PENDANT DOMINATION IN GRAPHS

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Abstract

Let G be any graph. The concept of paired domination was introduced having gaurd backup concept in mind. We introduce pendant domination concept, for which at least one guard is assigned a backup. A dominating set S in G is called a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. The least cardinality of a pendant dominating set is called the pendant domination number of G denoted by $\gamma_{pe}(G)$. In this article, we initiate the study of this parameter. The exact value of $\gamma_{pd}(G)$ for some families of standard graphs are obtained and some bounds are estimated. We also study the proprties of the parameter and interrelation with other invarients.

1 Introduction

Let $G = (V, E)$ be any graph. The paired domination is an interesting concept introduced by Haynes [2]. Recently, Sahul Hamid [3] Introduced a new domination parameter called isolated domination. Motivated by this concept and having the guard backup application in mind, we are now introducing the concept of pendant domination in graphs, which requires minimum back up. The pendant domination concept is also studied by binding it with concept of complement called as complementary pendant domination number of a graph [4].

Let G be any graph and v be any vertex in G. Then, the open neighborhood of a vertex v is denoted by $N(v)$ and is defined by $N(v) = \{u \in V | uv \in E\}$, the set of all vertices adjacent to v. The closed neighborhood of v is denoted $N[v]$ and defined by $N[v] = N(v) \cup \{v\}$. For any subset S of G, the open and closed neighborhoods of S in G is defined by $N(S) = \cup_{v \in S} N(v)$ and $N(S) = \cup_{v \in S} N[v]$. A subset S of the vertex set V is called a dominating set of G if any vertex not in S is adjacent to a vertex in S. The least cardinality of a dominating set in G is called the domination number of G, denoted by $\gamma(G)$. The maximum cardinality of a minimal dominating set is called the upper domination number, denoted by $\Gamma(G)$. A dominating set S of a graph G is a paired dominating set if $\langle S \rangle$ contains at least one perfect matching. Any paired dominating set with minimum cardinality is called a minimum paired dominating set. The cardinality of a minimum paired dominating is called the paired domination number of G denoted by $\gamma_{pd}(G)$. A dominating set S is a total dominating set if $\langle S \rangle$ contains no isolated vertex. The cardinality of a minimum total dominating set is the total domination number of G, denoted by $\gamma_t(G)$.

A set S of vertices is called an irredundant set if each vertex $v \in S$ has at least one private neighbor with respect to S. The minimum cardinality of maximal irredundant set is called the irredundance number, denoted by $ir(G)$. The maximum cardinality of a maximal irredundant set is called the upper irredundance number denoted by $IR(G)$. A subset S of V is called an independent set if no two vertices of S are adjacent in G. A dominating set S of a graph G is an independent dominating set if $\langle S \rangle$ has no edges. The minimum cardinality of an independent dominating set is called the independent domination number, denoted by $i(G)$ and the independence number $\beta_0(G)$ is the maximum cardinality of an independent set of G.

The corona of two disjoint graphs G_1 and G_2 is defined to be the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. If G and H are disjoint graphs, then the join of G and H denoted by $G \vee H$ is the graph such that $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$. The $n-$ Pan graph is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge.The ladder graph is the Cartesian product of P_2 and P_n where P_n is a path. The multi-star graph $K_m(a_1, a_2, \ldots, a_m)$ is a graph of order $a_1 + a_2 + \cdots + a_m + m$ formed by joining $a_1, a_2, \ldots, a_m$ end-edges to m vertices of K_m. In this paper by a graph, we mean a simple, finite and undirected graph without isolated vertices.

We recall the following results required for our study:

Theorem 1.1. *[1] A dominating set S is a minimal dominating set if and only if for each vertex $u \in S$, one of the following condition holds.*

1. *u is an isolate of S,*

2. *there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$.*

Theorem 1.2. *[1] For any graph G, $\gamma(G) = i(G)$ if G is a claw-free graph.*

Theorem 1.3. *[5] If a graph G has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

2 The Pendant Domination Number of a Graph

Let S be a dominating set in G. Then S is called a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. The pendant dominating set of least cardinality is called the pendent domination number, denoted by $\gamma_{pe}(G)$. Any pendant dominating set of cardinality $\gamma_{pe}(G)$ is called a $\gamma_{pe}-$set.

Example 2.1. *1. For a graph G of order n with $\Delta = n-1$, $\gamma_{pe}(G) = 2$.*

2. For a complete multipartite graph $K_{m_1,m_2,\ldots m_k}$, the value of γ_{pe} is 2.

3. Let G be a multi star graph $K_m(a_1, a_2, \ldots, a_m)$. Then

$$\gamma_{pe}(G) = \begin{cases} 2, & \text{if } m=2; \\ 3, & \text{if } a_i = 1 \text{ for some } i \text{ and } m = 3; \\ m+1, & \text{otherwise.} \end{cases}$$

Observation 2.1. *The parameter γ_{pe} is not defined for a totally disconnected graph. Therefore, throughout this paper, by a graph we assume that G has at least one edge.*

Observation 2.2. *If there exists a γ-set S of G such that $\langle S \rangle$ has an isolated vertex, then $\gamma_{pe}(G) = \gamma(G)$ or $\gamma(G) + 1$.*

Theorem 2.1. *Let G be a cycle or a path with n vertices. Then*

$$\gamma_{pe}(G) = \begin{cases} \frac{n}{3} + 1, & \text{if } n \equiv 0 \ (mod\ 3); \\ \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1 \ (mod\ 3); \\ \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 2 \ (mod\ 3). \end{cases}$$

Proof. Let $G \cong P_n$ be a path and let $V(G) = \{v_1, v_2, ..., v_n\}$. We consider the following possible cases here:
Case 1: Suppose $n \equiv 0\,(mod\ 3)$. Then $n = 3k$, for some integer $k > 0$. Then the set $S = \{v_1, v_{3i-1} | 1 \leq i \leq k\}$ is a pendant dominating set of G. Hence $\gamma_{pe}(G) \leq |S|$. i.e., $\gamma_{pe}(G) \leq \frac{n}{3} + 1$. On the other hand, we have $\gamma(G) = \frac{n}{3}$ and any minimum dominating set of G contains only isolated vertices. Hence $\gamma_{pe}(G) \geq \frac{n}{3} + 1$ and so we get, $\gamma_{pe}(G) = \frac{n}{3} + 1$.
Case 2: Suppose $n \equiv 1(mod\ 3)$. Then it is easy to check that any $\gamma-$set in G contains a pendant vertex. Hence, the $\gamma-$set in G itself a pendant dominating set in G. Therefore $\gamma_{pe}(G) = \gamma(G) = \lceil \frac{n}{3} \rceil$.
Case 3: Proof of this case is similar to Case 1. □

Theorem 2.2. *Let G be a Barbell graph. Then $\gamma_{pe}(G) = 2$.*

Theorem 2.3. *Let G be a Pan Graph. Then $\gamma_{pe}(G) = 2 + \lceil \frac{n-5}{3} \rceil$.*

Theorem 2.4. *Let G be Ladder graph. Then $\gamma_{pe}(G) = 2 + \lfloor \frac{n-1}{2} \rfloor$.*

Theorem 2.5. *Let G be a disconnected graph with components G_1, G_2, $\ldots$, G_m. Then $\gamma_{pe}(G) = min_{1 \leq i \leq m}\{\gamma_{pe}(G_i) + \sum_{j=1, j \neq i}^{m} \gamma(G_j)\}$.*

Proof. We prove this result by using mathematical induction. Since G is disconnected, $m \geq 2$. Suppose $m = 2$. Then $G = G_1 \cup G_2$. Let S_1, S_2 be the $\gamma_{pe}-$sets of G_1 and G_2 respectively. Then, $S_1 \cup S_2'$ and $S_2 \cup S_1'$ are pendant dominating sets in G, where S_i' denotes the $\gamma-$set of G_i, $i = 1, 2$. Therefore $\gamma_{pe}(G) \leq min\{\gamma_{pe}(G_1) + \gamma(G_2), \gamma_{pe}(G_2) + \gamma(G_1)\}$. On the other hand, let S be any pendant dominating set in G. Then S has to dominate both $V(G_1)$ and $V(G_2)$ and $\langle S \rangle$ should contain at least one pendant vertex. Moreover, the set S should contain the pendant dominating set of G_1 or G_2. Otherwise $\langle S \rangle$ contains no pendant vertex which is a contradiction. This contradiction shows that $|S| \geq min\{\gamma_{pe}(G_1) + \gamma(G_2), \gamma_{pe}(G_2) + \gamma(G_1)\}$. Hence, $|S| = min\{\gamma_{pe}(G_1) + \gamma(G_2), \gamma_{pe}(G_2) + \gamma(G_1)\}$, proving the result for $m = 2$.

Next, suppose $m \geq 3$ and assume that the result is true for $m = k - 1$. Let G be any graph with the components $G_1, G_2, ..., G_{k-1}, G_k$. Let G' be a graph with $k-1$ components, say $G_1, G_2, ..., G_{k-1}$. Then from the induction hypothesis we have $\gamma_{pe}(G') = min_{1 \leq i \leq k-1}\{\gamma_{pe}(G_i) + \Sigma_{j=1, j \neq i}^{k-1} \gamma(G_j)\}$. Now, we have $G = G' \cup G_m$. Now, from the case $m = 2$, we obtain that $\gamma_{pe}(G) = min_{1 \leq i \leq k}\{\gamma_{pe}(G_i) + \Sigma_{j=1, j \neq i}^{m} \gamma(G_j)\}$. Therefore the result is true for $m = k$ and hence true for any positive integer m. Thus we have $\gamma_{pe}(G) = min_{1 \leq i \leq r}\{\gamma_{pe}(G_i) + \sum_{j=1, j \neq i}^{m} \gamma(G_j)\}$. □

Theorem 2.6. *Let G_1 and G_2 be any two graphs. Then $\gamma_{pe}(G_1 \vee G_2) = 2$.*

As a consequence of above theorem, the value of $\gamma_{pe}(G)$ is 2 if G is a wheel, fan graph or a cone graph. In the following theorem, we determine γ_{pe} for corona of two graphs.

Theorem 2.7. *Let G be a graph connected with n vertices and H be any graph. Then*

$$\gamma_{pe}(G \circ H) = \begin{cases} n+1, & \text{if } G \text{ is a cycle and } \gamma(H) \geq 2; \\ n, & \text{otherwise.} \end{cases}$$

Theorem 2.8. *A dominating set S is a minimal pendant dominating set if and only if for each vertex $u \in S$ one of the following condition holds.*

1. *u is either an isolate or a pendant vertex of S.*
2. *each vertex of $S - \{u\}$ belongs to some cycle in G.*
3. *there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$.*

Proof. Let S be a minimal pendant dominating set of G. Then for every vertex $u \in S$, the set $S - \{u\}$ is not a pendant dominating set in G. So, we have two possible cases here:
Case 1: $S - \{u\}$ is not a dominating set of G.
Then from Theorem 1.1, it follows that either u is an isolated vertex of S or there exists a vertex $v \in V - S$ such that $N(v) \cap S = \{u\}$.
Case 2: $S - \{u\}$ is a dominating set, but contains no pendant vertex.

Then each vertex of $S - \{u\}$ is either an isolated vertex of S or has degree at least 2. If all the vertices of $S - \{u\}$ are isolated vertices of S then u will be the pendant vertex of S. For if each vertex has degree at least two, then each vertex of $S - \{u\}$ belongs to some cycle in G.

Conversely, assume that S is a pendant dominating set satisfying the above stated three conditions. For the purpose of contradiction, assume S is not a minimal pendant dominating set. Then there is a vertex $u \in S$ such that $S - \{u\}$ is also a pendant dominating set. Hence, u must be adjacent to at least one vertex $v \in S - \{u\}$, so $\{u\}$ is not an isolate of S and if v is the pendant vertex of $S - \{u\}$, then v is not a pendant vertex of S, hence condition (1) fails to hold. Clearly, condition 2 does not hold since $S - \{u\}$ contains a pendant vertex. Finally, every vertex in $V - S$ must be adjacent to at least one vertex in $S - \{u\}$ so the condition 3 fails to hold. Hence none of the above conditions holds, which is a contradiction to our assumption. So, this contradiction proves that at least one of the condition should hold. □

Corollary 2.1. *If S is a pendant dominating set of G which is minimal with respect to pendant domination, then there exists a vertex $v \in S$ such that $S - \{v\}$ is a minimal dominating set of G.*

Let G be any connected graph and $S \subseteq V(G)$ be a minimal dominating set of G. Then $V - S$ is also a dominating set of G. But, generally this is not true in the case of pendant dominating set. Next theorem gives the condition under which complement of a pendant dominating set is a dominating set.

Theorem 2.9. *Let G be any graph with $n \geq 3$ vertices. Then complement $V - S$ of any pendant dominating set is a dominating set if S contains no induced path P_3.*

Proof. Suppose S is any pendant dominating set in G. If S contains no induced path P_3, then every vertex in G will be either a vertex of S adjacent to some vertex in S. Therefore, $V - S$ will be a dominating set in G. □

Theorem 2.10. *Let T be any tree. Then $\gamma_{pe}(T) = \gamma(T)$ if and only if there is a γ-set which is not independent in T.*

Proof. Let T be a tree. Assume $\gamma_{pe}(T) = \gamma(T)$. Let S be any γ_{pe}-set in T. Then S is a γ-set in T. If S is independent, then $\langle S \rangle$ contains no pendant vertex, a contradicting our assumption. Hence $\langle S \rangle$ must be independent in T.

Conversely, suppose S' is any γ-set in T. If S' has at least one pendant vertex, we are done. Otherwise, assume that each vertex $v \in S'$ is of degree either zero or at least two. If $deg\, v = 0$, for all $v \in S'$, then S' is an independent set in T, which is a contradiction. For otherwise, set of all vertices of S' having degree at least two forms a cycle in $\langle S\rangle$, which is not possible since T is acyclic. This proves that $\langle S\rangle$ must contain a pendant vertex and hence S itself the $\gamma_{pe}-$set of T, proving that $\gamma_{pe}(T) = \gamma(T)$. □

In the next theorem, we extend the above result for all graphs:

Theorem 2.11. *Let G be any graph. Then $\gamma_{pe}(G) = \gamma(G)$ if and only if G contains a $\gamma-$set which is neither an independent set in G nor each vertex of S has degree zero or belongs to a cycle in S.*

Proof. Let G be any graph. If G is acyclic then we are done. Assume that G is a cyclic graph and let $\gamma_{pe}(G) = \gamma(G)$. On contrary, suppose that every $\gamma-$set S in G is either independent or each vertex of S has degree zero or belongs to a cycle in S, then $\gamma_{pe}-$set will be obtained by adding one vertex $u \in V - S$ to a $\gamma-$set in G. Hence $\gamma(G) < \gamma_{pe}(G)$, a contradiction. Conversely, if every $\gamma-$set in G fails to satisfy the above stated conditions, then $\langle S\rangle$ must contain at least one pendant vertex. Hence, S itself a pendant dominating set and so $\gamma_{pe}(G) = \gamma(G)$. □

3 Bounds for $\gamma_{pe}(G)$

Theorem 3.1. *For any graph G of order n, we have $2 \leq \gamma(G) \leq \gamma_{pe}(G) \leq n$. Further $\gamma_{pe}(G) = 2$ if and only if G contains an edge of degree at least $n-2$.*

Theorem 3.2. *Let G be any graph with n vertices. Then $\gamma_{pe}(G) = n$ if and only if G is a path P_2. Further, if G contains a subgraph non-isomorphic to P_2, then $n - m \leq \gamma_{pe}(G) \leq n-1$.*

Theorem 3.3. *Let G be a connected graph of order n. Then $\gamma_{pe}(G) = n-1$ if and only if G is one of the graphs P_3, K_3.*

Proof. First, assume $\gamma_{pe}(G) = n-1$. Suppose there exist two adjacent vertices u and v in G of degree at least two. Then the set $S = V - \{u, v\}$ is a dominating set in G. Suppose S contains no edge. Then S should have exactly one vertex, since otherwise $\gamma_{pe}(G) \leq n-2$. If $S = \{w\}$, then $G \cong K_3$. Suppose S contains an edge, then S will be a pendant dominating set in G. Therefore $\gamma_{pe}(G) \leq n-2$, a contradiction. Hence either u or v must be a pendant vertex in G and so $G \cong K_{1,n-1}$. But we have $\gamma_{pe}(K_{1,n-1}) = 2$, from which it follows that $n = 3$ showing that $G \cong P_3$. □

Theorem 3.4. *Let G be a connected graph. Then $\gamma_{pe}(G) \leq \frac{n}{2}+1$. Further, for every positive integer a, there is graph G of order 2a that contains* *a minimal pendent dominating set of cardinality $a+k$ for each k with $1 \leq k \leq n-1$.*

Proof. Let G be a connected graph. From Theorem 1.3 and observation 6, it follows that $\gamma_{pe}(G) \leq \frac{n}{2}+1$. Next, let a be any positive integer. Consider a path P_a with a vertices. Then $G = P_a \circ \overline{K}_2$. Then G is a graph of order $2a$. Moreover for any edge $e = uv$, the set $S = \{v_i | v_i$ is an isolated vertex of G, $1 \leq i \leq k-1\} \cup \{u, v\}$ will be a minimal pendent dominating set of size $a+k$. □

Let $\mathcal{G}$ be the collection of graphs of following types. A cycle, path, star, wheel and a complete graph and pan graph each of order 4 and a path, cycle of order 5.

Theorem 3.5. *Let G be a connected graph of order n. Then $\gamma_{pe}(G) = n-2$ if and only if $G \in \mathcal{G}$.*

Proof. Suppose $\gamma_{pe}(G) = n-2$ and S is a $\gamma_{pe}-$ set, then $\langle V-S \rangle$ either K_2 or $\overline{K}_2$. We first prove that $n \leq 5$. Clearly $V-S$ is will be a dominating set of G. Now if $\langle V-S \rangle = K_2$, then $V-S$ is itself a pendant dominating set of G and hence $n \leq 5$. Assume $\langle V-S \rangle = \overline{K}_2$ and assume $V(\overline{K}_2) = \{u, v\}$. Suppose u and v have at least two neighbors in S. Then the set $S' = (S - N(u,v)) \cup \{u, v\}$ is a dominating set in G of cardinality less than $n-3$. Hence G contains a pendant dominating set of size less than $n-2$, a contradiction. Therefore each vertex in $V-S$ has at most one neighbor in S, from which it follows that $|S| \leq 3$ and consequently $n \leq 5$. Thus G must be one among the graphs in $\mathcal{G}$. Converse is obvious. □

Theorem 3.6. *Let G be any graph. Then $\lceil \frac{n}{1+\Delta(G)} \rceil \leq \gamma_{pe}(G) \leq n - \Delta(G)+1$. Further if G is a tree, then $\gamma_{pe}(G) = n - \Delta(G)$ if and only if G is a wounded spider obtained by subdividing even number of edges of a star.*

Proof. We have $\lceil \frac{n}{1+\Delta(G)} \rceil \leq \gamma(G) \leq \gamma_{pe}(G)$. On the other hand, let u be a vertex of degree $\Delta(G)$ in G. Then for $v \in N(u)$, the set $(V - N(u)) \cup \{v\}$ will be a pendant dominating set in G and so $\gamma_{pe}(G) \leq n - \Delta(G) + 1$.

Let G be any tree such that $\gamma_{pe}(G) = n - \Delta(G)$. Choose a vertex u of degree $\Delta(G)$. If $N[u]$ itself a pendant dominating set, then G is a star without subdividing any edges. Otherwise, there is at least one vertex in G which is not dominated by u. Let D be the maximal independent in the graph induced by $V - N[u]$. Then D dominates $V - N[u]$ and so $D \cup \{u\}$ is a dominating set in G containing no pendent vertex. Thus for $v \in N(u)$, the set $D \cup \{u, v\}$ is a pendent dominating set in G. Therefore, we have $|D| = n - \Delta(G)$ and so $V - N(u)$ is an independent set in G. Moreover, $N(u)$ must be an independent set in G, otherwise G must contain a cycle, which is not possible. Further, $V - N[u]$ is a dominating set and no vertex in $V - N[u]$ can have more than one neighbor in $N(u)$ as G is acyclic. Hence, every vertex in $N(u)$ must have at most one neighbor different from v and so at least $\Delta(G)$ vertices are necessary dominate G. Therefore G is a wounded spider. □

Theorem 3.7. *For any triangle free graph of order at least 3, we have* $\gamma_{pe}(\overline{G}) = 2$ *or* 3.

Theorem 3.8. *For any graph G of order n,* $\gamma_{pe}(G) + \gamma_{pe}(\overline{G}) \leq n + 2$.

Proof. Let G be any graph. Suppose G contains an isolated vertex, then $\gamma_{pe}(G) \leq n$ and $\gamma_{pe}(\overline{G}) \leq 2$. Therefore, $\gamma_{pe}(G) + \gamma_{pe}(\overline{G}) \leq n+2$. Similarly, if $\overline{G}$ contains an isolated vertex, we obtain that $\gamma_{pe}(G) + \gamma_{pe}(\overline{G}) \leq n + 2$. Suppose G and $\overline{G}$ contains no isolated vertices. Then from Theorem 1.3, we have $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$ and $\gamma(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor$. Since $\gamma_{pe}(G) \leq \gamma(G) + 1$ always, it follows that $\gamma_{pe}(G) \leq \lfloor \frac{n}{2} \rfloor + 1$ and $\gamma_{pe}(\overline{G}) \leq \lfloor \frac{n}{2} \rfloor + 1$. Therefore $\gamma_{pe}(G) + \gamma_{pe}(\overline{G}) \leq 2\lfloor \frac{n}{2} \rfloor + 2 \leq n + 2$. Therefore, for any graph G, we have $\gamma_{pe}(G) + \gamma_{pe}(\overline{G}) \leq n + 2$. □

For a graph G, we have $\gamma_{pe}(G) \leq diam(G)$ always. But this bound is not sharp. For instance, if G is a path on n vertices, then $diam(G)$ is $n - 1$ whereas for large value of n, the difference will also be large.

4 Relation with other domination parameters

A chain of inequalities called domination chain that connects different parameters was established in [1] which is given by $ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G)$. I. S. Hamid [3], extended this chain by introducing isolate domination parameter, which is given by: $ir(G) \leq \gamma(G) \leq \gamma_0(G) \leq i(G) \leq \beta_0(G) \leq \Gamma_0(G) \leq \Gamma(G) \leq IR(G)$. A sequence (a, b, c, d, e, f) of integers with $1 \leq a \leq b \leq c \leq d \leq e \leq f$ is called a domination sequence if there is a graph G such that $1 \leq ir(G) = a \leq \gamma(G) = b \leq i(G) = c \leq \beta_0(G) = d \leq \Gamma(G) = e \leq IR(G) = f$.

From the definition of pendant domination, it may be observed that such chain may not be extended for this parameter. But it can be extended partially which is proved in following results. We try to establish similar chain involving other domination parameters.

Lemma 4.1. *Let T be any tree. Then* $\gamma_{pe}(T) \leq \gamma_t(T) \leq \gamma_{pd}(T)$. *Equality holds if T is either a cycle or a path of order $4k$.*

Proof. Let G be a tree. Since any total dominating set contains a pendant vertex and further any paired dominating set contains no isolated vertices, it follows that $\gamma_{pe}(G) \leq \gamma_t(G) \leq \gamma_{pd}(G)$. Suppose G is a path or a cycle with $4k$ vertices. Then $\gamma_{pe}(G) = \gamma_t(G) = \gamma_{pd}(G) = 2k$. □

The above result may not be true for a cyclic graph. For example, if $G \cong C_3 \circ \overline{K}_2$, then $\gamma_{pe}(G) = 4$ whereas $\gamma_t(G) = 3$. But, for any graph G, it is always true that $\gamma_{pe}(G) \leq \gamma_{pr}(G)$.

Theorem 4.1. *For any tree T, $ir(T) \leq \gamma(T) \leq \gamma_{pe}(T) \leq \gamma_t(T) \leq \gamma_{pr}(T)$.*

Proof. Let T be any tree. Since every pendant dominating set is also a dominating set and every paired dominating set is a total dominating set, first and third inequalities are trivial. Let S be any total dominating set in T. Since T is acyclic, $\langle S \rangle$ contains at least one pendant vertex. Therefore, S is a pendant dominating set and so $\gamma_{pe}(T) \leq \gamma_t(T)$. □

Corollary 4.1. *For any graph G, $ir(G) \leq \gamma(G) \leq \gamma_{pe}(G) \leq \gamma_{pr}(G)$.*

Given an integer sequence $1 \leq a \leq b \leq c \leq d \leq e$ is called a pendant domination sequence if there exists a graph G such that $1 \leq ir(G) = a \leq \gamma(G) = b \leq \gamma_{pe}(G) = c \leq \gamma_t(G) = d \leq \gamma_{pr}(G) = e$. For example, $(3, 3, 4, 5, 6)$ is a pendant dominating sequence as the path on 7 vertices satisfies above condition.

Lemma 4.2. *For any graph G, $\gamma_{pe}(G) \leq i(G) + 1$. Equality holds if G is a claw-free graph. Further, for any positive integer k, there exists a graph H such that $i(H) - \gamma_{pe}(H) = k$.*

Proof. Let G be any graph and let S be an $i(G)$−set of G. Then S is a dominating set in G and $\langle S \rangle$ contains only isolated vertices. Now, for any vertex $u \in V - S$, the set $S \cup \{u\}$ will be a pendant dominating set in G. Hence, $\gamma_{pe}(G) \leq i(G)+1$. Let G be a claw-free graph. Then from Theorem 1.2, it follows that any γ−set in G is independent and so $\gamma_{pe}(G) = \gamma(G)+1$. Hence $\gamma_{pe}(G) = i(G) + 1$ and so the equality.

Let k be any positive integer. Consider a path P_2 and let H be the graph obtained by attaching $k + 1$ vertices to each vertex of P_2. Then $\gamma(G) = \gamma_{pe}(G) = 2$ whereas $i(G) = k + 2$. Hence $i(G) - \gamma_{pe}(G) = k$. □

Lemma 4.3. *Let G be any graph. Then $\gamma_{pe}(G) \leq \gamma_0(G) + 1$.*

Since any independent dominating set in G contains only isolated vertices, it will be an isolated dominating set in G. Hence, we have the following lemma:

Proof. Let G be any graph and S be an isolated dominating set in G. If $\langle S \rangle$ contains a pendant vertex then we are done. If $\langle S \rangle$ contains no pendant vertex, then by adding a vertex v to S from its complement, the set $S \cup \{v\}$ will be a pendant dominating set. Hence $\gamma_{pe}(G) \leq \gamma_0(G) + 1$. □

Lemma 4.4. *Let G be any graph. Then* $\gamma_0(G) \leq i(G)$.

In light of Lemma 4.2, 4.3 and 4.4, we have the following result.

Theorem 4.2. *Let G be any graph. Then* $ir(G) \leq \gamma(G) \leq \gamma_{pe}(G) \leq \gamma_0(G) + 1 \leq i(G) + 1 \leq \beta_0(G) + 1 \leq \Gamma(G) + 1 \leq IR(G) + 1$.

5 Conclusion

Nowadays, study of domination related parameters is an important area in graph theory and many scholars are working in this area. In this article, we have introduced a new domination invariant called pendant domination. We have calculated the exact values for some standard families of graphs and established some bounds for this parameter in terms degree, order etc. Further, we have studied some important properties of this parameter, an attempt has been made to find the relation with other domination invariants and also we have studied some properties of the new parameter in the complement of graphs.

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