Fekete-Szegö Problem for Certain Class of Bi-stralike Functions involving q-differential Operator

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Abstract

Making use of q-derivative operator, in this paper, we introduce new subclasses of the function class Σ of normalized analytic and bi-starlike functions defined in the open disk $\mathbb{U}$. Furthermore, we find estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Moreover, we obtain Fekete-Szegö inequalities for the new function classes.

1 Introduction

Let $\mathcal{A}$ denote the functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

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which are analytic and univalent in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}; |z| < 1\}$. The function $f \in \mathcal{A}$ is said to be univalent in $\mathbb{U}$ if f is one-to-one in $\mathbb{U}$. We also denote by S the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$ (also see, [4], [8]).

For $0 \leq \gamma < 1$, a function $f \in \Sigma$ is in the class $\mathcal{S}_q^*(\gamma)$ of bi-starlike functions of order γ if both f and its inverse map f^{-1} are starlike. In fact, the Koebe one-quarter theorem [4] ensures that image of $\mathbb{U}$ under every univalent function $f \in S$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{A}$ has an inverse f^{-1}, which is define by

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w, \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$

Deniz et al. [5] and Altınkaya and Yalçın [1] studied the bi-starlike functions and obtain upper bounds for the second Hankel determinant.

In 1967, Lewin [12] showed that $|a_2| < 1.51$ using Grunsky inequalities. In 1981, Styer and Wright [16] studied the class of bi-univalent functions and obtain the bound $4/3$ for the second coefficient $|a_2|$. Subsequently, in 1985, Branges [3] conjectured Bieberbach conjecture which showed that

$$|a_n| \leq n; \quad (n \in \mathbb{N} - 1),$$

$\mathbb{N}$ being positive integer. We owe the revival of these topics to Srivastava et al. ([15]) for finding initial coefficient bounds for bi-univalent functions. Recently, several authors obtained initial coefficient bounds $|a_2|$ and $|a_3|$ for bi-univalent functions (also see, for e.g. [2], [6], [7], [9], [10], [11], [14]).

In this connection, we define two new subclasses $\mathcal{M}_\Sigma^q(\gamma)$ and $\mathcal{M}_\Sigma^q(\zeta)$ of the class Σ and obtain estimates on the initial coefficients $|a_2|$ and $|a_3|$ of new subclasses using q-differential operator.

2 Main Results

Definition 2.1 : *A function* $f(z) \in \mathcal{M}_{\Sigma}^{q}(\gamma)$ *if it holds:*

$$f \in \Sigma \text{ and } \left| arg\left(\frac{zD_qf(z)}{f(z)}\right)\right| < \frac{\gamma\pi}{2};\ (0 < \gamma \leq 1, z \in \mathbb{U}) \tag{2.1}$$

and

$$\left| arg\left(\frac{zD_qg(w)}{g(w)}\right)\right| < \frac{\gamma\pi}{2};\ (0 < \gamma \leq 1, w \in \mathbb{U}). \tag{2.2}$$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for function in the class $\mathcal{M}_{\Sigma}^{q}(\gamma)$.

Theorem 2.1 *Let the function* $f(z) \in \mathcal{M}_{\Sigma}^{q}(\gamma)$ *and* $0 < \gamma \leq 1; 0 < q < 1$. *Then*

$$|a_2| \leq \frac{2\gamma}{\sqrt{2\gamma(q_3 - q_2) + (1-\gamma)q_2^2}} \tag{2.3}$$

and

$$|a_3| \leq \frac{4\gamma^2}{q_2^2} + \frac{2\gamma}{q_3}. \tag{2.4}$$

Proof: It follows from (2.1) and (2.2) that

$$\frac{zD_qf(z)}{f(z)} = [\phi(z)]^{\gamma} \tag{2.5}$$

and

$$\frac{zD_qg(w)}{g(w)} = [\varphi(w)]^{\gamma}. \tag{2.6}$$

Comparing the coefficients of z and z^2 in (2.5) and (2.6), we obtain

$$q_2a_2 = \gamma\phi_1 \tag{2.7}$$

$$q_3a_3 - q_2a_2^2 = \gamma\phi_2 + \frac{\gamma(\gamma-1)}{2}\phi_1^2 \tag{2.8}$$

$$-q_2a_2 = \gamma\varphi_1 \tag{2.9}$$

and

$$q_3(2a_2^2 - a_3) - q_2a_2^2 = \gamma\varphi_2 + \frac{\gamma(\gamma-1)}{2}\varphi_1^2. \tag{2.10}$$

From (2.7) and (2.9), we arrive at

$$\phi_1 = -\varphi_1 \tag{2.11}$$

and

$$2q_2^2 a_2^2 = \gamma^2 \left(\phi_1^2 + \varphi_1^2\right). \tag{2.12}$$

Now from (2.8), (2.10) and (2.12), we find

$$2(q_3 - q_2)a_2^2 = \gamma\left(\phi_2 + \varphi_2\right) + \frac{(\gamma - 1)q_2^2 a_2^2}{\gamma}. \tag{2.13}$$

Thus, we establish

$$a_2^2 = \frac{\gamma^2\left(\phi_2 + \varphi_2\right)}{2\gamma(q_3 - q_2) + (1-\gamma)q_2^2}. \tag{2.14}$$

In view of Lemma 1 ([13], p.226), for the coefficients ϕ_2 and φ_2, we get

$$|a_2| \leq \frac{2\gamma.}{\sqrt{2\gamma(q_3 - q_2) + (1-\gamma)q_2^2}} \tag{2.15}$$

which gives us the desire estimate on $|a_2|$ as asserted in (2.3).

Next, in order to find the bound on $|a_3|$, by subtracting (2.10) and (2.8); using (2.12), we get

$$a_3 = \frac{\gamma\left(\phi_2 - \varphi_2\right)}{2q_3} + \frac{\gamma^2\left(\phi_1^2 + \varphi_1^2\right)}{2q_2^2} \tag{2.16}$$

which upon applying Lemma 1 ([13], p.226), immediately yields

$$|a_3| \leq \frac{4\gamma^2}{q_2^2} + \frac{2\gamma}{q_3}. \tag{2.17}$$

This completes the proof.

Corollary 2.1 *Let $f(z) \in \mathcal{M}_\Sigma(\gamma)$, $(0 < \gamma \leq 1)$. Then*

$$|a_2| \leq \frac{2\gamma}{\sqrt{\gamma + 1}} \tag{2.18}$$

and

$$|a_3| \leq \frac{2\gamma(4\gamma + 1)}{2}. \tag{2.19}$$

Definition 2.2 : *A function $f(z) \in \mathcal{M}_{\Sigma}^{q}(\zeta)$ if it holds:*

$$f \in \Sigma \text{ and } Re\left(\frac{zD_qf(z)}{f(z)}\right) > \zeta; \ (0 \leq \zeta < 1, z \in \mathbb{U}) \tag{2.20}$$

and

$$Re\left(\frac{zD_qg(w)}{g(w)}\right) > \zeta; \ (0 \leq \zeta < 1, w \in \mathbb{U}). \tag{2.21}$$

Theorem 2.2 *Let the function $f(z) \in \mathcal{M}_{\Sigma}^{q}(\zeta)$ and $0 \leq \zeta < 1; 0 < q < 1$. Then*

$$|a_2| \leq \min\left\{\frac{2(1-\zeta)}{q_2}, \sqrt{\frac{2(1-\zeta)}{(q_3-q_2)}}\right\} \tag{2.22}$$

and

$$|a_3| \leq \frac{4(1-\zeta)^2}{q_2^2} + \frac{2(1-\zeta)}{q_3}. \tag{2.23}$$

Proof: It follows from (2.20) and (2.21) that

$$\frac{zD_qf(z)}{f(z)} = \zeta + (1-\zeta)\phi(z) \tag{2.24}$$

and

$$\frac{zD_qg(w)}{g(w)} = \zeta + (1-\zeta)\varphi(w). \tag{2.25}$$

Comparing the coefficients of z and z^2 in (2.24) and (2.25), we get

$$q_2a_2 = (1-\zeta)\phi_1, \tag{2.26}$$

$$q_3a_3 - q_2a_2^2 = (1-\zeta)\phi_2, \tag{2.27}$$

$$-q_2a_2 = (1-\zeta)\varphi_1 \tag{2.28}$$

and

$$q_3(2a_2^2 - a_3) - q_2a_2^2 = (1-\zeta)\varphi_2. \tag{2.29}$$

From equation (2.26) and (2.28), we get

$$\phi_1^2 = \varphi_1^2 \tag{2.30}$$

and

$$2q_2^2a_2^2 = (1-\zeta)^2(\phi_1^2 + \varphi_1^2). \tag{2.31}$$

Now we add (2.27) and (2.29), we get the following

$$2(q_3-q_2)a_2^2=(1-\zeta)(\phi_2+\varphi_2). \tag{2.32}$$

In view of Lemma 1 ([13], p.226), equations (2.31) and (2.32), yields

$$|a_2|\le \min\left\{\frac{2(1-\zeta)}{q_2},\sqrt{\frac{2(1-\zeta)}{(q_3-q_2)}}\right\}. \tag{2.33}$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.29) from(2.27) and using (2.31), we get

$$a_3=\frac{(1-\zeta)^2(\phi_1^2+\varphi_1^2)}{2q_2^2}+\frac{(1-\zeta)(\phi_2-\varphi_2)}{2q_3}. \tag{2.34}$$

Applying Lemma 1 ([13], p.226) ϕ_1, φ_1, ϕ_2 and φ_2, immediately gives

$$|a_3|\le\frac{4(1-\zeta)^2}{q_2^2}+\frac{2(1-\zeta)}{q_3}. \tag{2.35}$$

This completes the proof.

Corollary 2.2 *Let $f(z)\in\mathcal{M}_\Sigma(\zeta)$, $(0\le\zeta<1)$. Then*

$$|a_2|\le\sqrt{2(1-\zeta)} \tag{2.36}$$

and

$$|a_3|\le 4(1-\zeta)^2+(1-\zeta). \tag{2.37}$$

3 Fekte-Szegö Inequalities

Making use of the a_2^2 and a_3, and motivated by recent work of Zaprawa [17] we prove the following Fekete-Szegö results for the function classes $\mathcal{M}_\Sigma^q(\gamma)$ and $\mathcal{M}_\Sigma^q(\zeta)$.

Theorem 3.1 *Let the function $f(z)\in\mathcal{M}_\Sigma^q(\gamma)$, $0<\gamma\le 1; 0<q<1$ and $\tau\in\mathbb{R}$. Then*

$$|a_3-\tau a_2^2|\le\begin{cases}\frac{4\gamma^2}{q_2^2+\gamma(2(q_3-q_2)-q_2^2)}|1-\tau| & \vartheta_1\ge\vartheta_2\\ \frac{2\gamma}{q_3} & \vartheta_1\le\vartheta_2,\end{cases} \tag{3.1}$$

where $\vartheta_1=q_3\gamma|1-\tau|$ and $\vartheta_2=q_2^2+\gamma((q_3-q_2)-q_2^2)$.

Proof: From equation (2.8) and (2.10), we get

$$a_3 = a_2^2 + \frac{\gamma}{2q_3}(\phi_2 - \varphi_2). \tag{3.2}$$

Using (2.14) and (3.2), we get

$$a_3 - \tau a_2^2 = \phi_2 \left[\eta(\gamma)(1-\tau) + \frac{\gamma}{2q_3}\right] + \varphi_2 \left[\eta(\gamma)(1-\tau) - \frac{\gamma}{2q_3}\right]$$

where

$$\eta(\gamma) = \frac{\gamma^2}{2\gamma(q_3 - q_2) - (\gamma - 1)q_2^2}.$$

From Lemma 1 ([13], p.226) and Lemma 7 ([17], p.2), we yields

$$|a_3 - \tau a_2^2| \leq \begin{cases} 4\eta(\gamma)|1-\tau| & \eta(\gamma)|1-\tau| \geq \frac{\gamma}{2q_3} \\ \frac{2\gamma}{q_3} & \eta(\gamma)|1-\tau| \leq \frac{\gamma}{2q_3}, \end{cases} \tag{3.3}$$

which completes the proof of Theorem 3.1.

Proceeding as in the above theorem one can easily prove the following result for $\mathcal{M}_\Sigma^q(\zeta)$, hence we state the following theorem without proof.

Theorem 3.2 *Let the function* $f(z) \in \mathcal{M}_\Sigma^q(\zeta)$, $0 \leq \zeta < 1; 0 < q < 1$ *and* $\tau \in \mathbb{R}$. *Then*

$$|a_3 - \tau a_2^2| \leq \begin{cases} \frac{2(1-\zeta)}{q_3 - q_2}|1-\tau| & q_3|1-\tau| \geq q_3 - q_2, \\ \frac{2(1-\zeta)}{q_3} & q_3|1-\tau| \leq q_3 - q_2. \end{cases} \tag{3.4}$$

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A Note On Equitable Total Coloring of Middle and Total Graph of Some Graphs

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Abstract

Among the varius coloring of graphs, the concept of equitable total coloring of graph G is the coloring of all its vertices and edges in which the number of elements in any two color classes differ by atmost one. The minimum number of colors required is called its equitable total chromatic number. In this paper, we obtained an equitable total chromatic number of middle graph of path, middle graph of cycle, total graph of path and total graph of cycle.

Keywords: Middle graph, Total graph, Equitable total coloring and Equitable total chromatic number.

1 Introduction

All the graphs considered here simple, finite and undirected graph. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$, respectively. A coloring of a graph G is an assignment of colors to the vertices or edges or both. A vertex-coloring(edge-coloring) is called proper coloring if no two vertices(edges) receive the same color. So many different proper colorings are available in graph theory such as a-coloring, b-coloring, acyclic coloring, star coloring, list coloring, harmonious coloring, total coloring, equitable total coloring. In the present work we focused an equitable total coloring of graph. The concept of total coloring was independently introduced by Behzad [1] and Vizing [10]. A total coloring of a graph G is an assignment of colors to both the vertices and edges of G, such that no two adjacent or incident vertices and edges of G are receive the same color. They both conjectured that for any graph G the following condition holds: $\Delta(G)+1 \leq \chi''(G) \leq \Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G. We observe that $\Delta(G)+1$ is the suitable lower bound. In general, an equitable total coloring problem is comparatively very difficult than the total coloring problem. In 1994, Fu[4] first introduce the concept of equitable total coloring and the equitable total chromatic number of a graph. A total coloring of a graph G is said to be equitable, if the number of elements(vertices and edges) in any two color classes differ by at most one, for which the required minimum number of colors is called the equitable total chromatic number and it is denoted by $\chi_{et}(G)$. Gong Kun et.al[3] proved some results on the equitable total chromatic number of $W_n \vee K_n$, $F_m \vee K_n$and $S_m \vee K_n$. In 2012, Ma Gang and Ma Ming[6] proved some results concerning the equitable total chromatic number of $P_m \vee S_n$, $P_m \vee F_n$ and $P_m \vee W_n$. Tong et.al[8] proved that the equitable total chromatic number of $C_m \square C_n$. Girija et.al[2] proved that equitable total chromatic number of Double star graph and fan graph. Gang et.al[7] proved that on the equitable total coloring of multiple join graph. Zhang Zhong-fu[12] proved that on the equitable total coloring of some join graphs. Veninstine vivik et.al[9] proved an algorithmic approach to equitable total chromatic number of wheel graph, Gear graph, Helm graph and sunlet graph.

2 Preliminaries

Definition 2.1. *The middle graph of a graph G, denoted by $M(G)$ is define as follows, the vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following condition holds: (i) x, y are in $E(G)$ and x, y is adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G. The total graph of a graph G, denoted by $T(G)$ is define as, the vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following condition holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y is adjacent in G (iii) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G. For a simple graph G, let f be a proper $k-$ total coloring of G, $||T_i| - |T_j|| \leq 1$, for $i, j = 1, 2, \ldots k$.*
The partition $\{T_i\} = \{V_i \cup E_i : 1 \leq i \leq k\}$ is called a $k-$ equitable total coloring and $\chi_{et}(G) = min\ \{k/$ k-equitable total coloring of $G\}$ is called the equitable total chromatic number of G, where for all $x \in T_i = V_i \cup E_i$,

$f(x) = i, \qquad for\ i = 1, 2, \ldots k.$

Conjecture 2.4([4]) For any simple graph $G(V, E)$, $\chi_{et}(G) \leq \Delta(G) + 2$.
Conjecture 2.5([13]) For any simple graph $G(V, E)$,

$$\chi_{et}(G) \geq \chi''(G) \geq \Delta(G) + 1.$$

Conjecture 2.6([11]) For every graph G, G has an equitable total $k-$ coloring for each $k \geq max\{\chi''(G), \Delta(G) + 2\}$
proposition 2.7([5]) Any four regular multi-graph can be total colored with 6 colors
proposition 2.8([7])For complete graph K_n with order n,

$$\chi_{et}(K_n) = \begin{cases} n, & \text{if } n \equiv\ 1\ (mod\ 2) \\ n+1, & \text{if } n \equiv\ 0\ (mod\ 2) \end{cases}$$

In this paper, we obtained an equitable total chromatic number of middle graph of path, Middle graph of cycle, Total graph of path and Total graph of cycle.

3 Main Results

Theorem 3.1. *For any positive integer $n \geq 3$, $\chi_{et}(M(P_n)) = 5$.*

Proof. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n) = \{e_i : 1 \leq i \leq n-1\}$, where $\{e_i : 1 \leq i \leq n-1\}$ be the edges $v_iv_{i+1}(1 \leq i \leq n-1)$. By the definition of middle graph, each edge $\{e_i : 1 \leq i \leq n-1\}$ is subdivided by the vertices $\{u_i : 1 \leq i \leq n-1\}$. In $M(P_n)$, the vertex set and the edge set is given by $V(M(P_n)) = \{v_i : 1 \leq i \leq n\} \bigcup \{u_i : 1 \leq i \leq n-1\}$ and $E(M(P_n)) = \{e_i : 1 \leq i \leq n-1\} \bigcup \{e_i^{'} : 1 \leq i \leq n-1\} \bigcup \{e_i^{''} : 1 \leq i \leq n-2\}$, where $e_i(1 \leq i \leq n)$ is an edge $v_iu_i(1 \leq i \leq n-1)$, $e_i{'}(1 \leq i \leq n-1)$ is an edge $u_iv_{i+1}(1 \leq i \leq n-1)$ and $e_i^{''}(1 \leq i \leq n-2)$ is an edge $u_iu_{i+1}(1 \leq i \leq n-2)$.
Define an equitable total coloring f, such that $f : S \rightarrow C$, where $S = V(M(P_n)) \cup E(M(P_n))$ and $C = \{1, 2, 3, 4, 5\}$. Now we assign an equitable total coloring to these vertices and edges as follows.

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv\ 1\ (mod\ 3) \\ 2, & \text{if } i \equiv\ 2\ (mod\ 3) \\ 3, & \text{if } i \equiv\ 0\ (mod\ 3) \end{cases} \quad \text{for } 1 \leq i \leq n$$

$$f(u_i) = \begin{cases} 3, & \text{if } i \equiv 1 \ (mod\ 3) \\ 1, & \text{if } i \equiv 2 \ (mod\ 3) \\ 2, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases} \quad \text{for } 1 \leq i \leq n-1$$

$$f(e_i) = 4, \quad for \ 1 \leq i \leq n-1$$

$$f(e_i') = 5, \quad for \ 1 \leq i \leq n-1$$

$$f(e_i'') = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 3) \\ 3, & \text{if } i \equiv 2 \ (mod\ 3) \\ 1, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases} \quad \text{for } 1 \leq i \leq n-2$$

Based on the above procedure of coloring, the graph $M(P_n)$ is equitable total colored with 5 colors, such that the color classes are $T(M(P_n)) = \{T_1, T_2, T_3, T_4, T_5\}$. We observe that these colors classes T_1, T_2, T_3, T_4, T_5 are independent sets of $M(P_n)$ and its satisfies the inequality $||T_i| - |T_j|| \leq 1$, for $i \neq j$. Hence $\chi_{et}(M(P_n)) \leq 5$. Further, since $\Delta = 4$, we have $\chi_{et}(M(P_n)) \geq \chi''(M(P_n)) \geq \Delta + 1 \geq 4 + 1 \geq 5$. Therefore $\chi_{et}(M(P_n)) = 5$. □

Theorem 3.2. *For any positive integer $n \geq 4$, $\chi_{et}(M(C_n)) = 5$.*

Proof. Let $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $E(C_n) = \{e_i : 1 \leq i \leq n\}$, where $\{e_i : 1 \leq i \leq n-1\}$ be the edges $v_i v_{i+1}, i+1$ taken modulo $n (1 \leq i \leq n)$. By the definition of middle graph, each edge $\{e_i : 1 \leq i \leq n\}$ is subdivided by the vertices $\{u_i : 1 \leq i \leq n\}$. In $M(C_n)$, the vertex set and the edge set is given by $V(M(C_n)) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(M(C_n)) = \{e_i : 1 \leq i \leq n\} \bigcup \{e_i^{'} : 1 \leq i \leq n\} \bigcup \{e_i^{''} : 1 \leq i \leq n\}$, where $e_i (1 \leq i \leq n)$ be an edge $v_i u_i (1 \leq i \leq n)$, $e_i' (1 \leq i \leq n-1)$ be an edge $u_i v_{i+1}, i+1$ taken modulo $n (1 \leq i \leq n)$ and $e_i^{''} (1 \leq i \leq n)$ be an edge $u_i u_{i+1}, i+1$ taken modulo $n (1 \leq i \leq n)$.
We define an equitable total coloring f, such that $f : S \to C$, where $S = V(M(C_n)) \cup E(M(C_n))$ and $C = \{1, 2, 3, 4, 5\}$. The equitable total coloring pattern is as follows. we consider the following two cases
Case(i): when n is even

$$f(v_i) = \begin{cases} 4, & \text{if } i \equiv 1 \ (mod\ 2) \\ 3, & \text{if } i \equiv 0 \ (mod\ 2) \end{cases} \quad \text{for } 1 \leq i \leq n$$

$$f(u_i) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 2) \\ 2, & \text{if } i \equiv 0 \ (mod\ 2) \end{cases} \quad \text{for } 1 \leq i \leq n$$

$$f(e_i) = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 2) \\ 1, & \text{if } i \equiv 0 \ (mod\ 2) \end{cases} \quad \text{for } 1 \leq i \leq n$$

$$f(e_i{}') = 5, \quad for\ 1 \le i \le n$$

$$f(e_i{}'') = \begin{cases} 3, & \text{if } i \equiv 1\ (mod\ 2) \quad \text{for } 1 \le i \le n \\ 4, & \text{if } i \equiv 0\ (mod\ 2) \end{cases}$$

Case(ii): when n is odd

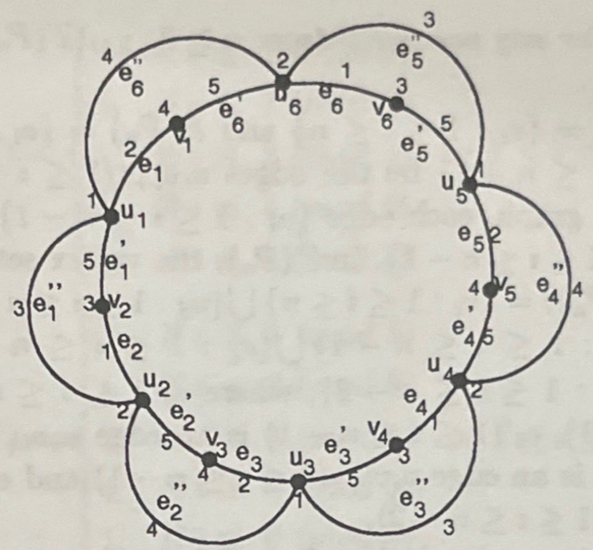

Figure 1: Equitable Total Coloring of middle graph of Cycle $M(C_6)$

$$f(v_1) = 2$$

$$f(v_i) = \begin{cases} 3, & \text{if } i \equiv 0\ (mod\ 2) \quad \text{for } 2 \le i \le n \\ 4, & \text{if } i \equiv 1\ (mod\ 2) \end{cases}$$

$$f(u_i) = \begin{cases} 1, & \text{if } i \equiv 1\ (mod\ 2) \quad \text{for } 1 \le i \le n \\ 2, & \text{if } i \equiv 0\ (mod\ 2) \end{cases}$$

$$f(e_i) = \begin{cases} 1, & \text{if } i \equiv 0\ (mod\ 2) \quad \text{for } 2 \le i \le n-2 \\ 2, & \text{if } i \equiv 1\ (mod\ 2) \end{cases}$$

$$f(e_1) = 4, \quad f(e_{n-1}) = f(e_n) = 1, \quad f(e_i{}') = 5, \quad for\ 1 \le i \le n$$

$$f(e_i{}'') = \begin{cases} 3, & \text{if } i \equiv 0\ (mod\ 2) \quad \text{for } 1 \le i \le n \\ 4, & \text{if } i \equiv 1\ (mod\ 2) \end{cases}$$

Using the above pattern of the coloring, we see that the graph $M(C_n)$ is equitable total colored with 5 colors. The color classes of $M(C_n)$ are grouped as $T(M(C_n)) = \{T_1, T_2, T_3, T_4, T_5\}$. We observe that these colors classes T_1, T_2, T_3, T_4, T_5 are independent sets of $M(C_n)$ and its satisfies the condition $||T_i|-|T_j|| \leq 1$, for $i \neq j$. For example consider the middle graph of cycle with 6 vertices (see Figure 1), for which $|T_1| = |T_2| = |T_3| = |T_4| = |T_5| = 7$ and it holds the inequality $||T_i| - |T_j|| \leq 1$, for $i \neq j$ and it is equitable total colored with 5 colors. Hence $\chi_{et}(M(C_n)) \leq 5$. Further, since $\Delta = 4$, we have $\chi_{et}(M(C_n)) \geq \chi''(M(C_n)) \geq \Delta + 1 \geq 4 + 1 \geq 5$. Therefore $\chi_{et}(M(C_n)) = 5$. □

Theorem 3.3. *For any positive integer $n \geq 3$, $\chi_{et}(T(P_n)) = 5$.*

Proof. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n) = \{e_i : 1 \leq i \leq n-1\}$, where $\{e_i : 1 \leq i \leq n-1\}$ be the edges $v_iv_{i+1}(1 \leq i \leq n-1)$. By the definition of total graph, each edge $\{e_i : 1 \leq i \leq n-1\}$ is subdivided by the vertices $\{u_i : 1 \leq i \leq n-1\}$. In $T(P_n)$, the vertex set and the edge set is given by $V(T(P_n)) = \{v_i : 1 \leq i \leq n\} \bigcup \{u_i : 1 \leq i \leq n-1\}$ and $E(T(P_n)) = \{e_i : 1 \leq i \leq n-1\} \bigcup \{e_i^{'} : 1 \leq i \leq n-1\} \bigcup \{e_i^{''} : 1 \leq i \leq n-1\} \bigcup \{e_i^{'''} : 1 \leq i \leq n-2\}$, where $e_i(1 \leq i \leq n-1)$ is an edge $v_iu_i(1 \leq i \leq n-1)$, $e_i'(1 \leq i \leq n-1)$ is an edge $u_iv_{i+1}(1 \leq i \leq n-1)$, $e_i^{''}(1 \leq i \leq n-1)$ is an edge $v_iv_{i+1}(1 \leq i \leq n-1)$ and $e_i^{'''}(1 \leq i \leq n-2)$ is an edge $u_iu_{i+1}(1 \leq i \leq n-2)$.
Define an equitable total coloring f, such that $f : S \to C$, where $S = V(T(P_n)) \cup E(T(P_n))$ and $C = \{1, 2, 3, 4, 5\}$. The equitable total coloring is obtained by coloring these vertices and edges as follows.

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 5) \\ 3, & \text{if } i \equiv 2 \ (mod\ 5) \\ 5, & \text{if } i \equiv 3 \ (mod\ 5) \\ 2, & \text{if } i \equiv 4 \ (mod\ 5) \\ 4, & \text{if } i \equiv 0 \ (mod\ 5) \end{cases} \quad \text{for } 1 \leq i \leq n$$

$$f(u_i) = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 5) \\ 4, & \text{if } i \equiv 2 \ (mod\ 5) \\ 1, & \text{if } i \equiv 3 \ (mod\ 5) \\ 3, & \text{if } i \equiv 4 \ (mod\ 5) \\ 5, & \text{if } i \equiv 0 \ (mod\ 5) \end{cases} \quad \text{for } 1 \leq i \leq n-1$$

$$f(e_i) = \begin{cases} 4, & \text{if } i \equiv 1 \ (mod\ 5) \\ 1, & \text{if } i \equiv 2 \ (mod\ 5) \\ 3, & \text{if } i \equiv 3 \ (mod\ 5) \\ 5, & \text{if } i \equiv 4 \ (mod\ 5) \\ 2, & \text{if } i \equiv 0 \ (mod\ 5) \end{cases} \quad \text{for } 1 \leq i \leq n-1$$

$$f(e_i') = \begin{cases} 5, & \text{if } i \equiv 1 \ (mod\ 5) \\ 2, & \text{if } i \equiv 2 \ (mod\ 5) \\ 4, & \text{if } i \equiv 3 \ (mod\ 5) \quad \text{for } 1 \leq i \leq n-1 \\ 1, & \text{if } i \equiv 4 \ (mod\ 5) \\ 3, & \text{if } i \equiv 0 \ (mod\ 5) \end{cases}$$

$$f(e_i'') = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 5) \\ 4, & \text{if } i \equiv 2 \ (mod\ 5) \\ 1, & \text{if } i \equiv 3 \ (mod\ 5) \quad \text{for } 1 \leq i \leq n-1 \\ 3, & \text{if } i \equiv 4 \ (mod\ 5) \\ 5, & \text{if } i \equiv 0 \ (mod\ 5) \end{cases}$$

$$f(e_i''') = \begin{cases} 3, & \text{if } i \equiv 1 \ (mod\ 5) \\ 5, & \text{if } i \equiv 2 \ (mod\ 5) \\ 2, & \text{if } i \equiv 3 \ (mod\ 5) \quad \text{for } 1 \leq i \leq n-2 \\ 4, & \text{if } i \equiv 4 \ (mod\ 5) \\ 1, & \text{if } i \equiv 0 \ (mod\ 5) \end{cases}$$

Based on the above method of coloring, it is clear that the graph $T(P_n)$

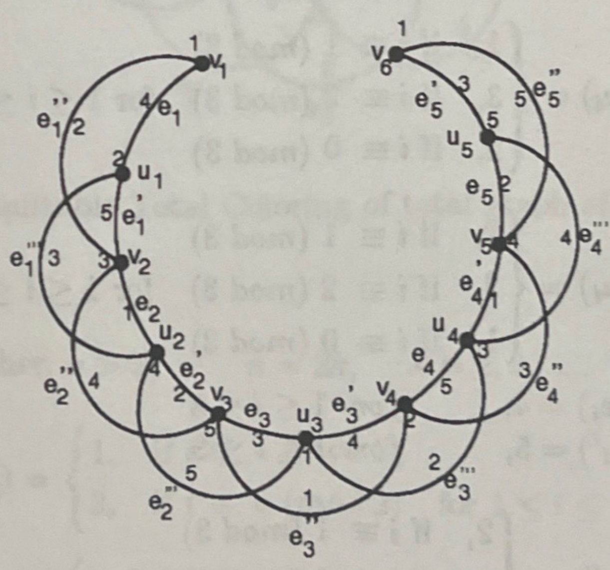

Figure 2: Equitable Total Coloring of total graph of Path $T(P_6)$

is equitable total colored with 5 colors, such that the color classes grouped are $T(T(P_n)) = \{T_1, T_2, T_3, T_4, T_5\}$. We observe that these color classes $\{T_1, T_2, T_3, T_4, T_5\}$are independent sets of $T(P_n)$ and its holds the inequality $||T_i| - |T_j|| \leq 1$, for $i \neq j$. For example consider the total graph of path with 6 vertices (see Figure 2), for which $|T_1| = |T_3| = |T_4| = |T_5| = 7$ and $|T_2| = 8$ and it satisfies the condition $||T_i| - |T_j|| \leq 1$, for $i \neq j$ and so it is equitable total colored with 5 colors. Hence $\chi_{et}(T(P_n)) \leq 5$. Further, since $\Delta = 4$, we have $\chi_{et}(T(P_n)) \geq \chi''(T(P_n)) \geq \Delta + 1 \geq 4 + 1 \geq 5$. Therefore $\chi_{et}(T(P_n)) = 5$. □

Theorem 3.4. *For any positive integer n,* $\chi_{et}(T(C_n)) = \begin{cases} 5, & n = 3 \\ 6, & n \neq 3 \end{cases}$

Proof. Let $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $E(C_n) = \{e_i : 1 \leq i \leq n\}$, where $\{e_i : 1 \leq i \leq n\}$ be the edges $v_i v_{i+1}, i+1$ taken modulo $n (1 \leq i \leq n)$. By the definition of total graph, each edge $\{e_i : 1 \leq i \leq n\}$ is subdivided by the vertices $\{u_i : 1 \leq i \leq n\}$. In $T(C_n)$, the vertex set and the edge set is given by $V(T(C_n)) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(T(C_n)) = \{e_i : 1 \leq i \leq n\} \bigcup \{e_i' : 1 \leq i \leq n\} \bigcup \{e_i'' : 1 \leq i \leq n\} \bigcup \{e_i''' : 1 \leq i \leq n\}$, where $e_i (1 \leq i \leq n)$ be an edge $v_i u_i (1 \leq i \leq n)$, $e_i' (1 \leq i \leq n)$ be an edge $u_i v_{i+1}, i+1$ taken modulo $n (1 \leq i \leq n)$, $e_i'' (1 \leq i \leq n)$ be an edge $v_i v_{i+1}, i+1$ taken modulo $n (1 \leq i \leq n)$ and $e_i''' (1 \leq i \leq n)$ be an edge $u_i u_{i+1}, i+1$ taken modulo $n (1 \leq i \leq n)$.
We define a function f, such that $f : S \to C$, where $S = V(T(C_n)) \cup E(T(C_n))$ and $C = \{1, 2, 3, 4, 5, 6\}$. Now we assign an equitable total coloring to these vertices and edges as follows. we consider following three cases
Case(i): When $n = 3$

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 3) \\ 3, & \text{if } i \equiv 2 \ (mod\ 3) \\ 2, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases} \quad \text{for } 1 \leq i \leq 3$$

$$f(u_i) = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 3) \\ 3, & \text{if } i \equiv 2 \ (mod\ 3) \\ 1, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases} \quad \text{for } 1 \leq i \leq 3$$

$$f(e_i) = 4, \qquad for \ 1 \leq i \leq 3$$
$$f(e_i') = 5, \qquad for \ 1 \leq i \leq 3$$

$$f(e_i'') = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 3) \\ 1, & \text{if } i \equiv 2 \ (mod\ 3) \\ 3, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases} \quad \text{for } 1 \leq i \leq 3$$

$$f(e_i''') = \begin{cases} 3, & \text{if } i \equiv 1 \ (mod\ 3) \\ 2, & \text{if } i \equiv 2 \ (mod\ 3) \\ 1, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases} \quad \text{for } 1 \leq i \leq 3$$

From the above rule of total coloring, it is easy to see that the graph $T(C_3)$ is equitable total colored with 5 colors. The color classes of $T(C_3)$ are grouped as $T(T(C_3)) = \{T_1, T_2, T_3, T_4, T_5\}$. We observe that these color classes $\{T_1, T_2, T_3, T_4, T_5\}$ are independent sets of $T(C_3)$ and its satisfies the inequality $||T_i| - |T_j|| \leq 1$, for $i \neq j$, for which $|T_1| = |T_2| = |T_3| = 4$ and $|T_4| = |T_5| = 3$ and it satisfies the inequality $||T_i| - |T_j|| \leq 1$, for $i \neq j$ and it is equitable total colored with 5 colors. Hence $\chi_{et}(T(C_3)) \leq 5$. Further, since $\Delta = 4$, we have $\chi_{et}(T(C_3)) \geq \chi''(T(C_3)) \geq \Delta + 1 \geq 4 + 1 \geq 5$. Therefore $\chi_{et}(T(C_3)) = 5$.

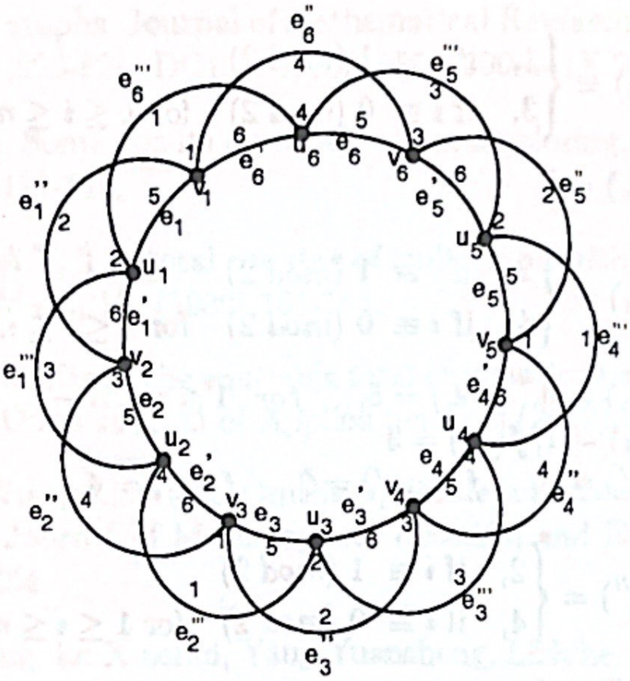

Figure 3: Equitable Total Coloring of total graph of Cycle $T(C_6)$

Case(ii): When $n > 3, \qquad n = 2k, \quad k = 2, 3.......$

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 2) \\ 3, & \text{if } i \equiv 0 \ (mod\ 2) \quad \text{for } 1 \leq i \leq n \end{cases}$$

$$f(u_i) = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 2) \\ 4, & \text{if } i \equiv 0 \ (mod\ 2) \quad \text{for } 1 \leq i \leq n-1 \end{cases}$$

$$f(e_i) = 5, \qquad for \ \ 1 \leq i \leq n$$
$$f(e_i{}') = 6, \ \ f(e_n{}') = 6 \qquad for \ \ 1 \leq i \leq n-1$$

$$f(e_i{}'') = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 2) \\ 4, & \text{if } i \equiv 0 \ (mod\ 2) \quad \text{for } 1 \leq i \leq n-1 \end{cases}$$

$f(e_n'') = 4$

$$f(e_i''') = \begin{cases} 3, & \text{if } i \equiv 1 \ (mod\ 2) \\ 1, & \text{if } i \equiv 0 \ (mod\ 2) \quad \text{for } 1 \leq i \leq n-1 \end{cases}$$

$f(e_n''') = 1$

Case(iii): When $n > 3, \qquad n = 2k+1, \qquad k = 2, 3......$

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 2) \\ 3, & \text{if } i \equiv 0 \ (mod\ 2) \quad \text{for } 1 \leq i \leq n-1 \end{cases}$$

$f(v_n) = 5$

$$f(u_i) = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 2) \\ 4, & \text{if } i \equiv 0 \ (mod\ 2) \quad \text{for } 1 \leq i \leq n-1 \end{cases}$$

$f(u_n) = 6, \quad f(e_i) = 5, \qquad for \ \ 1 \leq i \leq n-2$
$f(e_{n-1}) = 1, f(e_n) = 3$
$f(e_i') = 6, \qquad f(e_{n-1}') = 2, \qquad f(e_n') = 4 \qquad\qquad for \ \ 1 \leq i \leq n-2$

$$f(e_i'') = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 2) \\ 4, & \text{if } i \equiv 0 \ (mod\ 2) \quad \text{for } 1 \leq i \leq n-1 \end{cases}$$

$f(e_n'') = 6$

$$f(e_i''') = \begin{cases} 3, & \text{if } i \equiv 1 \ (mod\ 2) \\ 1, & \text{if } i \equiv 0 \ (mod\ 2) \quad \text{for } 1 \leq i \leq n-2 \end{cases}$$

$f(e_{n-1}''') = 5, \qquad f(e_n''') = 6$

The above pattern of total coloring, the graph $T(C_n)$ is equitable total colored with 6 colors. The color classes of $T(C_n)$ are grouped as $T(T(C_n)) = \{T_1, T_2, T_3, T_4, T_5, T_6\}$. We observe that these color classes $\{T_1, T_2, T_3, T_4, T_5, T_6\}$ are independent sets of $T(C_n)$ and its satisfies the inequality $||T_i| - |T_j|| \leq 1$, for $i \neq j$. For example consider the total graph of path with 6 vertices (see Figure 3), for which $|T_1| = |T_2| = |T_3| = |T_4| = |T_5| = |T_6| = 7$ and it satisfies the inequality $||T_i| - |T_j|| \leq 1$, for $i \neq j$. When $n \neq 3$, in this case every vertex is adjacent to exactly four vertices of same degree and due to the incidence and adjacency of elements, five colors can not be suffice for the total coloring. Thus $\chi_{et}(T(C_n)) \geq \chi''(T(C_n)) \neq 5$. Hence $\chi_{et}(T(C_n)) = 6$. □

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