order in which the degree 1 vertices are joined is subjective, hence for any tree with more than three pendant vertices, many Halin graphs can be constructed. Halin graphs are 3-connected, hamiltonian and planar.

Consider a graph G. Let $S \subset V(G)$ satisfy the following two conditions:

1. Each vertex in S has a private neighbor with respect to $V(S)$ and
2. If $|V - S| > 1$, then $< V - S >$ is a disconnected graph.

Then S is called a split irredundant set. The minimum cardinality of a maximal split irredundant set is called the split irredundant number of G and is denoted by $ir_S(G)$.

We now define two special subfamilies of Halin graphs. Let T_ℓ denote a tree with a root vertex with degree 1 whose adjacent vertex is degree 2 and each of these two vertices have degree 3 and so on. The branches extend till we have $\ell!$ vertices of degree 1. An example of T_4 is given in Figure 1. Clearly the order of a T_ℓ tree is $n = \sum_{j=0}^{k} j!$ and $m = n - 1 = \sum_{j=1}^{k} j!$.

Let $T(p, \ell)$ denote a tree with a central path with $p+2$ vertices ($p \geq 1$)

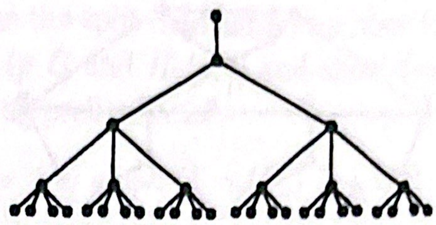

Figure 1: T_4

and each internal vertex of the central path is the root vertex of a tree T_ℓ. The tree $T(3,3)$ is given in Figure 2. Now the pendant vertices of this tree $T(p,\ell)$ can be joined in many ways to form Halin graphs. Two such ways are defined here for $p \geq 2$ and shown that they are non-isomorphic.

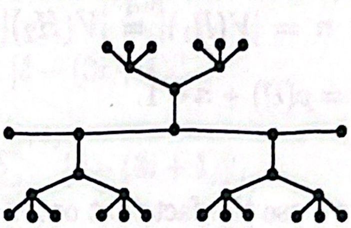

Figure 2: $T(3,3)$

Given $T(p,\ell)$, $p \geq 2$, make the branches lie in the same side of the central path and join all their pendant vertices as the order of their root vertices in the central path and finally form the cycle by joining with the initial and terminal vertices of the central path. The resulting graph is denoted by $H_1(p,\ell)$. The Halin graph $H_1(3,3)$ is given in Figure 3.

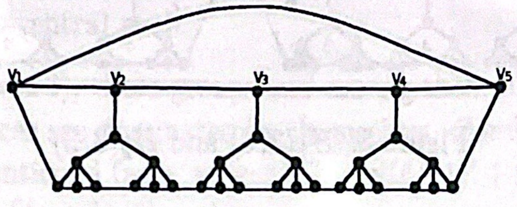

Figure 3: $H_1(3,3)$

In the tree $T(p,\ell)$, let $v_1, v_2, v_3, \ldots v_{p+2}$ denote the vertices in the central path. Let the branches of those internal vertices with even suffices be kept to one side of the central path and the remaining branches to the other side. Now form the Halin graph by joining the pendant vertices at each side separately and then these two paths are joined through vertices v_1 and v_{p+2}. This graph is denoted by $H_2(p,\ell)$, $p \geq 2$. The Halin graph $H_2(3,3)$ is given in Figure 5.

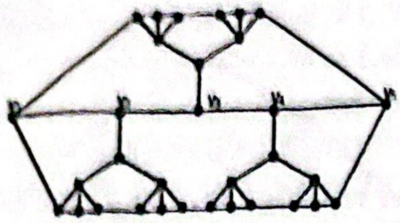

Figure 4: $H_2(3,3)$

2 Main Results

Proposition 2.1. *The order and size of the Halin graphs $H_1(p,\ell)$ and $H_2(p,\ell)$ are as follows:* $n = |V(H_1)| = |V(H_2)| = p+2+p\sum_{k=0}^{\ell} k!$ *and* $m = |E(H_1)| = |E(H_2)| = p(\ell!)+n-1$.

Proof. To determine n, we use the fact that order of a T_ℓ tree is $\sum_{k=0}^{\ell} k!$ and each of the p internal vertices represents a T_ℓ tree. Further the central path has $p+2$ vertices. The number of pendant vertices in each T_ℓ is $\ell!$. Hence there are $p(\ell!)+2$ pendant vertices in $T(p,\ell)$ which form a cycle. These $p(\ell!)+2$ edges constitute the outer cycle. The remaining edges form the tree $T(p,\ell)$. This proves that $m{=}p(\ell!)+n-1$. □

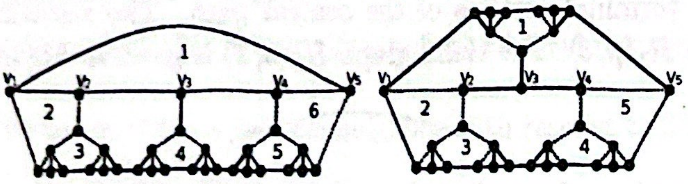

Figure 5: $H_1(3,3)$ and $H_2(3,3)$

Consider the graphs $H_1(3,3)$ and $H_2(3,3)$. If they are isomorphic, then the number of cycles of fixed length will be the same in both of them. Consider the number of cycles of length 5 in both the graphs. The cycles are shown by dotted edges in Figure (5). Clearly there are six such cycles in H_1 and only five such cycles in H_2. This shows that $H_1(3,3) \not\cong H_2(3,3)$. This argument can be extended in general when $p \geq 2$ and hence the theorem follows.

Theorem 2.2. *Given $p \geq 2$, the Halin graphs $H_1(p,\ell)$ and $H_2(p,\ell)$ are non-isomorphic.* □

We now determine the split domination number for the Halin graphs in both the families $H_1(p,\ell)$ and $H_2(p,\ell)$ and show that they are same. We first need the following preliminaries.

Lemma 2.3. *[8] For any graph G, $\gamma_s(P_n) = \lceil n/3 \rceil$, $n > 2$ where P_n is a path of length $n-1$.* □

Lemma 2.4. *[7] For any graph G, $ir_s(G) \leq \gamma_s(G)$.* □

Theorem 2.5. *The split domination number of the Halin graph H, which belongs to the family $H_1(p,\ell)$, is given by*

$$\gamma_s(H) = \begin{cases} \lceil (p+2)/3 \rceil + p \sum_{i=0}^{\lfloor (\ell-1)/3 \rfloor} [\ell - (3i+1)]!, & \text{if } \ell \equiv 0 (mod\, 3) \\ p \sum_{i=0}^{\lfloor (\ell-1)/3 \rfloor} [\ell - (3i+1)]!, & \text{if } \ell \equiv 1 (mod\, 3) \\ 2 + p \sum_{i=0}^{\lfloor (\ell-1)/3 \rfloor} [\ell - (3i+1)]!, & \text{if } \ell \equiv 2 (mod\, 3). \end{cases}$$

Proof. The proof is by finding a split dominating set and showing that this set is minimum. Let v be an internal vertex of the central path of H. Consider the vertices that lie in the ℓ, $\ell-1$ and $\ell-2$ stages of the tree T_ℓ that is adjacent to v. These vertices are dominated by the vertices in $(\ell-1)$-stage. There are totally $(\ell-1)!$ vertices at this stage in T_ℓ. For the vertices that lie in $(\ell-3)$, $(\ell-4)$ and $(\ell-5)$ stages, the $(\ell-4)!$ vertices that lies in the $(\ell-4)$-stage act as dominating vertices. In this same way, we can show that the dominating vertices dissect the stages of T_ℓ into groups of three. It now remains to determine whether these vertices also dominate the vertices of the central path.

When $\ell \equiv 0 (mod\, 3)$, these groups end among themselves and hence the central path vertices are dominated by themselves. Combining Lemma 2.3, and the above mentioned facts, we get $|S| = p[(\ell-1)! + (\ell-4)! + \ldots + (\ell-(3i+1))! + \ldots] + \lceil (p+2)/3 \rceil$ and hence

$$\gamma_s(H) = \lceil (p+2)/3 \rceil + p \sum_{i=0}^{\lfloor (\ell-1)/3 \rfloor} [\ell - (3i+1)]!.$$

When $\ell \equiv 1 (mod\, 3)$, there is no need for any vertex from the central path to be included in the dominating set. Hence

$$\gamma_s(H) = p \sum_{i=0}^{\lfloor (\ell-1)/3 \rfloor} [\ell - (3i+1)]!.$$

When $\ell \equiv 2(mod\,3)$, all the internal vertices of the central path are dominated and the origin and terminal vertices of the central path are to be included in the dominating set. We thereby prove that

$$\gamma_s(H) = 2 + p \sum_{i=0}^{\lfloor(\ell-1)/3\rfloor} [\ell - (3i+1)]!$$

This set of dominating vertices also behaves as a minimum split irredundant set due to the arrangement of the vertices in stages. Any two vertices in i, j stages, where $i \neq j$, will have private neighbors in $i \pm 1$ and $j \pm 1$ stages. Applying Lemma 2.4, we hence conclude that this set S is a minimum split dominating set. The split property is satisfied due to the fact that when the stages of the tree T_ℓ are put in groups of 3, they are isolated when the set $|S|$ is deleted from H. Hence the theorem follows. □

Theorem 2.6. *Both the families of graphs $H_1(p, \ell)$ and $H_2(p, \ell)$ have the same split domination number.*

Proof. In the course of the proof of Theorem 2.5, the position of the branches does not play a role while determining the split dominating set. Hence the family of graphs $H_2(p, \ell)$ also have the same split domination numbers as that of $H_1(p, \ell)$. □

3 Conclusion

In this paper we have constructed a pair of non-isomorphic Halin graph families whose are split-domination number is an invariant. This concept can be extended to other families of graphs.

References

[1] R. Balakrishnan and K. Ranganathan. A Textbook of Graph Theory. Second Edition, Springer, New York (2012).

[2] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs-Advanced Topics, Marcel Dekker, Inc., New York (1998).

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, (1998).

[4] T. P. Kirkman, "XVII. On the enumeration of x-edra having triedral summits, and an (x-1)-gonal base." Philosophical Transactions of the Royal Society of London 146 (1856): 399-411.

[5] V. R. Kulli, B. Janakiram, The split domination number of a graph, Graph Theory Notes, New York (1997).

[6] Preeti Gupta, Domination in graph with application, Indian Journal of Research, ISSN - 2250-1991.

[7] S.T. Hedetniemi, F. Knoll, R. Laskar, Split Domination, Independence, And Irredundance In Graphs, arXiv:1605.03151v2 [math.CO] 16 May (2016).

[8] T.Tamizh Chelvam, S. R. Chellathurai, A note on split domination number of a graph. Journal of Discrete Mathematical Sciences and Cryptography, 12(2) (2009): 179-86.