On the Nonsplit Monophonic Number of a Graph

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Abstract

In this paper, we introduced a new concept called nonsplit monophonic set and its relative parameter nonsplit monophonic number $m_{ns}(G)$. Some certain properties of nonsplit monophonic sets are discussed. The nonsplit monophonic number of standard graphs are investigated. Some existence theorems on nonsplit monophonic number are established.

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1 Introduction

Throughout this paper, we consider the graph $G = (V, E)$ which is simple, finite, undirected connected graph of order n and size m. For basic notations, we refer to Harary [4]. The length of a shortest x-y path in G is the distance [1] $d(x, y)$ between two vertices x and y in a connected graph G. An x-y path of length $d(x, y)$ is called an x-y geodesic. "The distance between v and a vertex farthest from v is the eccentricity $e(v)$ of v". It is defined that *radius* as $rad\ G = min(e(v))$ and *diameter* as $diam\ G = max(e(v))$ of G. "The set $N(v)$ consisting of all vertices which are adjacent with v is the *neighborhood* of a vertex v". "The vertex v is said to be an *extreme vertex* of G if the subgraph induced by its neighbors is complete". "For a cutvertex x in a connected graph G and a component H of $G - x$, the subgraph H and the vertex x together with all edges joining x and $V(H)$ is called a *branch* of G at x. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the cardinality of a minimum geodetic set. The geodetic number of a graph was introduced and studied in [2, 3, 5]. A set of vertices S of G is a nonsplit geodetic set if S is a geodetic set and the subgraph induced by $V - S$ is connected. The minimum cardinality of a nonsplit geodetic set is called a nonsplit geodetic number of G. The concept of nonsplit geodetic number studied in [8]. A vertex x is said to *lie* on a *u-v geodesic* P if x is a vertex of P and x is called an *internal vertex* of P if $x \neq u, v$. A chord of a path $u_1, u_2, \ldots, u_n$ in G is an edge $u_i u_j$ with $j \geq i + 2$. For two vertices u and v in a connected graph G, a u-v path is called a *monophonic path* if it contains no chords. A *monophonic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in S. The *monophonic number* $m(G)$ of G is the cardinality of a minimum monophonic set. The concept of monophonic set was introduced in [6, 7]".

The following theorem is referred in this paper.

Theorem 1.1 "Every extreme vertex of a connected graph G belongs to each monophonic set of G. In particular, if the set S of all extreme vertices of G is a monophonic set of G, then S is the unique minimum monophonic set of G".

2 Nonsplit monophonic number

Definition 2.1 A set S of vertices in a connected graph G is a *nonsplit monophonic set* if S is a monophonic set and the subgraph induced by $V - S$ is connected. A nonsplit monophonic set of minimum cardinality is a *minimum nonsplit monophonic set* and this cardinality is the *nonsplit monophonic number* $m_{ns}(G)$ of G.

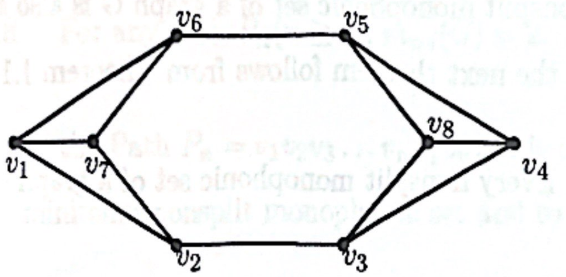

Figure 2.1: G

It is clear that the set $S = \{v_2, v_5\}$ is a monophonic set and so $m(G) = 2$. It is noticed that the subgraph induced by $V - S$ is disconnected. It is checked that no 3-element subsets of vertices is a nonsplit monophonic set. It is easily verified that the sets $S_1 = \{v_1, v_5, v_6, v_8\}$, $S_2 = \{v_1, v_5, v_6, v_7\}$ are the minimum nonsplit monophonic set and $m_{ns}(G) = 4$.

For the graph G shown in Figure 2.1, it is observed that the monophonic number of a graph is different from nonsplit monophonic number of a graph.

It clear that an monophonic set needs at least two vertices and so $m(G) \geq 2$. Also the nonsplit monophonic set of G is a monophonic set of G and the set of all vertices of G is a nonsplit monophonic set, so that $m_{ns}(G) \leq n$. Hence we have the following theorem.

Theorem 2.2 For any connected graph G of order n, $2 \leq m(G) \leq m_{ns}(G) \leq n$.

Remark 2.3 We observe that the bounds in 2.2 are sharp. For the complete graph $K_n (n \geq 2)$, $m_{ns}(K_n) = n$. The set of two end vertices of a path $P_n (n \geq 2)$ is its unique minimum monophonic set so that $m(P_n) = 2$. Thus the complete graph K_n has largest possible nonsplit monophonic number n and that non-trivial paths have the smallest monophonic number 2. For the graph G shown in Figure 2.1, $m(G) = 2$ and $m_{ns}(G) = 4$ and hence the Theorem 2.2 is strict.

Since every nonsplit monophonic set of a graph G is also a monophonic set of a graph G, the next theorem follows from Theorem 1.1.

Theorem 2.4 Every nonsplit monophonic set of a graph G contains its extreme vertices.

Corollary 2.5 For the complete graph $K_n (n \geq 2)$, $m_{ns}(K_n) = n$.

Theorem 2.6 For a connected graph G of order n, $m_{ns}(G) = n$ if and only if $G = K_n$.

Proof. If $G = K_n$, then by Theorem 2.4, $m_{ns}(G) = n$. Conversely, if $m_{ns}(G) = n$, then either $G = K_n$ or there exists a graph G for which the set of all vertices forms minimum nonsplit monophonic set. If $G = K_n$, then there is nothing to prove. Suppose if there exists a graph G for which the set $|S|$ of all vertices forms minimum nonsplit monophonic set, then each vertex in $|S|$ must be either extreme or not lies on any $x - y$ monophonic path for $x, y \in S$. With out loss of generality, let u be only one vertex which is not lies on any $x - y$ monophonic path for $x, y \in S$. Then it must be adjacent to all other vertices in S. It is clear that u is an extreme vertex. This implies that the set S contains only extreme vertices. Hence $G = K_n$. □

Theorem 2.7 For any cycle $G = C_n (n \geq 4)$, $m_{ns}(G) = 3$

Proof. Let the cycle $G = C_n (n \geq 6)$ be $C_n : v_1, v_2, \ldots, v_n, v_1$. It is clear that $S = \{v_1, v_3\}$ is a minimum monophonic set of G so that $m(G) = 2$. It is noticed that the subgraph induced by $V - S$ is not connected. Let $S_1 = \{v_1, v_2, v_3\}$. It is easily verified that the set S_1 is a minimum nonsplit monophonic set and so $m_{ns}(G) = 3$. □

Theorem 2.8 For any path $P_n (n \geq 3)$, $m_{ns}(G) = 2$.

Proof. Consider the Path $P_n = v_1 v_2 v_3 \ldots v_{n-1} v_n$. It is clear that set $S = \{v_1, v_n\}$ forms minimum nonsplit monophonic set and so $m_{ns}(G) = 2$. □

Now, we discuss that the connected monophonic number of a graph G is different from nonsplit monophonic number of a graph G. For the Path P_n, connected monophonic number is n varies as the nonsplit monophonic number is 2.

Theorem 2.9 For the complete bipartite graph $G = K_{r,s} (2 \leq r \leq s)$,

$$m_{ns}(G) = \begin{cases} 3 & \text{if } 2 = r = s \\ 4 & \text{if } 2 = r < s \cdot \\ 4 & \text{if } 3 \leq r \leq s. \end{cases}$$

Proof. Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the partite sets of G.

For $r = s = 2$, we have $G = K_{r,s} = C_4$, the set $S = \{u_1, u_2, w_1\}$ form a minimum nonsplit monophonic set and $m_{ns}(G) = 3$.

For $2 = r < s$, the set $S = \{u_1, u_2, w_1, w_2, \ldots, w_{s-1}\}$ forms a minimum nonsplit monophonic set. Hence $m_{ns}(G) = s + 1$.

For $3 \leq r \leq s$, it is clear that no 3-element subset of vertices of G is a nonsplit monophonic set of G so that $m_{ns}(G) \geq 4$. Let S be any set of four vertices formed by taking two vertices from each of U and W. Then it is easily verified that S is a nonsplit monophonic set of G so that $m_{ns}(G) = 4$. □

Theorem 2.10 For any wheel $W_n = K_1 + C_{n-1} (n \geq 5)$, $m_{ns}(W_n) = 2$

Proof. Let $W_n = K_1 + C_{n-1} (n \geq 5)$. Since the set $S = \{v_i, v_j\} (j \geq i + 1)$ is a nonsplit monophonic set of W_n, it follows that $m_{ns}(W_n) = 2$. □

Problem 2.11 Characterize the graph G for which $m_{ns}(G) = 2$.

Theorem 2.12 If G is a connected graph with a cutvertex v, then every nonsplit monophonic set of G contains at least one vertex from each component of $G - v$.

Proof. Let v be a cutvertex of G. Let $G_1, G_2, \ldots, G_k (k \geq 2)$ be the components of $G - v$. Let S be an nonsplit monophonic set of G. Suppose that S contains no vertex from a component, say $G_i (1 \leq i \leq k)$. Let u be a vertex of G_i. Then by Theorem 2.4 u is not an extreme vertex of G. Since S is an open monophonic set of G, there exist vertices $x, y \in S$ such that u lies on a x-y monophonic path $P : x = u_0, u_1, u_2, \ldots, u, \ldots, u_l = y$ with $u \neq x, y$.Then the x - u subpath of P and the u - y subpath of P both contain v. Hence it follows that P is not a path, which is a contradiction. Thus every nonsplit monophonic set of G contains at least one vertex from the component of $G - v$. □

Corollary 2.13 For any tree T, the nonsplit monophonic number $m_{ns}(T)$ equals the number of end vertices of T. In fact, the set of all end vertices of T is the unique minimum nonsplit monophonic set of T.

Proof. This follows from Theorem 2.4 and 2.12. □

Theorem 2.14 For any integer $n \geq 4$, there exists a connected graph G of order n such that $m(G) = n - 2$ and $m_{ns} = n - 1$.

Proof. Let us take the cycle $C_4 : v_1v_2v_3v_4$. Now, join the new set of vertices $\{w_1, w_2, \ldots, w_{n-4}\}$ with the vertex v_2 and also join the same set of vertices with v_3. Hence the graph G of order n shown in Figure 2.2 is obtained.

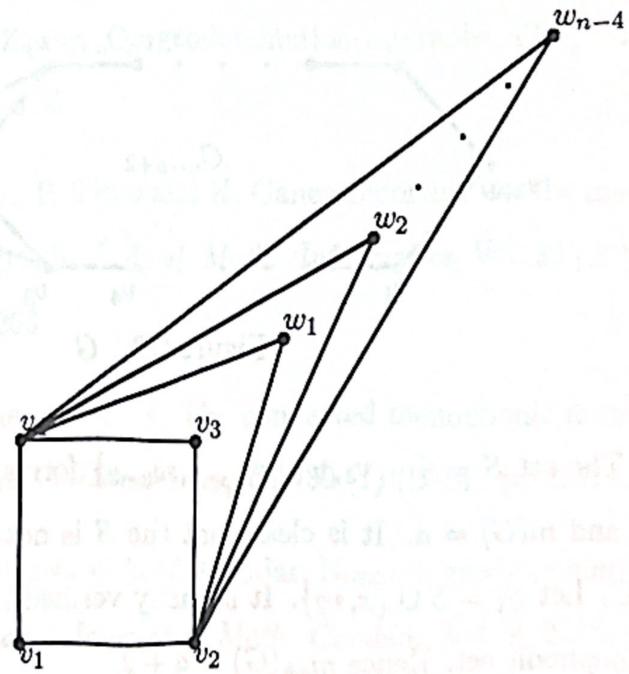

Figure 2.2: G

The set $S = \{v_1, v_4, w_1, w_2, \ldots, w_{n-4}\}$ forms a minimum monophonic set and $m(G) = n - 2$. It is easily verified the subgraph induced by $V - S$ is not connected. S is not nonsplit monophonic set. Now, the sets $S_1 = S \cup \{v_2\}$ and $S_2 = S \cup \{v_3\}$ are the minimum nonsplit monophonic sets of G and $m_{ns}(G) = n - 1$.

□

Theorem 2.15 For any integer $n, a \geq 3$ and $n - a \geq 2$, there exists a graph G of order n with $m(G) = a$ and $m_{ns}(G) = a + 2$.

Proof. Let $C_{n-a+2} : v_1v_2, v_3 \ldots, v_{n-a+2}v_1$ be the cycle of order $n - a + 2$. Join a each vertex x to the vertices v_1 and v_3 and add the vertices $w_1, w_2, \ldots, w_{a-2}$ to x. Thus the graph G of order n shown in Figure 2.3 is obtained.

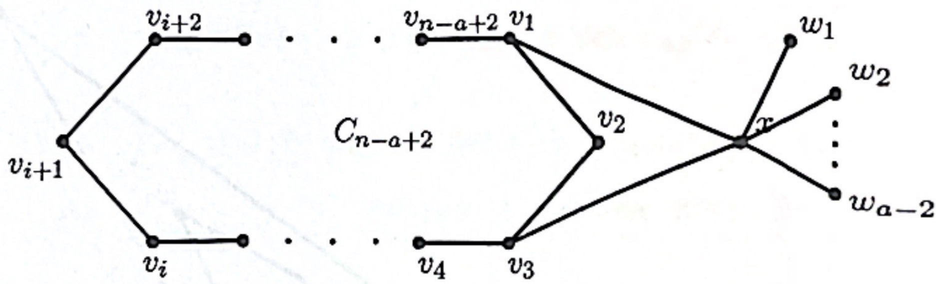

Figure 2.3 : G

The set $S = \{v_1, v_3, w_1, w_2, \ldots, w_{a-2}\}$ forms a minimum monophonic set and $m(G) = a$. It is clear that the S is not nonsplit monophonic set of G. Let $S_1 = S \cup \{x, v_2\}$. It is easily verified that the set S_1 is nonsplit monophonic set. Hence $m_{ns}(G) = a + 2$.

□

We leave the following problem as an open question.

Problem 2.16 Characterize the class of graphs G for which $m_{ns}(G) = n$.

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