On Hadwiger Conjecture for Certain Families of Graphs with Restricted Number of Cycles

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Abstract

Let G_k, $(k \geq 0)$ be the family of graphs that have exactly k cycles. For $0 \leq k \leq 3$, we compute the Hadwiger number for graphs in G_k and further deduce that the Hadwiger Conjecture is true for such families of graphs.

1 Introduction

All the graphs dealt in this paper are connected, finite and simple[1]. The notion of forming new graphs from existing graphs plays a crucial role in determining isomorphic graphs and subgraphs. When two or more graphs are considered, graph operations such as union,join and graph products such as Cartesian, tensor, strong and lexicographic products[1] and graph compositions[4] are used to form new families of graphs. Properties of newly formed resultant graphs are determined, in a majority case, using the basic graphs that were considered initially. New graphs can be formed from a single graph too. The operations vary from subdivision of edges to double duplication of graphs. All the above mentioned operations usually result in a graph that is superior to the original graph considered. The reverse case is possible too, i.e., given a graph we can reduce it to a smaller graph of our choice or need. Two elementary operations used are deletion of vertices and edges. Deletion may result in disconnecting a connected graph. To avoid this, we can employ another operation on edges called contraction of edges[1, 3].

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Let $e = \{u, v\}$ be an edge of a graph G. When the edge e is contracted, its end vertices are also deleted and a new vertex w is introduced wherein those vertices that were initially adjacent with either u or v (or both) are now made adjacent with w. The resulting graph is denoted by $G \bullet e$. If G is connected, then $G \bullet e$ will also be connected. Clearly $n(G \bullet e) = n(G) - 1$, where $n(X)$ denotes the order of the graph X and $m(G \bullet e) \leq m(G) - 1$, where $m(X)$ denotes the size of the graph X.

In 1943, Hugo Hadwiger[5, 6] defined that the maximum order of the complete subgraph resulting due to any/all of the following three operations - deletion of a vertex; deletion of an edge and/or contraction of an edge - is called as the Hadwiger number of the given graph G and it is denoted by $\eta(G)$. Hadwiger further conjectured[5, 6] that for any graph G, $\eta(G) \geq \chi(G)$, where $\chi(G)$ denotes the chromatic number of G. Chromatic number of a graph is defined as the number of distinct colours used to colour the vertices of G such that adjacent vertices receive distinct colours. Bela Bollobas[2], in 1993 stated that the Hadwiger Conjecture is one of the outstanding problems in Graph Theory.

2 Preliminary Results

Proposition 2.1. *The Hadwiger number for a tree is* 2.

Proof. Let $\mathbb{T}$ denote the families of all trees. For a tree $T \in \mathbb{T}$, there is a unique path between any two pair of vertices, deletion of either vertices or edges result in a forest and contraction of edges will result in a tree again. Hence $\eta(T) = 2$ for all $T \in \mathbb{T}$. □

Corollary 2.2. *Trees satisfy Hadwiger conjecture.*

Proof. Trees are bipartite and hence their chromatic number is 2. The result follows from Proposition 2.1. □

Proposition 2.3. *The Hadwiger number for a cycle* C_n, $n \geq 3$, *is* 3.

Proof. Consider a cycle C_n, $n \geq 3$. Deletion of an edge or a vertex of C_n will result in a tree. When an edge is contracted, it results in a cycle of order $n - 1$. This process, upon successive application, will result in C_3. Hence the Hadwiger number for any cycle C_n, $n \geq 3$ is 3. □

Corollary 2.4. *Hadwiger conjecture holds true in the family of cycles.*

Proof. We know that the chromatic number $\chi(C_n)$ for a cycle is 2 when n is even and 3 whenever n is odd. Hence it follows from Proposition 2.3 that Hadwiger conjecture is satisfied by all cycles. □

3 Main Results

Given $k \geq 0$, let $\mathbb{G}_k$ denote the family of graphs that contain exactly k cycles. The following theorems are the main results of our paper.

Theorem 3.1. *For each* $0 \leq k \leq 3$, *the family of graphs* $\mathbb{G}_k$ *has Hadwiger number atmost* 3.

Proof. **Case 1:** $k = 0$. The result is obvious as all the trees form $\mathbb{G}_0$.
Case 2: $k = 1$. The class of unicyclic graphs forms $\mathbb{G}_1$. In unicyclic graphs, the edges that are not in the cycle are all cut edges. Contracting a cut edge leads to a unicyclic graph again. Contracting an edge from the cycle will also yield a unicyclic graph only. When these two processes are repeated successively, the graph finally obtained is a triangle and hence $\eta(G) = 3$ for all $G \in \mathbb{G}_1$.
Case 3: $k = 2$. All bicyclic graphs satisfy the following property: The cycles C_1 and C_2 in the bicyclic graph $G \in \mathbb{G}_2$ may have at most one vertex in common. The proof is similar to that of the case when $k = 1$ and hence $\eta(G) = 3$ for all $G \in \mathbb{G}_2$.
Case 4: $k = 3$. Any graph $G \in \mathbb{G}_3$, with exactly three cycles falls into one of the following two categories:

1. G has exactly three internally (edge) disjoint cycles
2. In G, let C_1, C_2 and C_3 be the three cycles. Then any pair of C_i, C_j $(i \neq j,\ 1 \leq i, j \leq 3)$ share a common path.

Subcase 4a: When the three cycles share at most one vertex in common, the contractions of edges result in a subgraph with the same property. Contracting edges successively results in three mutually (edge) disjoint triangles. Hence $\eta(G) = 3$.
Subcase 4b: Let the graph $G \in \mathbb{G}_3$ have three cycles C_1, C_2, C_3 where, without loss of generality, C_1 and C_2 share a common path P_m. Then $C_3 = (C_1 \cup C_2) - P_m$. Let edge $e_1 \in (C_1 \cup C_2 \cup C_3)$. Then contraction of e_1 will again result in a graph G' which contains three cycles and hence $G' \in \mathbb{G}_3$. Let $e_2 \notin (C_1 \cup C_2 \cup C_3)$. Then e_2 will be a cut edge and contraction of e_2 will also result in a graph that is a member of G_3. Applying this process successively will result in the graph $K_4 - e$ which has Hadwiger number 3. Therefore $\eta(G) = 3$ for all $G \in \mathbb{G}_3$. □

Theorem 3.2. *Hadwiger Conjecture is true for the family* $\mathbb{G}_k$, $0 \leq k \leq 3$, *of graphs.*

Proof. Proof is trivial for $k = 0, 1$. When $k = 2$, if $G \in \mathbb{G}_2$ contains an odd cycle, then $\chi(G) = 3$ and 2 otherwise. Similarly if C_1, C_2, C_3 are three cycles of a graph $H \in \mathbb{G}_3$, then

$$\chi(H) = \begin{cases} 3, & \text{if atleast one of } C_1, C_2, C_3 \text{ is of odd length} \\ 2, & \text{otherwise.} \end{cases}$$

In all the cases, by theorem 3.1, $\eta(H) \geq \chi(H)$. □

4 Conclusion

In this paper we have determined the Hadwiger number for graphs with at most 3 cycles and concluded that Hadwiger Conjecture is true for these graphs. This work can be extended for graphs with exactly k cycles for $k \geq 4$.

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