Embedding Complete Bipartite Graph into Sibling Trees with Optimum Wirelength

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Abstract

In this paper, we determine the wirelength of embedding complete bipartite graphs $K_{2^{n-1},2^{n-1}}$ into 1-rooted sibling tree ST_n^1 and Cartesian product of 1-rooted sibling trees and paths.

Keywords: Embedding, congestion, edge isoperimetric problem, complete bipartite graph, tree.

1 Introduction

An interconnection networks has an ability in efficiently simulating the programs written for other architectures. For the simulation of different interconnection networks, the powerful tool used is Graph Embedding. It is an important technique used in the study of computational capabilities of processor interconnection networks and task distribution. The embeddings of graphs from one class of graphs into another class is increasingly pervasive in the area of fixed interconnection parallel architectures. In general, the embedding problem is NP-complete [1]. This inspires the study of congestion and wirelength problems for graph embeddings.

The rest of the paper is organized as follows. Section 2 gives definitions and other preliminaries. Section 3 gives the main results. Finally, Section 4 concludes the paper.

2 Basic concepts

In this section we give the basic definitions and preliminaries required for our subsequent work.

Definition 2.1. [2] *Let G and H be finite graphs. An embedding $\phi = (f, P_f)$ of G into H is defined as follows:*

1. f is a one-to-one map from $V(G) \to V(H)$
2. P_f is a one-to-one map from $E(G)$ to $\{P_f(u,v) : P_f(u,v)$ is a path in H between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$.

For brevity, we denote the pair (f, P_f) as f.

Definition 2.2. *Let $f : G \to H$ be an embedding. For $e \in E(H)$, let $EC_f(e)$ denote the number of edges (u,v) of G such that e is in the path $P_f(u,v)$ between $f(u)$ and $f(v)$ in H. In other words,*

$$EC_f(e) = |\{(u,v) \in E(G) : e \in P_f(u,v)\}|.$$

Then the edge congestion of $f : G \to H$ is $EC_f(G,H) = max EC_f(e)$ where the maximum is taken over all edges e of H. The edge congestion of G into H is defined as $EC(G,H) = min EC_f(G,H)$, where the minimum is taken over all embeddings $f : G \to H$.

If S is any subset of $E(H)$, then $EC_f(S) = \sum_{e \in S} EC_f(e)$.

The congestion problem is to determine that embedding whose congestion is $EC(G,H)$. The congestion problem is NP-complete [1].

There are several results on the congestion problem of various architectures such as hypercubes into n-dimensional grid [3], hypercubes into 2-dimensional grid [4], complete binary trees into grids and extended grids [5] and generalized wheels into arbitrary trees [6].

Definition 2.3. [4] *The wirelength of an embedding f of G into H is given by*

$$WL_f(G,H) = \sum_{(u,v) \in E(G)} d_H(f(u), f(v)) = \sum_{e \in E(H)} EC_f(e)$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(u,v)$ in H.

The wirelength of G into H is defined as

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all embeddings f of G into H.

The *wirelength problem* [4, 2, 3, 5] of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G, H)$. The wirelength problem is NP-complete [1]

Definition 2.4. [7] *Let G be a graph and $A \subseteq V(G)$. Denote*

$$I_G(A) = \{(u, v) \in E(G) : u, v \in A\},\ I_G(m) = \max_{A \subseteq V(G), |A|=m} |I_G(A)|$$

and

$$\theta_G(A) = \{(u, v) \in E(G) : u \in A, v \notin A\},\ \theta_G(m) = \min_{A \subseteq V(G), |A|=m} |\theta_G(A)|.$$

For a given m, where $m = 1, 2, \ldots, n$, we consider the problem of finding a subset A of vertices of G such that $|A| = m$ and $|\theta_G(A)| = \theta_G(m)$. Such subsets are called optimal [8, 9]. Moreover, for a regular graph G, I_G and θ_G are equivalent in the sense that a solution for one also becomes a solution for the other [8]. The problem of finding I_G is called *maximum subgraph problem* [1].

Lemma 2.5. (Modified Congestion Lemma) [10] *Let G and H be any arbitrary graphs and let f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let G_1 and G_2 be subgraphs of G induced by $f^{-1}(H_1)$ and $f^{-1}(H_2)$ respectively. Furthermore, suppose S satisfies the following conditions:*

(i) *For every edge $(a, b) \in E(G_i)$, $i = 1, 2, P_f(a, b)$ has no edges in S.*

(ii) *For every edge (a, b) in $E(G)$ with $a \in V(G_1)$ and $b \in V(G_2)$, $P_f(a, b)$ has exactly one edge in S.*

(iii) *$V(G_1)$ and $V(G_2)$ are optimal sets.*

Then $EC_f(S)$ is minimum, that is, $EC_f(S) \leq EC_g(S)$ for any other embedding g of G into H. Further $EC_f(S) = \sum_{v \in V(G_1)} deg_G(v) - 2|E(G_1)| = \sum_{v \in V(G_2)} deg_G(v) - 2|E(G_2)|$.

Lemma 2.6. (k-Partition Lemma) [11] *Let $f : G \to H$ be an embedding. Let $[kE(H)]$ denote a collection of edges of H repeated exactly k times. In other words, $[kE(H)]$ comprises of k copies of the edge set of H. Let $\{S_1, S_2, \ldots, S_m\}$ be a partition of $[kE(H)]$ such that each S_i is an edge cut of H satisfying the conditions of modified congestion lemma. Then*

$$WL_f(G, H) = \frac{1}{k}\sum_{i=1}^{m} EC_f(S_i).$$

3 Sibling Trees

In this section, we compute the wirelength of embedding complete bipartite graph into sibling trees.

Definition 3.1. [12] *A bipartite graph is one whose vertex-set can be partitioned into two subsets X and Y, so that each edge has one end vertex in X and another in Y. If each vertex of X is joined to all vertices of Y and vice-versa, we call it as a complete bipartite graph and denote it by $K_{m,n}$, where $|X| = m$ and $|Y| = n$.*

Lemma 3.2. *A set of l consecutive vertices of a hamiltonian cycle on $2n$ vertices, HC_{2n} induces a maximum subgraph of $K_{n,n}$ on l vertices, $n \geq 2$.*

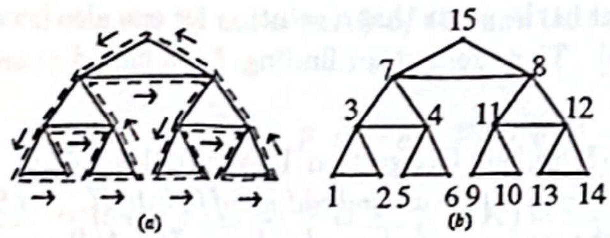

Figure 1: Traversal of Sibling tree

Lemma 3.3. [13] *$K_{\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor}$ is the maximum subgraph of $K_{m,n}$ on r vertices.*

For any non-negative integer n, the complete binary tree of height n, denoted by T_n, is the binary tree where each internal vertex has exactly two children and all the leaves are at the same level. Clearly, a complete binary tree T_n has n levels. Level i, $1 \leq i \leq n$, contains $2^i - 1$ vertices. Thus T_n has exactly $2^n - 1$ vertices. The 1-rooted complete binary tree T_n^1 is obtained from a complete binary tree T_n by attaching to its root a pendant edge. The new vertex is called the root of T_n^1 and is considered to be at level 0.

Definition 3.4. *The sibling tree ST_n is obtained from the complete binary tree T_n by adding edges called sibling edges between left and right children of the same parent node. See Figure 1(b).*

Remark 3.5. *The sibling tree ST_n has $2^n - 1$ vertices and $3(2^{n-1} - 1)$ edges.*

A Sibling Tree Traversal follows the usual pattern of binary tree traversal with an additional condition that the traversal does not cut any region, but travels along sibling edges. See Figure 1.

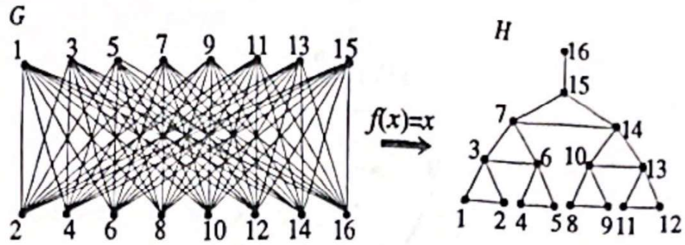

Figure 2: Embedding complete bipartite graph $K_{8,8}$ into 1-rooted sibling tree ST_4^1

Definition 3.6. *The 1-rooted sibling tree ST_n^1 is obtained from ST_n by attaching to its root a pendant edge.*

Algorithm A

Input : The complete bipartite graph $K_{2^{n-1},2^{n-1}}$ and the 1-rooted sibling tree ST_n^1.

Algorithm : Label the consecutive vertices of the hamiltonian cycle HC_{2^n} in the complete bipartite graph $K_{2^{n-1},2^{n-1}}$ in the clockwise sense from 1 to 2^n. Label the vertices of ST_n^1 using sibling tree traversal from 1 to 2^n.

Output : An embedding f of $K_{2^{n-1},2^{n-1}}$ into ST_n^1 given by $f(x) = x$ with minimum wirelength. See Figure 2.

Proof of correctness : For $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, let $S_i^{2^j-1}$ be an edge cut of the 1-rooted sibling tree ST_n^1 consisting of edges induced by the $\lceil i/2 \rceil^{\text{th}}$ parent vertex from left to right in level $n-j$ and its left child if i is odd, and its right child if i is even, together with the corresponding sibling edge which is the same edge in either case, such that $S_i^{2^j-1}$ disconnects ST_n^1 into two components $H_{i1}^{2^j-1}$ and $H_{i2}^{2^j-1}$ where $V(H_{i1}^{2^j-1})$ is consecutively labeled. See Figure 3. Let $G_{i1}^{2^j-1}$ and $G_{i2}^{2^j-1}$ be the inverse images of $H_{i1}^{2^j-1}$ and $H_{i2}^{2^j-1}$ under f respectively. By Lemma 3.3, $G_{i1}^{2^j-1}$ is a maximum subgraph of $K_{2^{n-1},2^{n-1}}$. Thus the edge cut $S_i^{2^j-1}$ satisfies conditions (i) and (ii) of the Congestion Lemma. Therefore $EC_f(S_i^{2^j-1})$ is minimum for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$.

For $j = 1, 2, \ldots, n-1$ and $i = 1, 2, \ldots, 2^{n-j-1}$, let $SS_i^{2(2^j-1)}$ be an edge cut of the 1-rooted sibling tree ST_n^1 consisting of the edges induced by the i^{th} parent vertex from left to right in level $n-j$ and its two children, such that $SS_i^{2(2^j-1)}$ disconnects ST_n^1 into two components $H_{i1}^{2(2^j-1)}$ and $H_{i2}^{2(2^j-1)}$ where $V(H_{i1}^{2(2^j-1)})$ is consecutively labeled. See Figure 3.

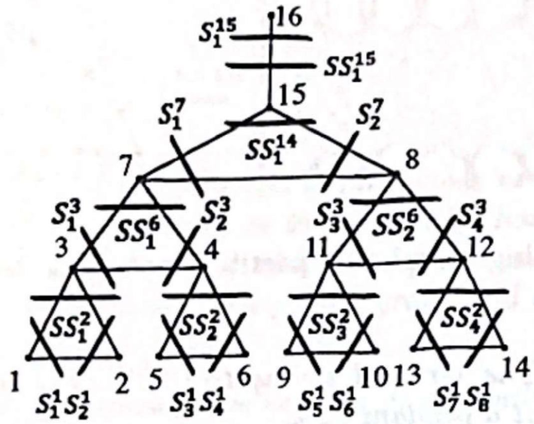

Figure 3: Edge cut of 1-rooted sibling tree ST_4^1

Let $G_{i1}^{2(2^j-1)}$ and $G_{i2}^{2(2^j-1)}$ be the inverse images of $H_{i1}^{2(2^j-1)}$ and $H_{i2}^{2(2^j-1)}$ under f respectively. By Lemma 3.3, $G_{i1}^{2(2^j-1)}$ is a maximum subgraph of $K_{2^{n-1},2^{n-1}}$. Thus the edge cut $SS_i^{2(2^j-1)}$ satisfies conditions (i) and (ii) of the Congestion Lemma. Therefore $EC_f(SS_i^{2(2^j-1)})$ is minimum for $j=1,2,\ldots,n-1$ and $i=1,2,\ldots,2^{n-j-1}$.

We note that the set $\{S_i^{2^j-1} : 1 \le j \le n, 1 \le i \le 2^{n-j}\} \cup \{SS_i^{2(2^j-1)} : 1 \le j \le n-1, 1 \le i \le 2^{n-j-1}\} \cup \{SS_1^{2^n-1}\}$ forms a 2-partition of $2E(ST_n^1)$. The 2 -Partition Lemma implies that $WL_f(K_{2^{n-1},2^{n-1}}, ST_n^1)$ is minimum.

Theorem 3.7. *Let G be the complete bipartite graph $K_{2^{n-1},2^{n-1}}$ and H be the 1-rooted sibling tree ST_n^1. Then the exact wirelength of G into H is given by*

$$WL(G,H) = 2^{n-2}(2^{n+1}n-3(2^n)+3) - \sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} I_G(2^j-1)$$
$$-\sum_{j=1}^{n-1}\sum_{i=1}^{2^{n-j-1}} I_G(2(2^j-1)).$$

Proof. . By Congestion Lemma, $EC_f(S_i^{2^j-1}) = (2^{n-1})(2^j-1) - 2I_G(2^j-1)$ for $j=1,2,\ldots,n$ and $i=1,2,\ldots,2^{n-j}$ and $EC_f(SS_i^{2(2^j-1)}) = (2^{n-1})2(2^j-1) - 2I_G(2(2^j-1))$, for $j=1,2,\ldots,n-1$ and $i=1,2,\ldots,2^{n-j-1}$. Then by 2-Partition Lemma,

$$
\begin{aligned}
WL(G,H) &= \frac{1}{2}\left[\sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} EC_f(S_i^{2^j-1}) + \sum_{j=1}^{n-1}\sum_{i=1}^{2^{n-j-1}} EC_f(SS_i^{2(2^j-1)})\right] \\
&= \frac{1}{2}\left[\sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} [(2^{n-1})(2^j-1) - 2I_G(2^j-1)]\right] \\
&\quad + \frac{1}{2}\left[\sum_{j=1}^{n-1}\sum_{i=1}^{2^{n-j-1}} [(2^{n-1})2(2^j-1) - 2I_G(2(2^j-1))]\right] \\
&= 2^{n-2}(2^{n+1}n - 3(2^n) + 3) - \sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} I_G(2^j-1) \\
&\quad - \sum_{j=1}^{n-1}\sum_{i=1}^{2^{n-j-1}} I_G(2(2^j-1)).
\end{aligned}
$$

□

4 Cartesian Product of 1-rooted Sibling Trees and Paths

In this section we embed complete bipartite graph onto the Cartesian product of 1-rooted sibling trees and paths to minimize the wirelength.

Level-labeling of $ST_n^1 \times P_m$:

Let $ST_1^1, ST_2^1, \ldots, ST_m^1$ be the rooted sibling trees in $ST_n^1 \times P_m$. Extra edges in $ST_n^1 \times P_m$ are cross edges. Label the trees $ST_1^1, ST_2^1, \ldots, ST_m^1$ with consecutive indices $\{(i-1)2^n+1, (i-1)2^n+2, \ldots, (i-1)2^n+2^n$ for $ST_i^1, 1 \le i \le n$, according to the following rule:

(i) When i is odd, label ST_i^1, level by level sequentially beginning with the last level, from left to right.

(ii) When i is even, label ST_i^1, level by level sequentially beginning with level 0, from right to left.

Algorithm B

Input : The complete bipartite graph $K_{2^{n-1},2^{n-1}}$ and the Cartesian product $ST_n^1 \times P_m$, $m = 2^n$, $n \ge 2$.

Algorithm : Label the consecutive vertices of the hamiltonian cycle HC_{2^n} in the complete bipartite graph $K_{2^{n-1},2^{n-1}}$ in the clockwise sense from 1 to 2^n. Let $ST_n^1, ST_n^2, \ldots, ST_n^m$ be the m vertex disjoint copies of sibling tree in $ST_n^1 \times P_m$. Label the vertices of $ST_n^1 \times P_m$, $m = 2^n$, $n \geq 2$ by Level-labeling. Let $f(x) = x$ for all $x \in V(K_{2^{n-1},2^{n-1}})$ and let $P_f(a,b)$ be a shortest path between $f(a)$ and $f(b)$ in $ST_n^1 \times P_m$ for $(a,b) \in E(K_{2^{n-1},2^{n-1}})$.

Output : An embedding f of $K_{2^{n-1},2^{n-1}}$ into $ST_n^1 \times P_m$ with optimal wirelength. See Figure 4.

Proof of correctness : For $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, let $S_i^{2^j-1}$ be the set of edges of the $ST_n^1 \times P_m$ induced by the $\lceil i/2 \rceil^{\text{th}}$ parent vertex from left to right in level $n-j$ and its left child if i is odd, and its right child if i is even, together with the corresponding sibling edge which is the same edge in either case. Removal of $S_i^{2^j-1}$ leaves $ST_n^1 \times P_m$ into two components $H_{i1}^{2^j-1}$ and $H_{i2}^{2^j-1}$ where $V(H_{i1}^{2^j-1})$ is consecutively labeled. Let $G_{i1}^{2^j-1}$ and $G_{i2}^{2^j-1}$ be the inverse images of $H_{i1}^{2^j-1}$ and $H_{i2}^{2^j-1}$ under f respectively. See Figure 5. By Lemma 3.3, $G_{i1}^{2^j-1}$ is an optimal set in $K_{2^{n-1},2^{n-1}}$.

For $j = 1, 2, \ldots, n-1$ and $i = 1, 2, \ldots, 2^{n-j-1}$, let $SS_i^{2(2^j-1)}$ be the set of edges of $ST_n^1 \times P_m$ consisting of the edges induced by the i^{th} parent vertex from left to right in level $n-j$ and its two children, such that $SS_i^{2(2^j-1)}$ disconnects $ST_n^1 \times P_m$ into two components $H_{i1}^{2(2^j-1)}$ and $H_{i2}^{2(2^j-1)}$ where $V(H_{i1}^{2(2^j-1)})$ is consecutively labeled. See Figure 5. Let $G_{i1}^{2(2^j-1)}$ and $G_{i2}^{2(2^j-1)}$ be the inverse images of $H_{i1}^{2(2^j-1)}$ and $H_{i2}^{2(2^j-1)}$ under f respectively. By Lemma 3.3, $G_{i1}^{2(2^j-1)}$ is a maximum subgraph of $K_{2^{n-1},2^{n-1}}$. Thus the edge cut $SS_i^{2(2^j-1)}$ satisfies conditions (i) and (ii) of the Congestion Lemma. Therefore $EC_f(SS_i^{2(2^j-1)})$ is minimum for $j = 1, 2, \ldots, n-1$ and $i = 1, 2, \ldots, 2^{n-j-1}$. Let $SS_0^n = S_0^n$.

For $k = 1, 2, \ldots, m-1$, let $EE_k = E_k$ be the set of edges of $ST_n^1 \times P_m$ such that each edge has one vertex in ST_n^{1k} and the other vertex in ST_n^{1k+1}. Removal of E_k leaves $ST_n^1 \times P_m$ into two components H_{k1} and H_{k2} where $V(H_{k1})$ contains equal number of odd and even labels. Let G_{k1} and G_{k2} be the inverse images of H_{k1} and H_{k2} under f respectively. See Figure 5. By Lemma 3.3, G_{k1} is an optimal set in $K_{2^{n-1},2^{n-1}}$.

We note that the sets $\{S_i^{2^j-1} : j = 1, 2, \ldots, n, i = 1, 2, \ldots, 2^{n-j}\} \cup \{SS_i^{2(2^j-1)} : j = 1, 2, \ldots, n-1, i = 1, 2, \ldots, 2^{n-j-1}\} \cup \{SS_0^n\} \cup \{EE_k, E_k : k = 1, 2, \ldots, m-1\}$ form a partition of $2E(ST_n^1 \times P_m)$. Moreover, each edge cut satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. The 2-Partition Lemma implies that $WL_f(K_{2^{n-1},2^{n-1}}, ST_n^1 \times P_m)$ is minimum.

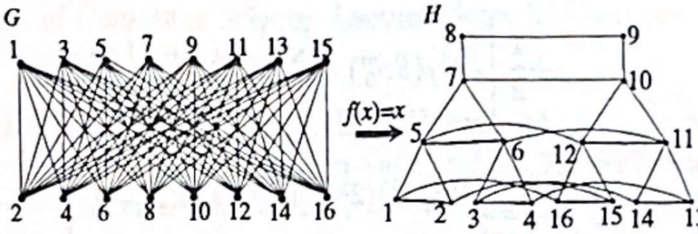

Figure 4: Embedding complete bipartite graph $K_{8,8}$ into $ST_3^1 \times P_2$

Theorem 4.1. *Let G be the complete bipartite graph, $K_{2^{n-1},2^{n-1}}$ and H be the Cartesian product $ST_n^1 \times P_m$, $m = 2^n$, $n \geq 2$. Then the exact wirelength of G into H is given by*

$$WL(G,H) = 2^{r-2}(2^r(m^2 - m + 4r - 8) + 10) - \frac{1}{2}\left[\sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} 2I_G(2^j - 1)\right]$$

$$-\frac{1}{2}\left[\sum_{j=1}^{n-1}\sum_{i=1}^{2^{n-j-1}} 2I_G(2(2^j - 1)) + n\right] - \frac{1}{2}\left[\sum_{k=1}^{m-1} 2I_G(k2^n)\right].$$

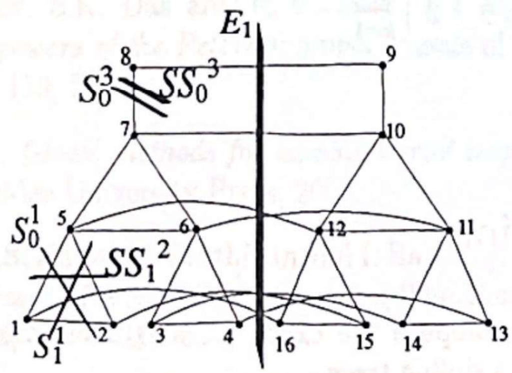

Figure 5: Edgecut of $ST_3^1 \times P_2$

Proof. By Congestion Lemma, $EC_f(S_i^{2^j-1}) = ((2^{n-1})(2^j-1)) - 2I_G(2^j-1)$ for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, 2^{n-j}$, and $EC_f(SS_i^{2(2^j-1)}) = ((2^{n-1})2(2^j - 1)) - 2I_G(2(2^j - 1))$, for $j = 1, 2, \ldots, n-1$ and $i = 1, 2, \ldots, 2^{n-j-1}$, $EC_f(SS_0^n) = 2^{n-1}$ and $EC_f(E_k) = (2^{n-1})(k2^n) - 2I_G(k2^n)$, for $k = 1, 2, \ldots, m-1$. Then by 2-Partition Lemma,

$$
\begin{aligned}
WL(G,H) &= \frac{1}{2}\left[\sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} EC_f(S_i^{2^j-1}) + \sum_{j=1}^{n-1}\sum_{i=1}^{2^{n-j-1}} EC_f(SS_i^{2(2^j-1)})\right] \\
&\quad + \frac{1}{2}\left[EC_f(SS_0^n) + \sum_{k=1}^{m-1} EC_f(E_k)\right] \\
&= \frac{1}{2}\left[\sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} ((2^{n-1})(2^j-1)) - 2I_G(2^j-1)\right] \\
&\quad + \frac{1}{2}\left[\sum_{j=1}^{n-1}\sum_{i=1}^{2^{n-j-1}} ((2^{n-1})2(2^j-1)) - 2I_G(2(2^j-1)) + 2^{n-1}\right] \\
&\quad + \frac{1}{2}\left[\sum_{k=1}^{m-1} (2^{n-1})(k2^n) - 2I_G(k2^n)\right] \\
&= 2^{r-2}(2^r(m^2-m+4r-8)+10) - \frac{1}{2}\left[\sum_{j=1}^{n}\sum_{i=1}^{2^{n-j}} 2I_G(2^j-1)\right] \\
&\quad - \frac{1}{2}\left[\sum_{j=1}^{n-1}\sum_{i=1}^{2^{n-j-1}} 2I_G(2(2^j-1)) + 2^{n-1}\right] \\
&\quad - \frac{1}{2}\left[\sum_{k=1}^{m-1} 2I_G(k2^n)\right].
\end{aligned}
$$

□

5 Conclusion

In this paper, we compute the exact wirelength of embedding complete bipartite graph into sibling trees.

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