Power Domination in Tree Derived Architectures

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Abstract

A set S of vertices in a graph G is called a dominating set of G if every vertex in $V(G)\backslash S$ is adjacent to some vertex in S. A set S is said to be a power dominating set of G if every vertex in the system is monitored by the set S following a set of rules for power system monitoring. A zero forcing set of G is a subset of vertices B such that if the vertices in B are colored blue and the remaining vertices are colored white initially, repeated application of the color change rule can color all vertices of G blue. The power domination number and the zero forcing number of G are the minimum cardinality of a power dominating set and the minimum cardinality of a zero forcing set respectively of G. In this paper, we obtain the power domination number, total power domination number, zero forcing number and total forcing number for m-rooted sibling trees, l-sibling trees and l-binary trees. We also solve power domination number for circular ladder, Möbius ladder, and extended cycle-of-ladder.

Keywords: Power domination, m-rooted sibling trees, l-sibling trees and l- binary trees, extended cycle-of-ladder.

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1 Introduction

We begin with the basic definition of power domination. For a vertex u in a graph G, let $N(u) = \{v \in V(G)/(u,v) \in E(G)\}$ and $N[u] = N(u) \bigcup \{u\}$. For a graph $G(V,E)$, $S \subseteq V$ is a *dominating set* of G if every vertex in $V \backslash S$ has at least one neighbour in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set S is called a *total dominating set* if each vertex v of G is dominated by some vertex $u \neq v$ of S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G.

In [1] authors introduced the related concept of power domination by presenting propagation rules by terms of vertices and edges in a graph. Let $G(V,E)$ be a graph and let $S \subseteq V(G)$. We define the sets $M^i(S)$ of vertices monitored by S at level i, $i \geq 0$, inductively as follows:
1. $M^0(S) = N[S]$.
2. $M^{i+1}(S) = \cup\{N[v] : v \in M^i(S) \text{ such that } |N[v] \backslash M^i(S)| = 1\}$.

If $M^\infty(S) = V(G)$, then the set S is called a *power dominating set* of G. The minimum cardinality of a power dominating set in G is called the power domination number of G written $\gamma_p(G)$. A power dominating set S is called a *total power dominating set* if S contains no isolated vertex. The total power domination number, denoted $\gamma_p^t(G)$ of G is the minimum cardinality of a total power dominating set of G [8].

The concept of zero forcing game introduced via color game on vertices of G. The color change rule is: If u is a blue vertex and exactly one neighbour w of u is white, then change the color of w to blue. We say that u forces w and denote it by $u \to w$. A zero forcing set of G is a subset of vertices B such that when the vertices in B are colored blue and the remaining vertices are colored white initially, repeated application of the color change rule can color all vertices of G blue. The zero forcing number, denoted $Z(G)$ of G is the minimum cardinality of a *zero forcing set* of G. A zero forcing set S is called a *total forcing set* if S contains no isolated vertex. The total forcing number, denoted $F_t(G)$ of G is the minimum cardinality of a total forcing set of G [8].

The power domination has been well studied for trees [1], product graphs [4], block graphs [5], interval graphs, circular-arc graphs [2], grids [3] and so on. In fact, the problem has been shown to be NP-complete even when restricted to bipartite graphs and chordal graphs [1].

2 Main Results

In this section, we solve the power domination number and zero forcing number for m-rooted sibling trees, l-sibling trees and l- binary trees.

A tree is a connected graph that contains no cycles. The most common type of tree is the binary tree. It is so named because each node can have at most two descendants. A binary tree is said to be a complete binary tree if each internal node has exactly two descendants. These descendants are described as left and right children of the parent node. Binary tree are widely used in data structures because they are easily stored, easily manipulated, and easily retrieved. Also many operations such as searching and storing can be easily performed on tree data structures. Furthermore, binary trees appear in communication pattern of divide-and-conquer type algorithms, functional and logic programming, and graph algorithms. A rooted tree represents a data structure with a hierarchical relationship among its various elements.

The basic skeleton of a m-rooted sibling trees, a l-sibling trees and a l-binary trees is a complete binary tree. Hence it is enough to consider level $r-1$ to determining the power dominating set or the zero forcing set of complete binary tree. Choosing the power dominating set or zero forcing set in level $r-1$ is the minimum power dominating set or the minimum zero forcing set for a complete binary tree.

2.1 Power Domination in m- Rooted Sibling Tree

Definition 2.1. [10] *1-rooted sibling tree ST_r^1 is obtained from the 1-rooted complete binary tree T_r^1 by adding edges (sibling edges) between left and right children of the same parent node. The m-rooted sibling trees ST_r^m is obtained from m number of vertex disjoint 1-rooted sibling tree ST_r^1 on 2^n vertices with roots say $r_1, r_2, \ldots, r_m$ and adding edges (r_i, r_{i+1}), $1 \leq i \leq m-1$. See Figure 1(a). The diameter of ST_r^m is $2n+m-1$.*

Theorem 2.2. *Let G be a m-rooted sibling tree ST_r^m $r \geq 2$. Then $\gamma_p(G) = \gamma_p^t(G) = m \times 2^{r-1}$.*

Proof. In ST_r^m, the vertices in level $r-1$ and level r induce $m \times 2^{r-1}$ vertex disjoint copies of 3-cycle. Any minimum power dominating set of G contains at least one vertex in each such 3-cycle. For, if not, even if all vertices in level $(r-1)$ are monitored, their children will be left unmonitored. Select all the vertices in level $r-1$ of ST_r^m in set S. The vertices in level $r-i$ monitor vertices in level $r-i-1$, $2 \leq i \leq r-1$. Hence S is a power

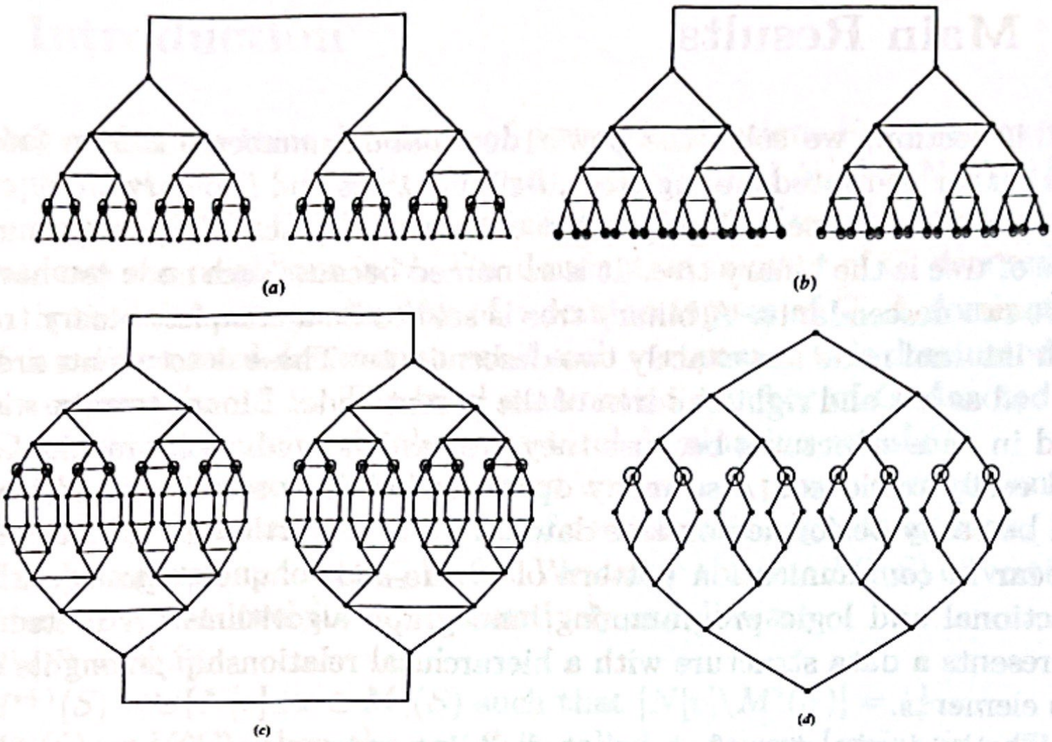

Figure 1: Circled vertices indicates a (a) Power dominating set of sbiling tree $ST_r^2(4)$ (b) Zero forcing set of sbiling tree $ST_r^2(4)$ (c) Power dominating set of sbiling tree l-$ST_r^2(4)$ (d) Power dominating set of l-T_4

dominating set of ST_r^m with $|S| = m \times 2^{r-1}$. Since the vertices in S induce a perfect matching in S, $\gamma_p^t(G) = |S| = m \times 2^{r-1}$. □

Theorem 2.3. *Let G be a m-rooted sibling tree ST_r^m, $r \geq 2$. Then $Z(G) = Z_t(G) = m \times 2^r$.*

Proof. In ST_r^m, the vertices in level $r-1$ and level r induce $m \times 2^{r-1}$ vertex disjoint copies of 3-cycle. Any minimum zero forcing set of G contains at least two vertices in each such 3-cycle. For, if not, even if all vertices in level $(r-1)$ are colored as blue, their children will be left colored white. Select all the vertices in level r of ST_r^m in set S. The vertices in level $r-i$ monitor vertices in level $r-i-1$, $1 \leq i \leq r-1$. Hence S is a zero forcing set of ST_r^m with $|S| = m \times 2^r$. Since the vertices in S induce a perfect matching in S, $Z_t(G) = |S| = m \times 2^r$.

2.2 Power Domination in l-Sibling Tree

Definition 2.4. [10] *The ST_r^m be a rooted sibling tree, $n \geq 1$, $m \geq 1$. A graph which is obtained from two copies of rooted sibling tree ST_r^m, say ST_1^m, ST_2^m by joining each vertex in the last level (i.e.,$(r-1)^{th}$ level) of*

ST_1^m *with the corresponding vertex of* ST_2^m *by an edge is called the l-sibling tree and is denoted by l-*ST_r^m*. See Figure* 1(*c*).

Theorem 2.5. *Let G be a l-sibiling tree l-*ST_r^m, $r \geq 2$. *Then* $\gamma_p(G) = \gamma_p^t(G) = m \times 2^{r-1}$.

Proof. In *l*-ST_r^m, the vertices in level $r-1$ and level r induce $m\times 2^{r-1}$ vertex disjoint copies of 3-cycle. Any minimum power dominating set of G contains at least one vertex in each such 3-cycle. For, if not, even if all vertices in level $(r-1)$ are monitored, their children will be left unmonitored. Select all the vertices in level $r-1$ of *l*-ST_r^m in set S. The vertices in level $r-i$ monitor vertices in level $r-i-1$, $2 \leq i \leq r-1$ both from top and bottom. Hence S is a power dominating set of *l*-ST_r^m with $|S| = m \times 2^{r-1}$. Since the vertices in S induce a perfect matching in S, $\gamma_p^t(G) = |S| = m \times 2^{r-1}$.

2.3 Power Domination in *l*-Complete Binary tree

Definition 2.6. [10] *Let* T_r *be a complete binary tree,* $r \geq 1$. *A graph which is obtained from two copies of complete binary tree* T_r, *say* T_1, T_2 *by merging each vertex in the last level (i.e.,* $(r-1)^{th}$*level) of* T_1 *with the corresponding vertex of* T_2 *is called the l-complete binary tree and is denoted by l-*T_r. *See Figure* 1(*d*)

Remark 2.7. *Number of vertices in l-*T_r *is* $3.2^{r-1} - 2$, $r \geq 1$.

Theorem 2.8. *Let G be a l-complete binary tree l-*T_r, $r \geq 2$. *Then* $\gamma_p(G) = 2^{r-1}$.

Proof. In *l*-T_r, the vertices in level $r-1$ and level r induce 2^{r-1} vertex disjoint copies of 4-cycle. Any minimum power dominating set of G contains at least one vertex in each such 4-cycle. For, if not, even if all vertices in level $(r-1)$ are monitored, their children will be left unmonitored. Select all the vertices in level $r-1$ of *l*-T_r in set S. The vertices in level $r-i$ monitor vertices in level $r-i-1$, $2 \leq i \leq r-1$ both from top and bottom. Hence S is a power dominating set of *l*-T_r with $|S| = 2^{r-1}$. Therefore, $\gamma_p(G) = 2^{r-1}$. □

3 Power Domination in Ladder-Like Networks

In this section, we solve the power domination number for cycle of ladder and extended cycle of ladder.

3.1 Circular Ladder

Definition 3.1. [13] *Cartesian product $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, two vertices (u, u') and (v, v') being adjacent if and only if either $u = v$ and $u'v' \in E(H)$ or $u' = v'$ and $uv \in E(G)$. The n-ladder graph L of length n is defined as $P_2 \times P_{n+1}$, where P_{n+1} is a path on $n+1$ vertices, $n \geq 1$.*

Definition 3.2. [11] *The circular ladder CL_n of length $n \geq 3$ is the Cartesian product $CL_n = C_n \square K_2$. Möbius Ladder graphs are constructed by introducing a twist in a circular ladder and is denoted by M_n.*

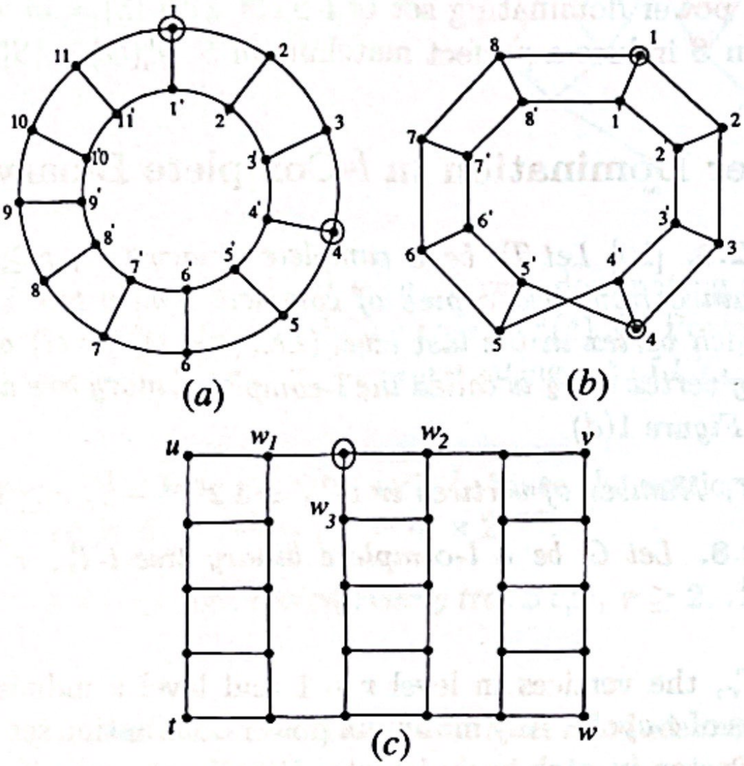

Figure 2: Circled vertices indicates power dominating set of (a) CL_{11} (b) M_8 (c) Sub graph H

Lemma 3.3. *Let G be a circular ladder CL_n of length $n \geq 3$. Then $\gamma_p(G) \geq 2$.*

Proof. Without loss of generality, let $S = \{v\}$. Then every vertex in $N(v)$ is adjacent to two vertices, a contradiction. □

Theorem 3.4. *Let G be a circular ladder CL_n, $n \geq 4$ or a Möbius ladder M_n, $n \geq 4$. Then $\gamma_p(G) = 2$.*

Proof. Select the vertices $\{1, 4\}$ in S as shown in Figure 2(a). Now for every vertex in $M^0(S)$ is adjacent to exactly one unmonitored vertices to it. Proceeding inductively, for every vertex $v \in M^i(S)$, $|N[v] \backslash M^i(S)| \leq 1$, $i \geq 1$. Thus $M^{\lceil \frac{n}{2} \rceil}(S) = V(G)$. Hence $|S| = 2$. Therefore, $\gamma_p(G) = 2$. □

3.2 Extended Cycle-of-Ladder

In 2008, Jywe-Fei Fang introduced a network called cycle-of-ladder and proved that it is a spanning subgraph of the hypercube network, thereby proving that hypercube network is bipancyclic [12]. The graph obtained looks like a ladder having two rails and $n + 1$ rungs between them. The length of the ladder is defined as n.

Definition 3.5. [13] *A cycle of ladder is a graph comprising of a cycle C_s of length $2l$ called the spine cycle such that removal of alternate edges on C_s leaves l components $L_1, L_2, \ldots, L_l$, each of which is isomorphic to a ladder. If $r_1, r_2, \ldots, r_l$ denote the number of rungs in the ladders $L_1, L_2, \ldots, L_l$ respectively, then the cycle of ladders is denoted by $CL(2l, r_1, r_2, \cdots, r_k)$. Let R^i_j, $1 \leq j \leq r_i$ denote the rungs of L_i such that the bottom rung R^i_1 is the edge of C_s in L_i, $1 \leq i \leq k$. For brevity, we denote $r_1, r_2, \ldots, r_k)$ as s and we denote the cycle-of-ladder as $CL(2l, s)$, where l and s represent the number of ladders and the length of each ladder respectively. For convenience, we label the vertices of L_i as x^i_j where $0 \leq j \leq s$ and $1 \leq i \leq l$ in $CL(2l, s)$.*

We add l number of edges to $CL(2l, s)$ to obtain a 3-regular graph and call it the extended cycle-of-ladder $ECL(2l, s)$.

Definition 3.6. [13] *The extended cycle-of-ladder $ECL(2l, s)$ is obtained from $CL(2l, s)$ by adding edges between $(x^l_j, x^l_{j+1}), 1 \leq j \leq s - 1$, the numbers taken modulo $2l$.*

Lemma 3.7. *Let H be as shown in Figure 2(c). Then $\gamma_p(H) \geq 2$.*

Let S be a power dominating set of H. We claim that $|S| \geq 2$. Suppose not, let $|S| = 1$. Each vertex of degree 2 is adjacent to two vertices, each of which is adjacent to two unmonitored vertices in H.

Theorem 3.8. *Let G be an extended cycle-of-ladder $ECL(2l, s)$, $l, s \geq 4$. Then*

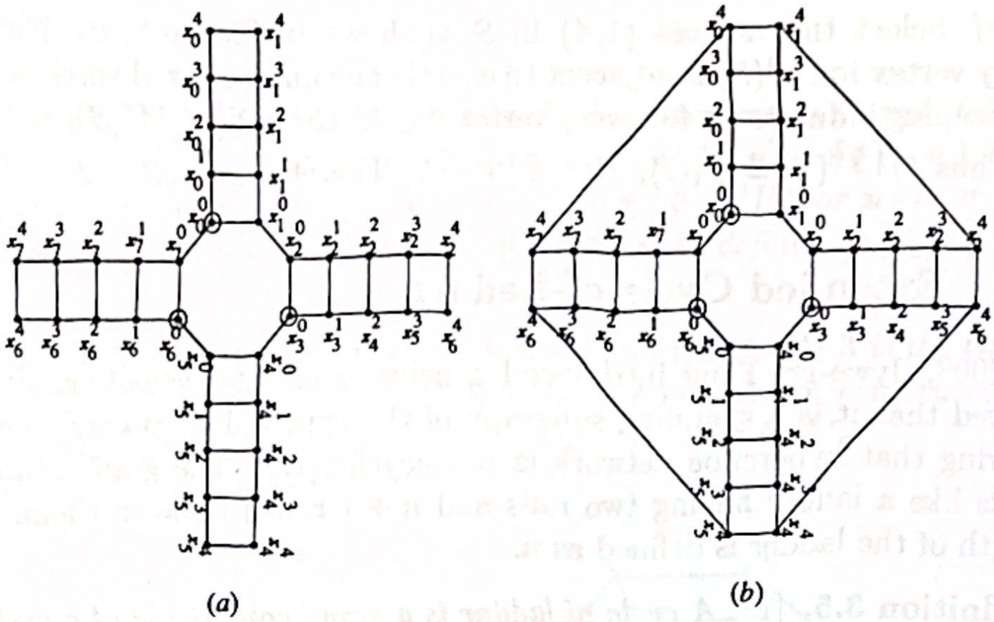

Figure 3: (a) $CL(8,5)$ and (b) Circled vertices constitute a power dominating set of $ECL(8,5)$.

$$\gamma_p(G) \geq \begin{cases} \frac{2l}{3} & if \;\; l \equiv 0 \; (mod\ 3) \\ \lceil \frac{2l}{3} \rceil & if \;\; l \equiv 1 \; (mod\ 3) \\ \lceil \frac{2l}{3} \rceil + 1 & if \;\; l \equiv 2 \; (mod\ 3) \end{cases}$$

Proof. We prove the result by induction on l.

Case (i): $l \equiv 0(mod\ 3)$

Suppose $l = 3$. Let S be a power dominating set of $ECL(2(3), s)$. It is easy to see from Lemma 3.7.

Assume the result is true for $l = k,\ l \geq 3$. That is, $\gamma_p(ECL(2(k), s)) \geq \frac{2k}{3}$. Now we prove that the result is true for $l = k+1$, that is $\gamma_p(ECL(2(k+1), s)) \geq \frac{2(k+1)}{3}$. Suppose $|S| < \frac{2(k+1)}{3}$. By Lemma 3.7, there are $\frac{l}{3}$ vertex disjoint copies of H in $(ECL(2(k+1), s))$. Consider any arbitrary H that contains exactly one vertex of S. Then for every vertex $u \in M^j(S)$, $|N[u] \backslash M^j(S)| > 2$ for some j, a contradiction. Thus $|S| \geq \frac{2(k+1)}{3}$. Therefore, $\gamma_p(ECL(2(k+1), s)) \geq \frac{2(k+1)}{3}$.

Case (ii): $l \equiv 1(mod\ 3)$

Suppose $l = 4$. Let S be a power dominating set of $ECL(2(4), s)$. We claim that $|S| \geq 3$. Suppose not, let $|S| = 2$. Then every vertex, $u \in N[S]$ is adjacent to 2 vertices, a contradiction.

Assume the result is true for $l = k,\ l \geq 3$. That is, $\gamma_p(ECL(2(k), s)) \geq \lceil \frac{2k}{3} \rceil$. Now we prove that the result is true for $l = k+1$, that is $\gamma_p(ECL(2(k+1), s)) \geq \left\lceil \frac{2(k+1)}{3} \right\rceil$. Suppose $|S| < \left\lceil \frac{2(k+1)}{3} \right\rceil$. By Lemma 3.7, there are $\frac{l}{3}$

vertex disjoint copies of H and one ladder say, l in $(ECL(2(k+1),s))$. To monitor vertices in l, we include either one vertex from l or one vertex from H. By the deletion of one vertex, say, x_j^0 the vertices adjacent to x_j^i, say x_{j-1}^0 or x_{j+1}^0 do not monitor by any member of S. Then for at least one vertex $u \in M^j(S)$, $|N[u] \backslash M^j(S)| > 2$ for some j, a contradiction. Thus $|S| \geq \left\lceil \frac{2(k+1)}{3} \right\rceil$. Therefore, $\gamma_p(ECL(2(k+1),s)) \geq \left\lceil \frac{2(k+1)}{3} \right\rceil$.

Case (iii): when $l \equiv 2(mod\ 3)$

Suppose $l = 5$. Let S be a power dominating set of $ECL(2(4),s)$. We claim that $|S| \geq 4$. Suppose not, let $|S| = 3$. Then for every vertex, $u \in N[S]$ is adjacent to 2 vertices, a contradiction.

Assume the result is true for $l = k$, $l \geq 3$. That is, $\gamma_p(ECL(2(k),s)) \geq \lceil \frac{2k}{3} \rceil + 1$. Now we prove that the result is true for $l = k+1$, that is $\gamma_p(ECL(2(k+1),s)) \geq \left\lceil \frac{2(k+1)}{3} \right\rceil + 1$. Suppose $|S| < \left\lceil \frac{2(k+1)}{3} \right\rceil + 1$. By Lemma 3.7, there are $\frac{l}{3}$ vertex disjoint copies of H and two more ladders, say, L_i, L_j in $(ECL(2(k+1),s))$. To monitor vertices in l, we include either one vertex from L_i or L_j . By the deletion of one vertex, say, x_j^0 the vertices adjacent to x_j^0, say x_{j-1}^0 or x_{j+1}^0 do not monitor by any member of S. Then for at least one vertex $u \in M^j(S)$, $|N[u] \backslash M^j(S)| > 2$ for some j, a contradiction. Thus $|S| \geq \left\lceil \frac{2(k+1)}{3} \right\rceil + 1$. Therefore, $\gamma_p(ECL(2(k+1),s)) \geq \left\lceil \frac{2(k+1)}{3} \right\rceil + 1$. Hence the proof. □

The following algorithm proves that the lower bound obtained in Theorem 3.8 is sharp.

Algorithm Power Domination in Extended Cycle-of-Ladder $ECL(2l,s)$

Input: Extended cycle-of-ladder $ECL(2l,s)$, $l, s \geq 4$.

Algorithm: Name the vertex in the bone cycle as $\{x_j^i : 0 \leq i \leq s-1,\ 0 \leq j \leq 2l-1\}$ and select the vertices $\{x_j^0 : j \equiv 0(mod\ 3)\}$ in S.

Output: $\gamma_p(ECL(2l,s)) = \begin{cases} \frac{2l}{3} & if\ \ l \equiv 0\ (mod\ 3) \\ \lceil \frac{2l}{3} \rceil & if\ \ l \equiv 1\ (mod\ 3) \\ \lceil \frac{2l}{3} \rceil + 1 & if\ \ l \equiv 2\ (mod\ 3) \end{cases}$

Proof of Correctness: Let S be a power dominating set of $ECL(2l,s)$. Now vertices in S monitor all the vertices in the bone cycle. Then every vertex $u \in M^0(S)$ is adjacent to exactly one vertex. Proceeding inductively, for every vertex $v \in M^i(S)$, $|N[v] \backslash M^i(S)| \leq 1$, $i \geq 1$. Thus $M^{s-1}(S) = V(G)$. Hence the proof.

Theorem 3.9. *Let G be the graph isomorphic to cycle-of-ladder as $CL(2l,s)$, $l, s \geq$ 4 or an extended cycle-of-ladder $ECL(2l,s)$, $l, s \geq 4$. Then*

$$\gamma_p(G) = \begin{cases} \frac{2l}{3} & if \ \ l \equiv 0 \ (mod \ 3) \\ \lceil \frac{2l}{3} \rceil & if \ \ l \equiv 1 \ (mod \ 3) \\ \lceil \frac{2l}{3} \rceil + 1 & if \ \ l \equiv 2 \ (mod \ 3) \end{cases}$$

4 Conclusion

In this paper, we have obtained the power domination number, total power domination number, zero forcing number and total forcing number for m-rooted sibiling trees, l-sibiling trees and l-complete binary trees. Further, the equality of the power domination parameters and forcing parameters has been studied for these graphs.

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