Mittag-Leffler Function Method For Solving Nonlinear Riccati Differential Equation With Fractional Order

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Abstract

In this Paper, we establish a new application of the Mittag-Leffler Function method that will enlarge the application to the non linear Riccati Differential equations with fractional order. This method provides results that converge promptly to the exact solution. The description of fractional derivatives is made in the Caputo sense. To emphasize the consistency of the approach, few illustrations are presented to support the outcomes. The outcomes declare that the procedure is very constructive and relavent for determining non linear Riccati differential equations of fractional order.

Key Words: Mittag-Leffler Function method, Riccati differential equation.

1 Fractional Calculus

The notation of differentiation to fractional order be done in different ways. For instance, renowned mathematicians like the Riemann-Lioulle,Grunwald-

Letnikov Caputo generalized functional approach. Though fractional derivative approach was used by many mathematical researchers, it is considered unsuitable to the practical problems as it demands a definition of the initial conditions of fractional order which have not been explained meaningfully so far. An alternative definition proposed by Capito gains greater significance in describing integer order primary circumstances for fractional order differential equation.

The greatest asset of Capito approach lies in using in the identical system for both integer order and fractional order. S.Z.Rida and A.A.M.Arafa was applied New Method for solving Linear Fractional Differential Equations in [1], A.A.M.Arafa and S.Z.Rida is used Generalized Mittag-Leffler Function Method (MLFM) for solving Nonlinear Fractional differential equations in [2], Abiodun A.Opanuga was used New approach forsolving Quadratic Riccati Differential Equationsin[3], P.L.Suresh and D.Piriadarshani were determined the numerical solution of various kinds of Riccati differential equation using Differential Transform Method in [4], T.Allahviranloo and Sh.S.Behzadi applied Iterative techniques to compute the solution of General Riccati Equation in [5], Farshid Mirzaee explained the main advantage of Differential Transform Method (DTM) to implement straightly to Linear and Nonlinear System of Ordinary Differential Equations in [6]. In this work, we enlarged the application of Mittag-Leffler Function Method (MLFM) to the solutions of Nonlinear Riccati Differential equations of fractional order.

$$\frac{d^{\mu}y(x)}{dx^{\mu}} + A(x)y(x) + B(x)y^2(x) + C(x) = 0,\ y(x_0) = y_0,\ 0 < \mu \leq 1,\ x \in R \quad (1)$$

Theorem 1.1 The fractional derivative of $f(x)$ in the Caputo sense is defined as $D^{\mu}f(x) = I^{p-\mu}D^{p}f(x) = \frac{1}{\Gamma(p-\mu)}\int_0^x (x-t)^{p-\mu+1} f^{(p)}(t)dt$ (2) for $p-1 < \mu \leq p$, $p \in N$, $x > 0$. The Caputo derivative, $D^{\mu}K = 0$, K is invariable

$$D^{\mu}t^{n} = \begin{cases} 0 & (n \leq \mu - 1) \\ \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}t^{n-\mu}, & (n > \mu - 1) \end{cases} \quad (3).$$

Theorem 1.2 For p to be the smallest integer that exceeds μ, the Caputo fractional derivative of order $\mu > 0$ is defined as $D^{\mu}u(x,t) = \frac{\partial^{\mu}u(x,t)}{\partial t^{\mu}}$

$$= \begin{cases} \frac{1}{\Gamma(p-\mu)} \int_0^x (t-\tau)^{p-\mu+1} \frac{\partial^p u(x,t)}{\partial \tau^p} d\tau & for \ \ p-1 \le \mu \le p \\ \frac{\partial^p u(x,t)}{\partial t^p}, & for \ \ \mu = p \in N \end{cases} \quad (4)$$

Analysis of the Method

The Mittag-Leffler functions E_μ and $E_{\mu,\sigma}$, defined by

$E_\mu(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k+1)}$, $E_{\mu,\sigma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k+\sigma)}, \mu > 0, \sigma > 0$ (5)

In this work, we discussed two different methods for solving nonlinear Riccati differential equations of fractional order through the generalized Mittag-Leffler Function Method (MFLM) and Generalized Differential Transform Method (GDTM). The generalization Mittag-Leffler function method proposed that y(x) isdisintegrated by limitless sequence of elements.

$y = E_\mu(ax^\mu) = \sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma(n\mu+1)}$ (6)

The definition of fractional calculus are given by

$D^\mu y = \sum_{n=1}^{\infty} \frac{a^n x^{n-1\mu}}{\Gamma((n-1)\mu+1)}$ (7)

$D^{2\mu} y = \sum_{n=2}^{\infty} \frac{a^n x^{n-2\mu}}{(n-2)\Gamma((n-2)\mu+1)}$ (8)

Applications and Results

Herein, we study few illustrations exhibit the attainment and productivity of the generalized Mittag-Leffler function approach for solving nonlinear Riccati differential equations of fractional order.

Example 1. When $C(x) = 0$ & $A(x)$ and $B(x)$ are real constants and $\mu = \frac{1}{4}$

Equation (1) implies that

$$\frac{d^\mu y(x)}{dx^\mu} + A(x)y(x) + B(x)y^2(x) = 0,\ y(0) = 0, \quad (9)$$

By applying equations (6) and (7) into (9), we get

$$\sum_{n=1}^{\infty} \frac{a^n x^{n-1\mu}}{\Gamma((n-1)\mu+1)} + \mathrm{A}\left(\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma(n\mu+1)}\right)^2 + B\left(\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma(n\mu+1)}\right) = 0 \ (10)$$

$\sum_{n=1}^{\infty} \frac{a^n x^{n-1\mu}}{\Gamma((n-1)\mu+1)} + A(\sum_{n=0}^{\infty} c^n x^{n\mu}) + B(\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma((n)\mu+1)}) = 0$ (11)

where $c^n = \sum_{k=0}^{n} \frac{a^k x^{n-k}}{\Gamma(k\mu+1)\Gamma((n-k)\mu+1)}$

Now Replace n by $n+1$ in the first term, we obtain

$\sum_{n=0}^{\infty} \frac{a^{n+1} x^{n\mu}}{\Gamma((n)\mu+1)} + A(\sum_{n=0}^{\infty} c^n x^{n\mu}) + B(\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma((n)\mu+1)}) = 0$ (12)

Combining the identical terms, we assume the form

$\sum_{n=0}^{\infty} (\frac{a^{n+1}}{\Gamma(n\mu+1)} + Ac^n + Ba^n) x^{n\mu} = 0$ (13)

$\frac{a^{n+1}}{\Gamma(n\mu+1)} + Ac^n + Ba^n = 0$

$a^{n+1} = (-Ac^n - Ba^n)\Gamma(n\mu+1)$ (14)

At $n = 0$, $a^1 = (Ac^0 - Ba^0)\Gamma(1) = -A - B$

At $n = 1$, $a^2 = (-Ac^1 - Ba^1)\Gamma(\frac{1}{3}+1) = (A+B)(2A + B(0.89))$

At $n = 2$, $a^3 = (-Ac^2 - Ba^2)\Gamma(\frac{2}{3}+1) = -A[(2.2)(A+B)2A + 0.89B + 1.25(A+B)^2] - B(A+B)[2A + 0.89B]$

Substituting into (5), we get

$y(x) = a^0 + a^1 \frac{x^{\mu}}{\Gamma(\mu+1)} + a^2 \frac{x^{2\mu}}{\Gamma(2\mu+1)} + a^3 \frac{x^{3\mu}}{\Gamma(3\mu+1)} + a^4 \frac{x^{4\mu}}{\Gamma(4\mu+1)} + \ldots$ (15)

The solution of equation (9) is

$y(x) = 1 + (-A-B)\frac{x^{(\frac{1}{4})}}{\Gamma(\frac{1}{4}+1)} + (A+B)[2A+B(0.89)]\frac{x^{(\frac{2}{4})}}{\Gamma(\frac{2}{4}+1)} - \{(A[(2.2)(A+B)2A + 0.89B + 1.25(A+B)^2] - B(A+B)[2A+0.89B]\}\frac{x^{(\frac{3}{4})}}{\Gamma(\frac{3}{4}+1)} + \ldots$

When $A > 0$ and $B < 0$ i.e A=1 and B = -2

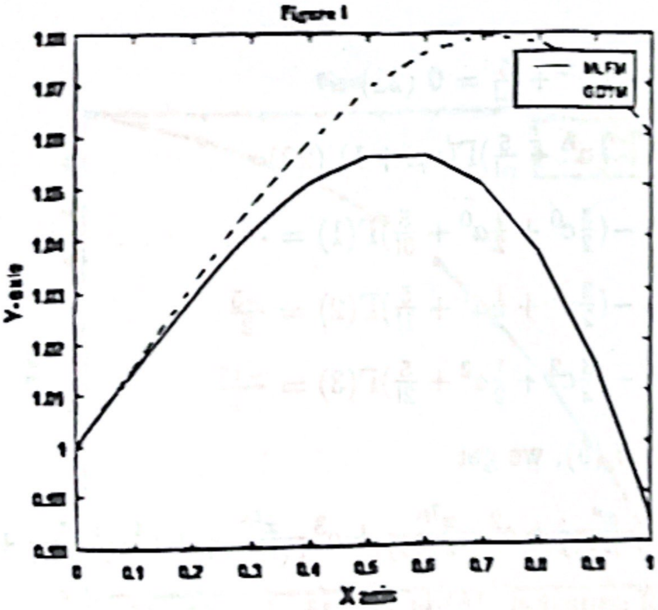
Then the solution of (9) is $y(x) = 1 + 1.1x^{\frac{1}{4}} - 0.24x^{\frac{1}{2}} - 0.68x^{\frac{3}{4}} + \ldots$

The solution of equation(9) by GDTM is $y(x) = 1 + 1.1x^{\frac{1}{4}} - 0.22x^{\frac{1}{2}} - 0.42x^{\frac{3}{4}} + \ldots$

Table 1:

x	MLFM	GDTM	Error
0	1.0000000	1.0000000	0.0000000
0.1	1.0153300	1.0163950	0.0010650
0.2	1.0296400	1.0321600	0.0025200
0.3	1.0419100	1.0466650	0.0047550
0.4	1.0511200	1.0592800	0.0081600
0.5	1.0562500	1.0693750	0.0131250
0.6	1.0562800	1.0763200	0.0200400
0.7	1.0501900	1.0794850	0.0292950
0.8	1.0369600	1.0782400	0.0412800
0.9	1.0155700	1.0719550	0.0563850
1	0.9850000	1.0600000	0.0750000

Figure 1



Example 2. When $A(x) = \frac{1}{2} \& B(x) = \frac{3}{2}$ and $C(x) = 5e^{x^\mu}$ with $\mu = 1$

Equation (1) implies that

$$\frac{d^\mu y(}{dx^\mu} + \frac{1}{2}y^2 + \frac{3}{2}y + 5e^{x^\mu} = 0,\ y(0) = 0, \quad (16)$$

By applying equations (5) and (6) into (16), we get $\sum_{n=1}^{\infty} \frac{a^n x^{n-1\mu}}{\Gamma((n-1)\mu+1)} + \frac{1}{2}(\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma(n\mu+1)})^2 + 3\frac{}{2}(\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma(n\mu+1)}) + 5\sum_{n=0}^{\infty} \frac{x^{n\mu}}{n!} = 0$ (17)

$\sum_{n=1}^{\infty} \frac{a^n x^{n-1\mu}}{\Gamma((n-1)\mu+1)} + \frac{1}{2}(\sum_{n=0}^{\infty} c^n x^{n\mu}) + \frac{3}{2}(\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma((n)\mu+1)}) + 5\sum_{n=0}^{\infty} \frac{x^{n\mu}}{n!} = 0$ (18)

where $c^n = \sum_{k=0}^{n} \frac{a^k x^{n-k}}{\Gamma(k\mu+1)\Gamma((n-k)\mu+1)}$

Now Replace n by $n+1$ in the first term, we obtain

$\sum_{n=0}^{\infty} \frac{a^{n+1} x^{n\mu}}{\Gamma((n)\mu+1)} + \frac{1}{2}(\sum_{n=0}^{\infty} c^n x^{n\mu}) + \frac{3}{2}(\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma((n)\mu+1)}) + 5\sum_{n=0}^{\infty} \frac{x^{n\mu}}{n!} = 0$ (19)

Combining the identical terms, we assume the form

$\sum_{n=0}^{\infty} (\frac{a^{n+1}}{\Gamma(n\mu+1)} + \frac{3}{2}c^n + \frac{1}{2}a^n + \frac{5}{n!}) x^{n\mu} = 0$ (20)

Equate to zero and identifying the coefficients, we obtain $x^{n\mu}$ recurrence relations

$\frac{a^{n+1}}{\Gamma(n\mu+1)} + \frac{3}{2}c^n + \frac{1}{2}a^n + \frac{5}{n!} = 0$ (21)

$a^{n+1} = -(\frac{3}{2}c^n + \frac{1}{2}a^n + \frac{5}{n!})\Gamma(n\mu+1)$ (22)

At $n = 0$, $a^1 = -(\frac{3}{2}c^0 + \frac{1}{2}a^0 + \frac{5}{0!})\Gamma(1) = -5$

At $n = 1$, $a^2 = -(\frac{3}{2}c^1 + \frac{1}{2}a^1 + \frac{5}{1!})\Gamma(2) = \frac{-5}{2}$

At $n = 2$, $a^3 = -(\frac{3}{2}c^2 + \frac{1}{2}a^2 + \frac{5}{2!})\Gamma(3) = \frac{-155}{2}$

Substituting into (5), we get

$$y(x) = a^0 + a^1 \frac{x^\mu}{\Gamma(\mu+1)} + a^2 \frac{x^{2\mu}}{\Gamma(2\mu+1)} + a^3 \frac{x^{3\mu}}{\Gamma(3\mu+1)} + a^4 \frac{x^{4\mu}}{\Gamma(4\mu+1)} + \ldots \quad (23)$$

The solution of equation (16) is

$$y(x) = -5x\frac{-5}{4}x^2 + \frac{-155}{12}x^3 + \ldots$$

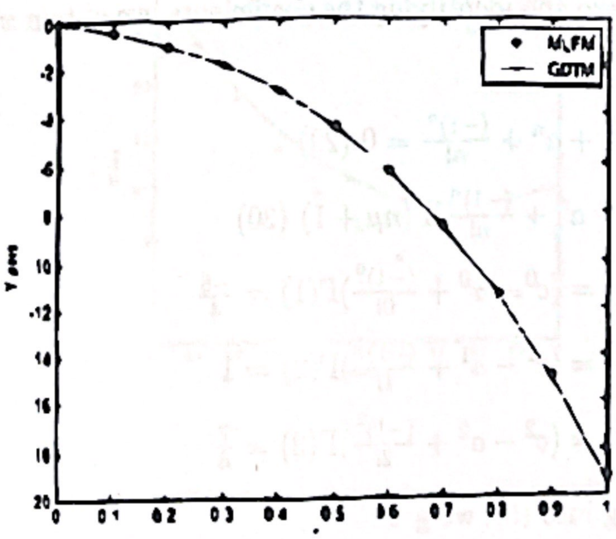
The solution of equation (16) by GDTM is

$$y(x) = -5x\frac{-5}{4}x^2 - \frac{-315}{24}x^3 + \ldots$$

Table 2:

x	MLFM	GDTM	Error
0	0.00000	0.00000	0.00000
0.1	-0.52542	-0.52563	0.00021
0.2	-1.15333	-1.15500	0.00167
0.3	-1.96125	-1.96688	0.00562
0.4	-3.02667	-3.04000	0.01333
0.5	-4.42708	-4.45313	0.02604
0.6	-6.24000	-6.28500	0.04500
0.7	-8.54292	-8.61438	0.07146
0.8	-11.41333	-11.52000	0.10667
0.9	-14.92875	-15.08063	0.15187
1	-19.16667	-19.37500	0.20833

Figure 2



Example 3. When $A(x) = -1 \& B(x) = 1$ and $C(x) = 5e^{-2x}$ with $\mu = 1$

Equation (1) implies that

$$\frac{d^\mu y(}{dx^\mu} + y^2 - y + 5e^{-2x} = 0, \; y(0) = \frac{1}{2}, \quad (24)$$

By applying equations (5) and (6) into (24), we get

$\sum_{n=1}^{\infty} \frac{a^n x^{n-1\mu}}{\Gamma((n-1)\mu+1)}$ - $(\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma(n\mu+1)})^2 + (\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma(n\mu+1)}) + \sum_{n=0}^{\infty} \frac{(-1)^{n\mu} x^{n\mu}}{n!} =$ 0 (25)

$\sum_{n=1}^{\infty} \frac{a^n x^{n-1\mu}}{\Gamma((n-1)\mu+1)}$ - $(\sum_{n=0}^{\infty} c^n x^{n\mu}) + (\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma((n)\mu+1)}) + \sum_{n=0}^{\infty} \frac{(-1)^{n\mu} x^{n\mu}}{n!}$ $= 0$ (26)

where $c^n = \sum_{k=0}^{n} \frac{a^k x^{n-k}}{\Gamma(k\mu+1)\Gamma((n-k)\mu+1)}$

Now Replace n by $n+1$ in the first term, we obtain

$\sum_{n=0}^{\infty} \frac{a^{n+1} x^{n\mu}}{\Gamma((n)\mu+1)}$ - $(\sum_{n=0}^{\infty} c^n x^{n\mu}) + (\sum_{n=0}^{\infty} \frac{a^n x^{n\mu}}{\Gamma((n)\mu+1)}) + \sum_{n=0}^{\infty} \frac{(-1)^{n\mu} x^{n\mu}}{n!}$ $= 0$ (27)

Combining the identical terms, we assume the form

$\sum_{n=0}^{\infty} (\frac{a^{n+1}}{\Gamma(n\mu+1)} - c^n + a^n + \frac{(-1)^n}{n!}) x^{n\mu} = 0$ (28)

Equate to zero and identifying the coefficients, we obtain $x^{n\mu}$ recurrence relations

$\frac{a^{n+1}}{\Gamma(n\mu+1)} - c^n + a^n + \frac{(-1)^n}{n!} = 0$ (29)

$a^{n+1} = (c^n - a^n + \frac{(-1)^n}{n!})\Gamma(n\mu + 1)$ (30)

At $n = 0$, $a^1 = (c^0 - a^0 + \frac{(-1)^0}{0!})\Gamma(1) = \frac{-5}{4}$

At $n = 1$, $a^2 = (c^1 - a^1 + \frac{(-1)^1}{1!})\Gamma(2) = 1$

At $n = 2$, $a^3 = (c^2 - a^2 + \frac{(-1)^2}{2!})\Gamma(3) = \frac{7}{8}$

Substituting into (5), we get

$y(x) = a^0 + a^1 \frac{x^\mu}{\Gamma(\mu+1)} + a^2 \frac{x^{2\mu}}{\Gamma(2\mu+1)} + a^3 \frac{x^{3\mu}}{\Gamma(3\mu+1)} + a^4 \frac{x^{4\mu}}{\Gamma(4\mu+1)} + \ldots$ (31)

The solution of equation (24) is

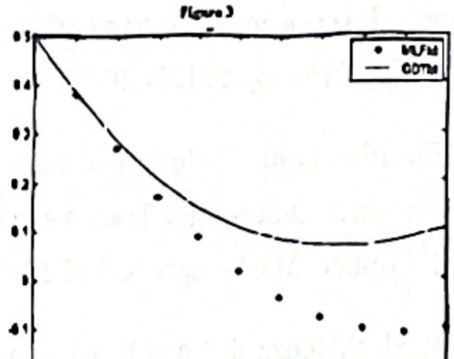
$y(x) = \frac{1}{2} - \frac{5}{4}x + \frac{1}{2}x^2 + \frac{7}{48}x^3 + \ldots$

The solution of equation (24) by GDTM is

$y(x) = \frac{1}{2} - \frac{5}{4}x + x^2 + \frac{7}{48}x^3 + \ldots$

Table 3:

x	MLFM	GDTM	Error
0	0.50000	0.50000	0.00000
0.1	0.38015	0.38485	-0.00471
0.2	0.27117	0.28883	-0.01767
0.3	0.17394	0.21106	-0.03713
0.4	0.08933	0.15067	-0.06133
0.5	0.01823	0.10677	-0.08854
0.6	-0.03850	0.07850	-0.11700
0.7	-0.07998	0.06498	-0.14496
0.8	-0.10533	0.06533	-0.17067
0.9	-0.11369	0.07869	-0.19238
1	-0.10417	0.10417	-0.20833



2 Conclusion

In this paper, we applied the Mittag-Leffler functiontechnique for reviewing the explanations of nonlinear Riccati differential equations of fractional order. This technique is very dominant and effective for finding the solution of nonlinear Riccati differential equations of fractional order than the Differential Transform Method. Results were proved through the numerical examples.

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