Edge Graceful Labeling of Paley Digraph

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Abstract

A digraph G is finite and is denoted as $G(V, E)$ with V as set of nodes and E as set of directed arcs which is exact. If (u, v) is an arc in a digraph D, we say vertex u dominates vertex v. A special digraph arises in round robin tournaments. Tournaments with a special quality $Q(n, k)$ have been studied by Ananchuen and Caccetta. Graham and Spencer defined tournament with q vertices where $q \equiv 3 \pmod 4$ is a prime. It was named suitably as Paley digraphs in respect deceased Raymond Paley, he was the person used quadratic residues to construct Hadamard matrices more than 50 years earlier. This article is based on a special class of graph called Paley digraph which admits odd edge graceful, super edge graceful and strong edge graceful labeling.

Keywords: Paley Digraph, Odd edge graceful labeling, Super edge graceful labeling and Strong edge graceful labeling.
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1 Introduction

This article deals with special directed graph called Paley digraph, which has named to tribute Raymond Paley. He used quadratic residues to derive the Hadamard Matrices. So Paley digraphs are also called as quadratic residue digraph. This digraph was introduced by Graham and Spencer in 1971 [5]. In this digraph each undersized subset of nodes is dominated by some other node. Since it is a tournament each pair of different vertices is connected by an edge in one and only one direction.

A graph labeling is defined as the set of integers assigned to the graph elements such as vertices and edges on some basic conditions.

Graph labeling that come into sight in graph theory has a quick development recently. Alex Rosa initiated graph labeling around 1967 as certain valuation to graphs [10]. Different labeling techniques have been introduced for the past 50 years and above 2000 papers were published [3].

Graph labeling have frequently been provoked by practical contemplation such as chemical isomers, but they are also of importance in their own right due to their abstract mathematical properties arising from various structural consideration of the fundamental graphs. Since then, research around the world have attempted and successfully developed mathematical analysis of graphs. Meanwhile, scientists and engineers were able to successfully apply these concepts in their own respective fields such as cloud computing, cryptography, software testing, data mining. Graph labeling is used for fast communication in sensor network. It is also used to represent the protein structure and pattern discovery using graph mining, to derive a complete complexity in computational techniques.

Application of graph labeling has been categorized as qualitative labeling and quantitative labeling. Bloom and Golomb have discussed in detail about graph labeling application papers

[1, 2].

Lo [8] introduced edge graceful labeling in the year 1985. For trees k-edge graceful labeling was described by Lee and Wang [7]. For directed graphs edge graceful labeling was extended by Tamizharasi. R, Rajeswari. R [14].

Solairaju et. al introduced edge odd graceful labeling [13]. The admissible of same labeling for some undirected graph is proved by Singhun [12]. Seoud and Salim exhibit edge-odd graceful labeling for W_n with the condition $n \equiv 1, 2$ and 3 (mod 4), $C_n \cdot K_{2m-1}$, P_n, even helms. K_{2m} and $K_{2,s}$. They also provide the non existence of edge-odd graceful labeling for some graph [11].

Mitchem and Simoson introduced Super edge-graceful labeling (SEGL) and they proved that super edge-graceful trees are edge-graceful [6]. Strong edge graceful labeling was initiated by Gayathri and Subbiah [4]. Thamizharasi. R and Rajeswari. R extended the above said labeling for directed graphs [15, 16].

Parameswari. R and Rajeswari. R [9] discussed about k edge graceful labeling of quadratic residue digraph. In this paper edge odd graceful labeling is extended for directed graph and proved that a Paley digraph admits edge odd graceful labeling, strong edge graceful labeling and super edge graceful labeling.

2 Basic Concepts

Let $G(V, E)$ be a directed graph with V as vertex set and E as edge set. In a graph if the vertices are labeled then the labeling is known as vertex labeling, edges alone labeled then it is edge labeling and it is denoted as total labeling if both vertices and edges are labeled.

Definition 1. *Paley digraph $P(q)$ is defined as a directed graph with vertices as finite field elements $|V| = q$ where q is a prime* *number such that* $q \equiv 3 \pmod 4$ *and the edge set is* $E(P) = \{(x, y) : x, y \in F_q, x - y \in (F_q^*)^2\}$.

Definition 2. *Let* G *be a directed graph with* $V = \{v_1, v_2, \ldots, v_p\}$ *as vertices and* $E = \{e_1, e_2, \ldots, e_q\}$ *as edges. If there exists a one to one mapping* $f : E(G) \to \{1, 3, \ldots, 2q - 1\}$ *induces a mapping* $f^* : V(G) \to \{0, 1, 2, \ldots, (2s - 1)\}$ *which is defined as* $f^*(u) \equiv \sum\{f(u, v)/uv \in E\} \pmod{2s}$ *where* u *and* v *are incident vertices and* $s = \max\{p, q\}$. *This labeling is called edge odd graceful labeling. A graph which admits this labeling is called edge-odd graceful graph.*

Definition 3. *A directed graph* $G(V, E)$ *with* V *as node set where* $|V| = p$ *and* E *as arc set where* $|E| = q$. *Let* $f : E \to \{0, \pm 1, \pm 2, \ldots, \pm(q-1)/2\}$ *is a bijection when the number of edges are odd and from* E *to* $\{\pm 1, \pm 2, \ldots, \pm q/2\}$ *when the number of edges are even induces the vertex labeling* f^* *defined by* $f^*(v_i) = f(e_{ij})$ *taken over the leaving edges of* v_i, $1 \leq i \leq p$ *is one-one and onto defined as* $f^* : V \to \{0, \pm 1, \pm 2, \ldots, \pm(p - 1)/2\}$ *when* p *is odd and* $f^* : V \to \{\pm 1, \pm 2, \ldots, \pm p/2\}$ *when* p *is even is called super edge graceful labeling [16].*

Definition 4. *A digraph is called a strong edge graceful graph if there is a one-one mapping* $f : E \to \{1, 2, 3 \ldots 3(|E|/2)\}$ *such that the induced mapping* $f^*(v_i) = (\sum f(e_{ij})) \pmod{2|V|}$ *taken over all the leaving edges of* v_i *is an injection, where* e_{ij} *is the* j*th outgoing arc of the vertex* v_i *[15].*

3 Main Result

Theorem 1. *Let* G *be a digraph with finite field elements as vertices* $v_1, v_2, \ldots, v_q$, q *is prime and* $q \equiv 3 \pmod 4$. *If each vertex of* G *has* p *incoming and outgoing arcs with* $|E| = pq$, *where* $p = \frac{q-1}{2}$ *then* G *admits Edge odd graceful labeling.*

Proof. Let $v_1, v_2, \ldots, v_q$, be the vertices of a digraph G where q is a prime number and $q \equiv 3$ (m0d 4). By definition 1 construct a Paley directed graph. We denote the vertex set $V(P)$ as $\{v_1, v_2, \ldots, v_q\}$ and the Edge set as $E = \{E_1 \cup E_2 \cup \cdots \cup E_p\}$, $p = \frac{q-1}{2}$ where $E_1 = \{e_{1j} 1 \leq j \leq q$, $E_2 = \{e_{2j} 1 \leq j \leq q$, $\ldots$, $E_p = \{e_{pj} 1 \leq j \leq q$. This graph has same number of incoming and outgoing edges for each vertex. Let us substantiate that there exists a function $f : E(G) \to \{1, 3, \ldots, 2E-1\}$ so that the induced map $f^* : V \to \{0, 1, \ldots, 2s-1\}$ defined by $f^*(u) = \sum f(u, v) \pmod{2s}$. (By definition 2). Define the bijective mapping as

$$f(e_{ij}) = (i-1)2q + 2j - 1 \quad \text{for } 1 \leq j \leq q$$

The induced mapping is

$$f^*(v_j) = \sum_{i=1}^{p} f(e_{ij}) \pmod{2s} \quad \text{for } 1 \leq j \leq q$$

$$\begin{aligned} f^*(v_j) &= f(e_{1j}) + f(e_{2j}) + \cdots + f(e_{pj}) \\ &= 2j - 1 + 2q + 2j - 1 + 4q + 2j - 1 + \cdots + (2p-2)q \\ &\qquad + 2j - 1 \\ &= p(2j-1) + 2q(1 + 2 + \cdots + (p-1) \\ &= p(2j-1) + p(p-1)q \pmod{2s} \end{aligned}$$

Hence all the labels of v_j are distinct for all $j = 1, 2, \ldots, q$. This shows that every quadratic residue digraph is odd edge graceful. □

Preposition 1. *Every Paley digraph is k edge graceful graph.*

Theorem 2. *Let G be a digraph with finite field elements as vertices $v_1, v_2, \ldots, v_q$, q is prime and $q \equiv 3 \pmod 4$. If each vertex of G has p incoming and outgoing arcs with $|E| = pq$, where $p = \frac{q-1}{2}$ then G is strong edge graceful graph.*

Proof. Paley digraph is a quadratic residue digraph whose vertices are finite field element $q \equiv 3 \pmod 4$. It is a strong regular graph which has same number of incoming and outgoing arcs. To prove that a Paley digraph is a strong edge graceful graph one has to show that it admits strong edge graceful labeling. Define the injective mapping as

For $1 \leq i \leq p$

$$f(e_{ij}) = \begin{cases} iq - 2j + 2 & \text{for } 1 \leq j \leq \frac{q+1}{2} \\ (i+1)q - 2j + 2 & \text{for } \frac{q+1}{2} + 1 \leq j \leq q \end{cases}$$

Define the induced map

$$f^*(v_j) = \sum_{j=1}^{p} f(e_{ij}) \pmod{2|V|} \text{ are distinct.}$$

Where e_{ij} is an outgoing arc of j^{th} vertex
For $1 \leq j \leq \frac{q+1}{2}$

$$\begin{aligned} f^*(v_j) &= f(e_{1j}) + f(e_{2j}) + \cdots + f(e_{pj}) \\ &= q - 2j + 2 + 2q - 2j + 2 + 3q - 2j + 2 + \cdots \\ &\quad + pq - 2j + 2 \\ &= p(2 - 2j) + q(1 + 2 + \cdots + p) \\ &= 2p(1 - j) + [\frac{p(p+1)}{2}]q \pmod{2|V|} \end{aligned}$$

For $\frac{q+1}{2} + 1 \leq j \leq q$

$$\begin{aligned} f^*(v_j) &= f(e_{1j}) + f(e_{2j}) + \cdots + f(e_{pj}) \\ &= 2q - 2j + 2 + 3q - 2j + 2 + \cdots + (p+1)q - 2j + 2 \\ &= p(2 - 2j) + q(1 + 2 + \cdots + (p+1)) - q \\ &= 2p(1 - j) - q + \left[\frac{(p+1)(p+2)}{2}\right] q \pmod{2|V|} \end{aligned}$$

Hence all v_j are distinct for all $j = 1, 2, \ldots, q$. This shows that every Paley digraph is strong edge graceful graph. □

Corollary 1. *Every odd regular digraph is strong edge graceful if it is edge graceful graph.*

Theorem 3. *Let G be a digraph with finite field elements as vertices $v_1, v_2, \ldots, v_q$, q is prime and $q \equiv 3 \pmod 4$. If each vertices of G has p incoming and outgoing arcs with $|E| = pq$, where $p = \frac{q-1}{2}$ then G admits super edge graceful labeling.*

Proof. Consider the finite field element $q \equiv 3 \pmod 4$. Construct the digraph with this finite field element as vertices and edges as $E = \{(u, v) : u, v \in F_q, x - y \in (F_q^*)^2\}$. It is also called a strong regular graph with same number of incoming and outgoing arcs. To substantiate that the directed Paley graph admits super edge graceful labeling define the bijective mapping as

For $1 \leq j \leq q$

$$f(e_{ij}) = \begin{cases} iq - p - j & \text{for } 1 \leq i \leq \frac{p+1}{2} \\ -\{[i - (\frac{p-1}{2})]q - p - j\} & \text{for } \frac{p+1}{2} + 1 \leq i \leq q \end{cases}$$

The induced mapping is

$$f^*(v_j) = \sum_{i=1}^{p} f(e_{ij})$$

For $1 \leq j \leq q$

$$\begin{aligned} f^*(v_j) &= f(e_{1j}) + f(e_{2j}) + \cdots + f(e_{pj}) \\ &= q - p - j + 2q - p - j + \cdots + \left(\frac{p+1}{2}\right) q - p - j - \\ &\quad \left(\left(\frac{p+1}{2} + 1 - \frac{p-1}{2}\right) q - p - j\right) - \cdots \\ &\quad - \left(\left(p - \frac{p-1}{2}\right) q - p - j\right) \\ &= q - p - j \end{aligned}$$

Hence all vertices are labeled with $\{0, \pm 1, \pm 2, \ldots, \pm(q-1)/2\}$ are distinct. Hence quadratic residue digraph is super edge graceful graph. □

4 Conclusion

In this article the extension of edge odd graceful labeling for directed graph was discussed and prove the same for Paley digraph. Also super edge graceful and strong edge graceful labeling were discussed for the same directed graph and conclude that every odd regular digraph is strong edge graceful graph if it is edge graceful graph.

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