Dominator Coloring Changing and Stable Graphs upon Vertex Removal

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Abstract

A dominator coloring is a proper vertex coloring of a graph G such that each vertex is adjacent to all the vertices of at least one color class or either alone in its color class. The minimum cardinality among all dominator coloring of G is a *dominator chromatic number* of G, denoted by $\chi_d(G)$. On removal of a vertex the dominator chromatic number may increase or decrease or remain unaltered. In this paper, we have characterized nontrivial trees for which dominator chromatic number is stable.

Keywords: dominator coloring, proper coloring, domination.

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1 Introduction

Let G be a simple graph, where V is the vertex set, E is the edge set, n is the order of G and m is the size of G. For graph-theoretic terminology we refer to [3].

The *open neighborhood* and *closed neighborhood* of $v \in V$ is the set $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$ respectively. The number of vertices adjacent to a vertex v is called the *degree* of v, denoted by $d_G(v)$. If $d_G(v) = 0$, then v is an *isolated vertex.* If $d_G(v) = 1$, then v is a *leaf* and its adjacent vertex a *support vertex.* A support vertex v is a *strong support vertex* (resp. *weak support*) if the number of leaves adjacent to v is at least two (resp. exactly one). For a set $S \subseteq V$, the *induced subgraph* $\langle S \rangle$ of G is a maximal subgraph such that two vertices in $\langle S \rangle$ are adjacent if and only if they are adjacent in G.

A *dominator coloring*, namely DC, of G is a proper vertex coloring of G in which each vertex dominates some color class or either alone in its color class. The minimum cardinality among all DC of G is a *dominator chromatic number* of G, denoted by $\chi_d(G)$. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a DC of a graph G, where each c_i is a color class. If $|\mathcal{C}| = \chi_d(G)$, then we say that the graph G has a χ_d-coloring. A vertex v is a *solitary vertex* if $\{v\} \in \mathcal{C}$ of G. The set of all vertices which dominates solely the color class $c_1 \in \mathcal{C}$ of G is denoted by $Pn(c_1, \mathcal{C})$. If $v \in Pn(c_1, \mathcal{C})$ and $v \in c_i, i \neq 1$, then $|c_i| \geq 2$. Let $v_i \in c_i$. The color class $\{c_i - \{v_i\}\}$ denotes the removal of a vertex v_i from the color class c_i and the color class $\{c_i \cup \{v_j\}\}$ denotes the inclusion of a vertex v_j in to the color class c_i. The concept of DC was introduced by Hedetniemi et al. [9] and studied further by [1, 5, 6, 7, 8, 12]. It has been proved in [6] that the decision problem for DC is NP-complete on arbitrary graphs. In this paper, we characterize nontrivial trees T for which χ_d-coloring is stable.

2 Preliminary results

On removal of a vertex, the dominator chromatic number may increase or decrease or remain unaltered. Hence we can partition $V(G)$ into subsets as follows.

- If $\chi_d(G - v) = \chi_d(G)$, then v is in V^0.
- If $\chi_d(G - v) < \chi_d(G)$, then v is in V^-.
- If $\chi_d(G - v) > \chi_d(G)$, then v is in V^+.

2.1 Properties of vertices in $V^0 \cup V^-$

Observation 2.1. *If a vertex* $v \in V^-$, *then* $\chi_d(G - v) = \chi_d(G) - 1$.

Proposition 2.2. *If* $\{v\} \notin \mathcal{C}$ *for any* χ_d*-coloring* $\mathcal{C}$ *of* G, *then* v *is in* V^0.

Proof. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a DC of G using χ_d colors such that $\{v\} \notin \mathcal{C}$ and let $v \in c_1$ such that $|c_1| \geq 2$. Now consider a coloring $\mathcal{C}_1 = (\mathcal{C} - c_1) \cup \{c_1 - \{v\}\}$ of $G - v$. Clearly the coloring $\mathcal{C}_1$ is a DC using at most $\chi_d(G)$ colors. Thus $v \in V^0 \cup V^-$. Suppose $\chi_d(G - v) = \chi_d(G) - 1$. Let $\mathcal{C}_1$ be a DC of $G - v$ using $\chi_d(G) - 1$ colors. Then the coloring $\mathcal{C}_2 = \mathcal{C}_1 \cup \{v\}$ to G is a DC using $|\mathcal{C}_1| + 1$ colors. Since $\{v\} \in \mathcal{C}_2$, it follows from hypothesis that $\mathcal{C}_2$ is a DC but not a χ_d-coloring of G. Hence

$\chi_d(G) \le |C_2|-1 = |C_1|+1-1 = |C_1| = \chi_d(G)-1$, which is a contradiction. Thus $\chi_d(G-v) = \chi_d(G)$ and hence $v \in V^0$. □

We now characterize the vertices $v \in V^-$ in a graph G.

Proposition 2.3. *A vertex $v \in V^-$ if and only if $\{v\} \in \mathcal{C}$ for some χ_d-coloring $\mathcal{C}$ of G and $Pn(\{v\}, \mathcal{C}) = \{v\}$ or empty set $\emptyset$.*

Proof. Suppose $v \in V^-$. Then by observation 2.1 we have $\chi_d(G-v) = \chi_d(G)-1$. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a DC of $G-v$ using χ_d colors. Then the coloring $\mathcal{C}_1 = \mathcal{C} \cup \{v\}$ is a χ_d-coloring of G. Hence $Pn(\{v\}, \mathcal{C}_1) = \{v\}$ or empty set $\emptyset$.

Conversely, Suppose $\{v\} \in \mathcal{C}$ for some DC $\mathcal{C}$ of G using χ_d colors and $Pn(\{v\}, \mathcal{C}) = \{v\}$ or empty set $\emptyset$. Then the coloring $\mathcal{C}_1 = \mathcal{C} - \{v\}$ of $G-v$ is a DC using at most $\chi_d(G)-1$ colors. Hence $\chi_d(G-v) \le \chi_d(G)-1$. Which implies that $v \in V^-$. □

2.2 Properties of vertices in V^+

Observation 2.4. *If $v \in V^+$, then v is a solitary vertex and is in every χ_d-coloring of G.*

Proof. The proof follows from proposition 2.2. □

Proposition 2.5. *If $v \in V^+$, then $\{v\} \in \mathcal{C}$ for any χ_d-coloring $\mathcal{C}$ of G and $Pn(\{v\}, \mathcal{C})$ contains at least two non-adjacent vertices.*

Proof. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a DC of G using $\chi_d(G)$ colors in which a vertex $v \in c_1$. By observation 2.4, we have $\{v\} \in \mathcal{C}$. Suppose $Pn(\{v\}, \mathcal{C}) = \emptyset$. Then by proposition 2.3, we have $v \in V^-$, which is a contradiction. Let $u \in Pn(\{v\}, \mathcal{C})$ and let $u \in c_2$. Clearly $|c_2| \ge 2$. If $\langle Pn(\{v\}, \mathcal{C})\rangle$ is complete, then $(\mathcal{C} - \{c_1, c_2\}) \cup \{u\} \cup \{c_2 - \{u\}\}$ of $G-v$ is a DC using at most $\chi_d(G)$ colors. Thus $\chi_d(G-v) \le \chi_d(G)$, which is a contradiction. Hence $Pn(\{v\}, \mathcal{C})$ contains at least two non-adjacent vertices. □

Corollary 2.5.1. $|V^+| \le \left\lfloor \frac{|V|}{3} \right\rfloor$.

Proof. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a DC of G using $\chi_d(G)$ colors and let $v \in V^+$. Then $\{v\} \in \mathcal{C}$ and $Pn(\{v\}, \mathcal{C})$ contains at least two non-adjacent vertices u_1 and u_2. Clearly, the color class containing $u_i, i = 1, 2$ is of cardinality at least two. Thus $\{u_1, u_2\} \subseteq V^0 \cup V^-$ and hence $|V^+| \le \left\lfloor \frac{|V|}{3} \right\rfloor$. □

The above bound is sharp for the graph $G \circ 2K_1$ and also for the graph $G \equiv 2P_6$.

Corollary 2.5.2. *There does not exist a graph G such that dominator chromatic number increases for all $v \in V$.*

3 Trees

In this section, we give a recursive characterization of all the trees for which the deletion of any vertex does not affect the value of the dominator coloring number. In [12] it was shown that "For a tree T of order $n \geq 3$, there exists a χ_d-coloring $\mathcal{C}$ of T such that all leaves of T receive the same color". Also, it has been proved that "For any χ_d-coloring $\mathcal{C}$ of T, either each support vertex or a leaf is a solitary vertex". Throughout this section, we consider a χ_d-coloring $\mathcal{C}$ of a tree T in which each leaf vertex is within the same color class, say c_1, and each support vertex is a solitary vertex.

Proposition 3.1. *Every tree T of order at least three has a vertex $v \in V^0$.*

Proof. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T and let $v \in V(T)$.

Case 1. *Suppose v is a strong support vertex.*

Let v_1 and v_2 be two leaves adjacent to the vertex v. We now claim that in any χ_d-coloring $\mathcal{C}$ of T, $\{v\} \in \mathcal{C}$ of T. Suppose not. Let $v \in c_1$ such that $|c_1| \geq 2$. Clearly $c_2 = \{v_1\} \in \mathcal{C}$ and $c_3 = \{v_2\} \in \mathcal{C}$ of T. Now consider a coloring $\mathcal{C}_1 = (\mathcal{C} - \{c_1, c_2, c_3\}) \cup \{v\} \cup \{c_1 \cup \{v_1, v_2\}\}$ of T using at most $\chi_d(T) - 1$ colors. Clearly, the vertices v, v_1 and v_2 dominates the color class $\{v\} \in \mathcal{C}_1$ of T. The remaining vertices dominate some color class as in $\mathcal{C}$ of T. In case a vertex, say x, dominates the color class $c_1 \in \mathcal{C}$ of T, then x dominates the color class $\{v\} \in \mathcal{C}_1$ of T. Hence $\mathcal{C}_1$ is a DC of T using at most $\chi_d(T) - 1$ colors, which is a contradiction. Thus $\{v\} \in \mathcal{C}$ and is in every χ_d-coloring of T. Then by proposition 2.2 we have $\{v_1, v_2\} \subseteq V^0$.

Case 2. *Suppose v is a weak support vertex.*

Let v_1 and v_2 be the leaf and non-leaf vertices adjacent to v respectively.

Case 2.1. *Suppose $\{v\} \in \mathcal{C}$ of T.*

Clearly $v_1 \in Pn(\{v\},C)$. Let $v_1 \in c_1$ such that $|c_1| \geq 2$. Suppose $v_2 \in Pn(\{v\},C)$. Consider the tree $T' = T - v_1$. Then the coloring $C_1 = (C - c_1) \cup \{c_1 - \{v_1\}\}$ is a DC of T' using at most $\chi_d(T)$ colors. Thus $\chi_d(T') \leq \chi_d(T)$. Suppose $v_1 \in V^-$. Then by proposition 2.3 we have $Pn(\{v_1\},C) = \{v_1\}$ or empty set $\emptyset$, which is a contradiction to our assumption that $v_1 \in c_1$ such that $|c_1| \geq 2$. Hence $\chi_d(T - v_1) = \chi_d(T)$. Suppose $v_2 \notin Pn(\{v\},C)$. Then the vertex v_2 dominates some color class in C of T. The subgraph $T' = T - v$ has exactly one isolated vertex, namely v_1. Then the coloring $C_1 = (C - \{c_1, \{v\}\}) \cup \{v_1\} \cup \{c_1 - \{v_1\}\}$ of T' is a DC using at most $\chi_d(T)$ colors. Thus $\chi_d(T') \leq \chi_d(T)$. Suppose $v \in V^-$. Then by proposition 2.3 we have $Pn(\{v\},C) = \{v\}$ or empty set $\emptyset$. Which is a contradiction. Hence $v \in V^0$.

Case 2.2. *Suppose $\{v\} \notin C$ of T.*

Then $v \in c_2$ such that $|c_2| \geq 2$. Now consider a coloring $C_1 = (C - c_2) \cup \{c_2 - \{v\}\}$ of $T - v$. Suppose there exists a vertex x dominating the color class $c_2 \in C$ of T. Then it still continues to dominate the color class $c_2 \in C_1$ of T. The remaining vertices of $T - v$ dominates some color class as in C of T. Hence $\chi_d(T - v) \leq \chi_d(T)$. Suppose $v \in V^-$. Then by proposition 2.3, we have $\{v\} \in C$ of T. Which is a contradiction to our assumption that $v \in C_2$ such that $|c_2| \geq 2$. Thus $\chi_d(T - v) = \chi_d(T)$. □

By proposition 3.1, it follows that there is no tree T of order $n \geq 3$ for which $V = V^-$ or $V = V^+$. We now characterize the families of trees for which χ_d-coloring is stable, that is $V = V^0$. Let $\mathcal{T}$ be the family of trees constructed as follows. Let $T_1 = P_3$ and $T_2 = P_5$. From $T_k, k \geq 2$, we can construct iteratively a tree T_{k+1} by one of the following operations.

1. Operation $\mathcal{O}_1$: Joining a vertex of path P_2 to a non-leaf vertex adjacent to a weak support vertex of T_k.

2. Operation $\mathcal{O}_2$: Joining a leaf of path P_3 to a non-leaf vertex of T_k, which is a weak support vertex or is adjacent to a path P_3.

Proposition 3.2. *If $V(T) = V^0$, then there exists a χ_d-coloring C of T in which the neighborhood vertices of a support vertex other than leaves can be assigned the color of leaf vertices.*

Proof. Let $C = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T in which all the leaves are assigned the same color class, say c_1, and the support vertices are solitary vertices. Let $v_1, v_2, \ldots, v_k$ be the longest path in T. Clearly $\{v_2\} \in C$ of T. Suppose v_3 is a support vertex in T. Then $\{v_3\} \in C$ of T. Let $T' = T - v_1$. Now consider a coloring $C_1 = (C - \{c_1 \cup \{v_2\}\}) \cup \{(c_1 - \{v_1\}) \cup \{v_2\}\}$

of T'. Clearly, the vertex v_2 dominates the color class $\{v_3\} \in \mathcal{C}_1$ of T' and the remaining vertices dominate some color class as in $\mathcal{C}$ of T. Hence $\mathcal{C}_1$ is a DC of T' using at most $\chi_d(T) - 1$ colors, which is a contradiction. Thus every child of v_3 is a support vertex. Since $V = V^0$, either $\{v_3\} \in \mathcal{C}$ or $\{v_4\} \in \mathcal{C}$ of T. Suppose $\{v_3\} \in \mathcal{C}$ of T. Then by above argument we give a contradiction to $V = V^0$. Hence $\{v_4\} \in \mathcal{C}$ of T. Thus v_3 can be assigned the color of leaf vertices. □

Lemma 3.3. *If a tree $T \in \mathcal{T}$, then $V = V^0$.*

Proof. We prove the result by induction on k operations. Suppose $T = T_1 = P_3$ or $T = T_2 = P_5$. One can easily observe that $V(T) = V^0$. Suppose the result holds for all trees $T' = T_{k+1}$ of $\mathcal{T}$ obtained by $k-1$ operations. Let a tree $T = T_{k+2}$ obtained by k operations be of family $\mathcal{T}$.

In the case of T being constructed from T' by operation $\mathcal{O}_1$. Let a path P_2 has vertices v_1 and v_2 such that v_1 is attached to a vertex of T', say u. Let ab denote path P_2 adjacent to u different from v_1v_2 in such a way that $'a'$ is adjacent to u. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T' in which $\{u\} \in \mathcal{C}$. Now consider a coloring $\mathcal{C}_1 = (\mathcal{C} - c_1) \cup \{v_1\} \cup \{c_1 \cup \{v_2\}\}$ of T. Clearly the vertices v_1 and v_2 dominates the color class $\{v_1\} \in \mathcal{C}_1$ of T and the remaining vertices of T dominates some color class as in $\mathcal{C}$ of T'. Thus $\chi_d(T) \leq \chi_d(T') + 1$. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T in which $\{v_1\} \in \mathcal{C}$. Now consider the graph $T' = T - \{v_1, v_2\}$. One can easily observe that $\mathcal{C}_1 = (\mathcal{C} - (c_1 \cup \{v_1\})) \cup \{c_1 - \{v_2\}\}$ is a DC of T' using at most $\chi_d(T) - 1$ colors. Then $\chi_d(T) = \chi_d(T') + 1$.

We now claim that $\chi_d(T - v) = \chi_d(T)$. In case there is at least one vertex $v \in V(T)$ such that $\chi_d(T - v) \neq \chi_d(T)$. Since the paths v_1v_2 and ab are similar, we may assume that $v \neq v_1, v_2$. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of $T' - v$. Now consider a coloring $\mathcal{C}_1 = (\mathcal{C} - c_1) \cup \{v_1\} \cup \{c_1 \cup \{v_2\}\}$ of $T - v$. Clearly the vertices v_1 and v_2 dominate the color class $\{v_1\} \in \mathcal{C}_1$ of $T - v$ and the remaining vertices dominates some color class as in $\mathcal{C}$ of $T' - v$. Thus $\mathcal{C}_1$ is a DC of $T - v$ using at most $\chi_d(T' - v) + 1$ colors. Hence $\chi_d(T - v) \leq \chi_d(T' - v) + 1$. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of $T - v$. One can easily observe that $\mathcal{C}_1 = (\mathcal{C} - (c_1 \cup \{v_1\})) \cup \{c_1 - \{v_2\}\}$ is a DC of $T' - v$ using at most $\chi_d(T - v) - 1$ colors. Therefore $\chi_d(T' - v) = \chi_d(T - v) - 1 \neq \chi_d(T) - 1 = \chi_d(T')$. Which is a contradiction to our assumption and hence $V = V^0$.

In the case of T being constructed from T' by operation $\mathcal{O}_2$. Let v_2 be the central vertex of P_3 and let v_1 and v_3 be its leaf vertices. Let v_1 of path P_3 be attached to a vertex of T', say u. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T' in which non-leaf vertices adjacent to the support vertices are assigned color 1. Now consider a coloring $\mathcal{C}_1 = (\mathcal{C} - c_1) \cup \{v_2\} \cup \{c_1 \cup \{v_1, v_3\}\}$

of T. Clearly the vertices v_1, v_2 and v_3 dominates the color class $\{v_2\} \in C_1$ of T and the remaining vertices dominate some color class as in C of T'. Thus $\chi_d(T) \leq \chi_d(T')+1$. Let $C = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T and let $v_1 \in c_1$. One can easily observe that $C_1 = (C-(c_1 \cup \{v_2\})) \cup \{c_1 - \{v_1, v_3\}\}$ is a DC of T' using at most $\chi_d(T) - 1$ colors. Then $\chi_d(T) = \chi_d(T') + 1$.

We now claim that $\chi_d(T-v) = \chi_d(T)$. In case there is a vertex $v \in V(T)$ such that $\chi_d(T-v) \neq \chi_d(T)$. Let $C = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of $T'-v$ in which non-leaf vertices adjacent to the support vertices are assigned color 1. If $v = v_1$, then consider a coloring $C_1 = (C - c_1) \cup \{c_1 - \{v_1\}\}$. Clearly the vertices of $T - v_1$ dominates some color class as in C of T. Hence $\chi_d(T - v_1) \leq \chi_d(T)$. Suppose $v_1 \in V^-$. Then by proposition 2.3 we have $\{v_1\} \in C$ and $Pn(\{v_1\}, C) = \{v_1\}$ or empty set $\emptyset$. Which is a contradiction to our assumption that $v_1 \in c_1$. Suppose $v = v_2$. Let $v_1 \in Pn(\{v_2\}, C)$ and $v_3 \in Pn(\{v_2\}, C)$. Now consider the tree $T - v_2$. Let $C_1 = (C - \{c_1 \cup \{v_2\}\}) \cup \{c_1 - \{v_1\}\} \cup \{v_1\}$ be a coloring of $T - v_2$ using at most $\chi_d(T)$ colors. Since $v_3 \in Pn(\{v_2\}, C)$ of T, the vertex v_3 does not dominate any color class of C_1 of T. Thus $\chi_d(T - v_2) > \chi_d(T)$. Which is a contradiction to our assumption. Suppose $Pn(\{v_2\}, C) = \{v_1\}$. Then consider a coloring $C_1 = (C - \{c_1 \cup \{v_2\}\}) \cup \{c_1 - \{v_1\}\} \cup \{v_1\}$. The vertex v_1 dominates itself and the rest of the vertices of $T - v_2$ dominates some color class as in C of T. Hence $\chi_d(T - v_2) \leq \chi_d(T)$. Suppose $v_2 \in V^-$. Then by proposition 2.3 we give a contradiction to our assumption that $Pn(\{v_2\}, C) = \{v_1\}$. Suppose $v = v_3$. Then the proof is similar to $v = v_1$. Now assume that $v \in V(T')$. Consider a tree $T' - v$. Then by above arguments we can show that $\chi_d(T - v) = \chi_d(T' - v) + 1$. Therefore $\chi_d(T' - v) = \chi_d(T - v) - 1 \neq \chi_d(T) - 1 = \chi_d(T')$. Which is a contradiction to our assumption and hence $V = V^0$. □

Lemma 3.4. *If $V = V^0$, then $T \in \mathcal{T}$.*

Proof. If $diam(T) \leq 3$, then one can easily observe that P_3 is the only tree for which $V = V^0$. Thus $T = P_3 \in \mathcal{T}$. Suppose $diam(T) \geq 4$. We prove the result by induction on n. Suppose the result holds for a tree T' of order less than n.

Let v be a support vertex of T adjacent to at least two leaves, say x and y. Let $C = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T in which all the leaves are assigned the same color class, say c_1, and each support vertex is a solitary vertex. Let $x \in c_1$ and $y \in c_1$. It follows from the proof of proposition 3.1 that $\{v\}$ is in every χ_d-coloring of T. Now consider a coloring $C_1 = (C - \{c_1 \cup \{v\}\}) \cup \{x\} \cup \{c_1 - \{x\}\}$ of $T - v$ using at most $\chi_d(T)$ colors. Clearly, the vertex y does not dominate any color class of C_1. Thus $\chi_d(T - v) > \chi_d(T)$, which is a contradiction. Hence each support vertex of T is weak.

Let $\mathcal{P} = (v_1, v_2, v_3, \ldots, v_k)$ be the longest path in T rooted at v_k and let v_1 be a leaf at the farthest distance from v_r, v_1 be the child of v_2, v_2 be the child of v_3 and v_3 be the child of v_4. By Proposition 3.2, every child of v_3 is a support vertex. Let T_v be a subtree induced by a vertex v and its successors in a rooted tree T.

In case $d_T(v_3) \geq 3$. Let $T' = T - T_{v_2}$ and let $C = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T'. Now consider the tree T. It follows from the proof of Lemma 3.3 that $\chi_d(T) = \chi_d(T') + 1$. We now show that each vertex $v \in V(T')$ is in V^0. In case there is at least one vertex $v \in V(T')$ such that $\chi_d(T' - v) \neq \chi_d(T')$. Let $T'' = T' - v$ and let $C_1 = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T''. Now consider a tree $T - v$. Again it follows from the proof of Lemma 3.3 that $\chi_d(T - v) = \chi_d(T' - v) + 1 \neq \chi_d(T') + 1 = \chi_d(T)$, which is a contradiction to our assumption that $V = V^0$. Therefore $V(T') = V^0$. Thus by inductive hypothesis we have $T' \in \mathcal{T}$ and using operation $\mathcal{O}_1$ we obtain T from T'. Which implies that $T \in \mathcal{T}$.

Assume that $d_T(v_3) = 2$ and $d_T(v_4) \geq 3$. Let $C = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T in which all the leaves are assigned the same color class, say c_1, and each support vertex is a solitary vertex. Let $v_3 \in c_1$ in C of T and let u_1 be the child of v_4, other than v_3. It is sufficient to examine the case when T_{u_1} is a path P_2, say u_1u_2. Assume that $\{v_4\} \in C$ of T. Clearly $\{v_2\} \in C$ and $\{u_1\} \in C$ of T. Now consider a tree $T' = T - u_2$. Let $C_1 = (C - (c_1 \cup \{u_1\})) \cup \{(c_1 - u_2) \cup \{u_1\}\}$ be a coloring of T' using at most $\chi_d(T) - 1$ colors. Then the vertex u_1 is adjacent to the color class $\{v_4\} \in C_1$ of T' and the remaining vertices of T' dominate some color class as in C of T. Thus $\chi_d(T') < \chi_d(T)$, which is a contradiction. Now assume that no χ_d-coloring of T contains v_4 as a solitary vertex. Let C be a DC of $T'' = T - v_2$ using χ_d colors in which each leaf is contained in a color class, say c_1 and each support vertex is a solitary vertex. Clearly $\{v_1\} \in C$ and $\{v_4\} \in C$ of T''. Now consider a tree T. Let $C_1 = (C - (c_1 \cup \{v_1\})) \cup \{v_2\} \cup \{c_1 \cup \{v_1\}\}$ be a coloring of T using at most $\chi_d(T'')$ colors. Since no χ_d-coloring of T contains v_4 as a solitary vertex, the coloring C_1 is not a χ_d-coloring of T. Therefore $\chi_d(T) \leq \chi_d(T'') - 1$, which is a contradiction.

Suppose $d_T(v_4) = 2$. Let $C = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T in which all the leaves are assigned the same color class, say c_1, and each support vertex is a solitary vertex. Since $V = V^0$, it follows from the proof of Proposition 3.2 that $\{v_4\} \in C$ of T. Now we claim that v_4 is a support vertex of T. Suppose not. Clearly $Pn(\{v_4\}, C) \neq \{v_4\}$ or empty set $\emptyset$. Then some non-leaf vertex $y \in Pn(\{v_4\}, C)$. Clearly $y \in c_i$ such that $|c_i| \geq 2$. Let $y_1 \in N(y)$. In case y_1 dominates itself,that is $\{y_1\} \in C$, then $\chi_d(T - y_1) < \chi_d(T)$. Since each vertex of T must dominate some color class, the vertex y_1 dominates a color class, say $c_j, j \neq 1$. Let $v_j \in c_j$.

case i. Suppose $|c_j| = 1$.

Let v_k be a leaf vertex adjacent to v_j. Then $\chi_d(T - v_j) > \chi_d(T)$. Which is a contradiction. Suppose v_j is a leaf. Then we can interchange the colors of y_1 and v_j. In this case, the vertex $y \notin Pn(\{v_4\}, \mathcal{C})$. This implies that $Pn(\{v_4\}, \mathcal{C}) = \{v_4\}$ or empty set $\emptyset$. Then by proposition 2.3, we have $v_4 \in V^-$. Which is a contradiction. Suppose v_j is neither a leaf nor a support vertex. Then there exists a vertex $v_l \in N(v_k)$ other than y_1 such that v_l dominates some color class $c_k, k \neq \{1, j\}$. Let $v_m \in c_k$. Suppose $|c_k| = 1$. Then by the above argument, we give a contradiction. Suppose $|c_k| \geq 2$. Then the vertex v_m cannot dominate itself. So the vertex v_m dominates some color class, say $c_l, l \neq \{1, k\}$. If $|c_l| = 1$, then by the above argument we give a contradiction. Suppose $|c_l| \geq 2$. Let $v_\alpha \in c_l$ and $v_\beta \in c_l$. Suppose any two vertices of c_l dominates the same color class, say c_m. Let $v_n \in c_m$. Then the induced subgraph $\langle v_m, v_\alpha, v_\beta, v_n \rangle$ produces a cycle. Which is a contradiction. Hence each vertex of c_l dominates distinct color classes. Suppose v_α dominates a color class, say c_n, such that $|c_n| \geq 2$. Then the process continues. Since T is finite, the process stops by providing a vertex v_i dominating the color class c_α with $|c_\alpha| \geq 2$ such that each $v \in c_\alpha$ dominates a distinct color class of size one. Let $v_{\alpha_1} \in c_\alpha$ and $v_{\alpha_2} \in c_\alpha$. Let v_{α_1} and v_{α_2} dominates a color class c_{α_1} and c_{α_2} respectively. Let $v_a \in c_{\alpha_1}$ and $v_b \in c_{\alpha_2}$. If either v_a or v_b is a support vertex, then by the above argument we give a contradiction. Suppose both v_a and v_b are leaf vertices. Then interchanging the colors of v_{α_1} and v_{α_2} with the colors of v_a and v_b respectively. Thus we assign colors α_1 and α_2 to v_{α_1} and v_{α_2} respectively and assign the color α to the vertices v_a and v_b. Hence $\{v_{\alpha_1}\}$ is dominated by two non-adjacent vertices v_i and v_a. Thus $\{v_{\alpha_1}\} \in V^+$. Which is a contradiction.

case ii. Suppose $|c_j| \geq 2$.

Then as in case i we give a contradiction to our assumption that $V = V^0$. Thus $v_4 \in V(T)$ is a support vertex.

In case no child of v_4 is a support vertex, say u_1. Let $T' = T - T_{v_3}$. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of T'. It follows from the proof of Lemma 3.3 that $\chi_d(T) = \chi_d(T') + 1$. We now show that each vertex $v \in V^0$ in T'. Suppose not. Then $\chi_d(T' - v) \neq \chi_d(T')$ for some vertex $v \in V(T')$. Let $\mathcal{C}_1 = \{c_1, c_2, \ldots, c_k\}$ be a χ_d-coloring of $T'' = T' - v$. Now consider a tree $T - v$. Again it follows from the proof of Lemma 3.3 that $\chi_d(T - v) = \chi_d(T' - v) + 1 \neq \chi_d(T - T_{v_3}) + 1 = \chi_d(T)$. Which is a contradiction to our assumption that $V = V^0$. Therefore each vertex $v \in V(T')$ is in V^0. By inductive hypothesis $T' \in \mathcal{T}$ and using operation

O_2 we obtain T from T'. Which implies that $T \in \mathcal{T}$. □

As a consequence of Lemma 3.3 and 3.4, we obtain the following characterization of trees for which χ_d-coloring is stable.

Theorem 3.5. *Every vertex of a tree T of order at least three is in V^0 if and only if $T \in \mathcal{T}$.*

4 Further research

The following are a few problems for further examination.

Problem 4.1. *Characterize graphs for which $V = V^-$.*

Problem 4.2. *Characterize graphs for which $|V^+| = \left\lfloor \frac{|V|}{3} \right\rfloor$.*

Problem 4.3. *Characterize trees T for which $|V^0| = 2$.*

Problem 4.4. *How the parameter carries on when an edge is expelled or included is an intriguing issue for further examination.*

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