Wiener index for Common Neighborhood Graphs of Special trees

A. Subhashini[1], J. Baskar Babujee[2]
Department of Mathematics
Anna University
Chennai - 600 044, India.
[1]subhachelvan2015@gmail.com
[2]baskarbabujee@yahoo.com

Abstract

Molecular graphs are models of molecules in which atoms are represented by vertices and chemical bonds by edges of a graph. Graph invariant numbers reflecting certain structural features of a molecule that are derived from its molecular graph are known as topological indices. A topological index is a numerical descriptor of a molecule, based on a certain topological feature of the corresponding molecular graph. One of the most widely known topological descriptor is the Wiener index. The Weiner index $W(G)$ of a graph G is defined as the half of the sum of the distances between every pair of vertices of G. The construction and investigation of topological is one of the important directions in mathematical chemistry. The common neighborhood graph of G is denoted by $con(G)$ has the same vertex set as G, and two vertices of $con(G)$ are adjacent if they have a common neighbor in G. In this paper we investigate the Wiener index of Y-tree, X-tree, $con(Y\text{-tree})$ and $con(X\text{-tree})$.

Keywords: Graph, degree, tree, neighbor, distance.
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1 Introduction

A topological representation of a molecule is called molecular graph. A molecular graph is a collection of points representing the atoms in the molecule and set of lines representing the covalent bonds. These points are named vertices and the lines are named edges in graph theory language. In mathematical terms a graph is represented as $G = (V, E)$, where V is the set of vertices and E is the set of edges. Let G be an undirected connected graph without loops or multiple edges with n vertices, denoted by $v_1, v_1, \ldots, v_n$. The topological distance between a pair of vertices v_i and v_j, which is denoted by $d(v_i, v_j)$, is the number of edges of a shortest path joining v_i and v_j. In 1947 Harold Wiener [1] defined the Wiener index $W(G)$ as the sum of distances between all vertices of the graph G as

$$W(G) = \sum_{i<j} d(v_i, v_j).$$

The Wiener index of complete graph K_n, path graph P_n, star $K_{1,n-1}$ and cycle graph C_n is given by the expressions

$$W(K_n) = \frac{n(n-1)}{2},$$
$$W(P_n) = \frac{n(n^2-1)}{6},$$
$$W(K_{1,n-1}) = (n-1)^2,$$
$$W(C_n) = \begin{cases} \frac{n(n^2-1)}{8}, & n \equiv 1 \bmod 2 \\ \frac{n^3}{8}, & n \equiv 0 \bmod 2 \end{cases}$$

The detail literature of topological indices are studied from [2, 3]. .

Among all the trees on n vertices, the star $K_{1,n-1}$ has the smallest Wiener index and the path P_n has the largest Wiener index and hence for any tree T on n vertices [1]

$$W(K_{1,n-1}) \leq W(T) \leq W(P_n).$$

Let G be a simple graph. The common neighborhood graph of G is denoted by $con(G)$ has the same vertex set as G, and two vertices of $con(G)$ are adjacent if they have a common neighbor in G. In [4, 5], the motivation for the consideration of congraphs came from the theory of graph energy. They have introduced the concept of common neighborhood energy ECN of a graph G and obtained an upper bound for ECN, when G is regular [6, 7].

If the Wiener index is defined, and if the graph G consists of disconnected components G_1 and G_2, then $W(G) = W(G_1) + W(G_2)$ [8, 9, 10, 11, 12]. When speaking of the Wiener indices of congraphs, this is important, because of the following result.

Theorem 1.1. *[6, 7] Let G be a connected bipartite graph, so that its vertex set is partitioned as $V(G) = V_a \cup V_b$. Then $con(G)$ consists of two disconnected components G_a and G_b, whose vertex sets are V_a and V_b, respectively. Both graphs G_a and G_b are connected.*

Now we have:

$$W(con(G)) = W(G_a) + W(G_b).$$

In this paper we investigate the Wiener index of Y-tree, X-tree, $con(Y$-tree) and $con(X$-tree).

2 Common Neighborhood Graphs of Y and X-tree

Generalized Y-tree and generalized X-tree are simply the trees obtained by subdividing the edges of $K_{1,3}$ and $K_{1,4}$ any number of times. In other words for calculation purpose we redefine Y-tree and X-tree as follows:

Definition 2.1 (Y-tree). *A generalized Y-tree is a tree in which there is exactly one vertex of degree three and three pendent vertices is called generalized Y-tree. It is a one point union of three paths $P_{n_1}, P_{n_2}, P_{n_3}$ denoted as $(P_{n_1}; P_{n_2}; P_{n_3} : K_1)$ which has $n_1 + n_2 + n_3 + 1$ vertices.*

Definition 2.2 (X-tree). *A generalized X-tree is a tree, in which there is exactly one vertex with degree four and four pendent vertices. Also a generalized X-tree can be redefined as one point union of four paths $P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}$ denoted as $(P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4} : K_1)$ which has $n_1 + n_2 + n_3 + n_4 + 1$ vertices.*

Definition 2.3. *$(C_3 \odot P_{n_1}; P_{n_2}; P_{n_3})$ is a graph obtained by attaching one of the pendent vertices of the paths $P_{n_1}, P_{n_2}, P_{n_3}$ to the 3 vertices of cycle C_3. The graph $(C_3 \odot P_{n_1}; P_{n_2}; P_{n_2})$ contains $n_1 + n_2 + n_3$ vertices and exactly one cycle C_3.*

Definition 2.4. *$(K_4 \odot P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4})$ is a graph obtained by attaching one of the pendent vertices of the paths $P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}$ to the 4 vertices of complete graphK_4. The graph $(K_4 \odot P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4})$ contains $n_1 + n_2 + n_3 + n_4$ vertices. The following two results are obvious.*

The following two results are obvious.

Theorem 2.1. *If $G = (P_{n_1}; P_{n_2}; P_{n_3} : K_1)$ is a Y-tree then $con(G) = G_1 \cup G_2$ where $G_1$is a Y-tree $G_1 = (P_{m_1}; P_{m_2}; P_{m_3} : K_1)$ and $G_2 = (C_3 \odot P_{r_1}; P_{r_2}; P_{r_3})$ where*

$$m_i = \begin{cases} \frac{n_i-1}{2} & \text{if } n_i \text{ is odd} \\ \frac{n_i}{2} & \text{if } n_i \text{ is even} \end{cases} \quad i = 1,2,3$$

$$r_i = \begin{cases} \frac{n_i+1}{2} & \text{if } n_i \text{ is odd} \\ \frac{n_i}{2} & \text{if } n_i \text{ is even} \end{cases} \quad i = 1,2,3$$

Let $G = (P_{n_1}; P_{n_2}; P_{n_3} : K_1)$ is a Y-tree in Figure 1. Then common neighborhood graph of G is disconnected graphs $G_1 = (P_{m_1}; P_{m_2}; P_{m_3} : K_1)$ and $G_2 = (C_3 \odot P_{r_1}; P_{r_2}; P_{r_3})$ in Figures 1(a) and 1(b).

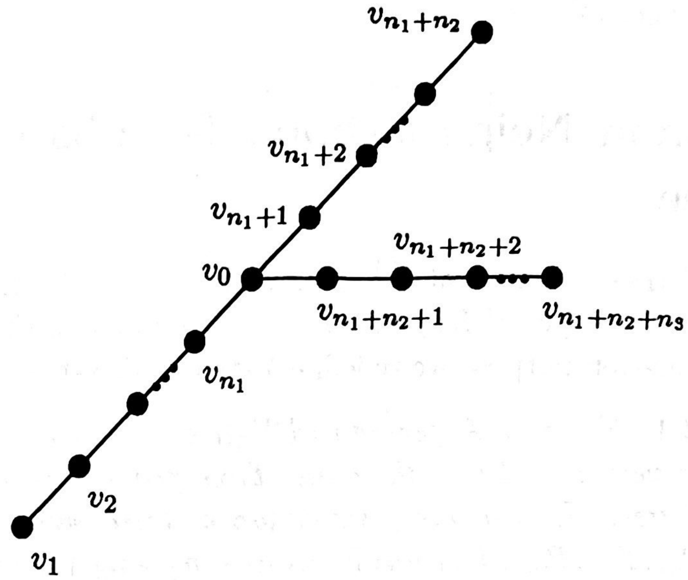


Figure 1: Y-tree $G = (P_{n_1}; P_{n_2}; P_{n_3} : K_1)$

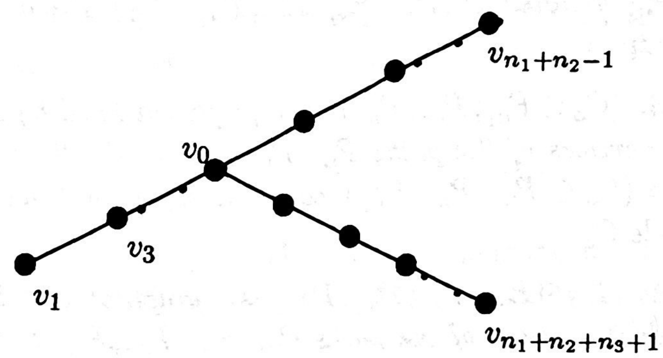


Figure 1(a): Y-tree $G_1 = (P_{m_1}; P_{m_2}; P_{m_3} : K_1)$

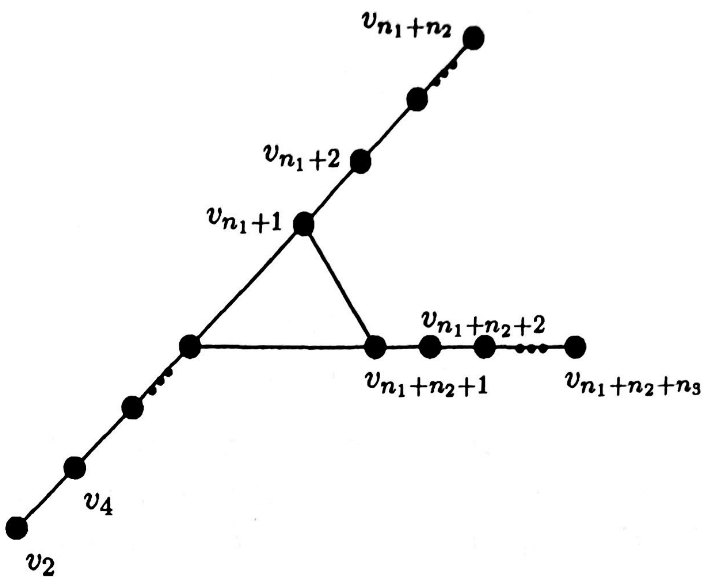
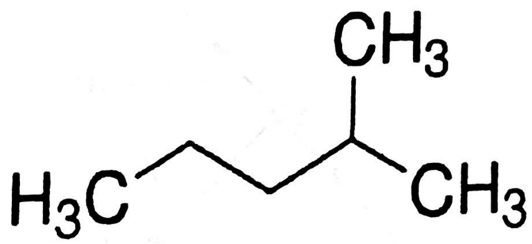


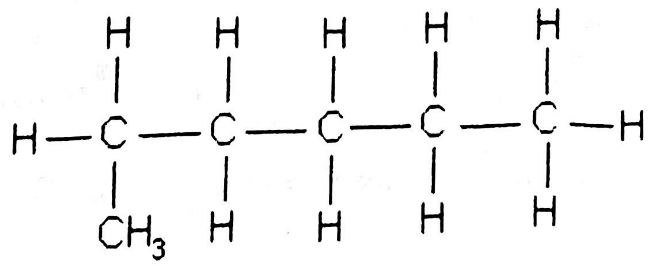
Figure 1(b): $G_2 = (C_3 \odot P_{r_1}; P_{r_2}; P_{r_3})$

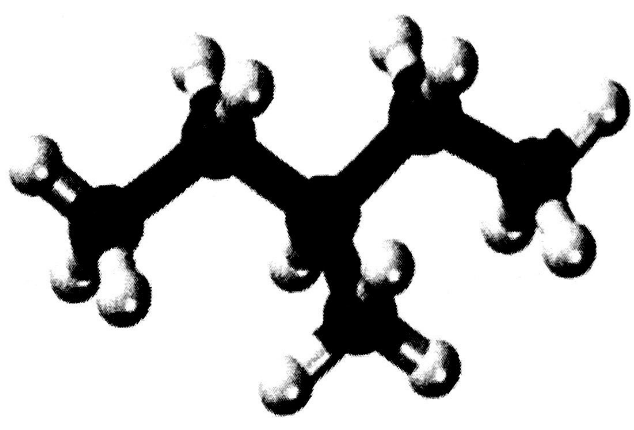
Example of Chemical Molecule of Y-tree

2-methylpentane



Molecular Formula: C_6H_{14}





Theorem 2.2. *If $G = (P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4} : K_1)$ is a X-tree then $con(G) = G_3 \cup G_4$ where G_3 is a X-tree $G_3 = (P_{m_1}; P_{m_2}; P_{m_3}; P_{m_4} : K_1)$ and $G_4 = (K_4 \odot P_{r_1}; P_{r_2}; P_{r_3}; P_{r_4})$ where,*

$$m_i = \begin{cases} \frac{n_i-1}{2} & \text{if } n_i \text{ is odd} \\ \frac{n_i}{2} & \text{if } n_i \text{ is even} \end{cases} \qquad i = 1,2,3,4$$

$$r_i = \begin{cases} \frac{n_i+1}{2} & \text{if } n_i \text{ is odd} \\ \frac{n_i}{2} & \text{if } n_i \text{ is even} \end{cases} \qquad i = 1,2,3,4$$

Let $G = (P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4} : K_1)$ be a X-tree in Figure 2. Then common neighborhood graph of G is disconnected graphs $G_3 = (P_{m_1}; P_{m_2}; P_{m_3}; P_{m_4} : K_1)$ and $G_4 = (K_4 \odot P_{r_1}; P_{r_2}; P_{r_3}; P_{r_4})$ in Figures 2(a) and 2(b).

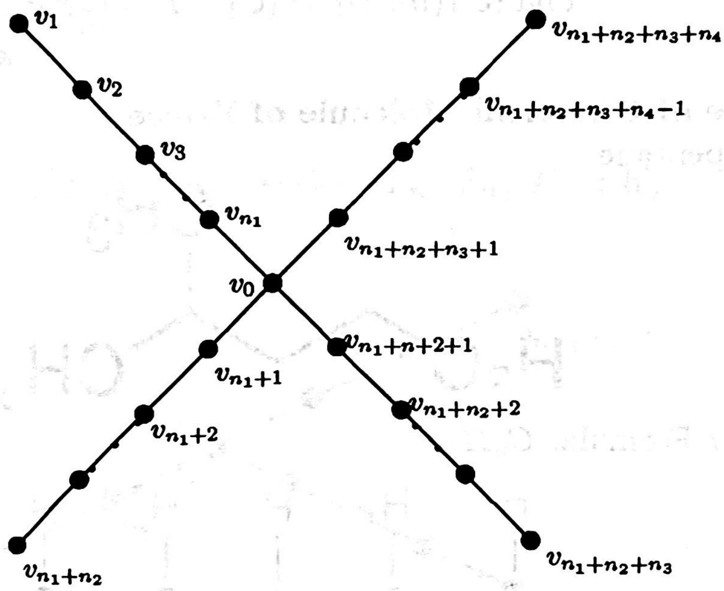


Figure 2: X-tree $G = (P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4} : K_1)$

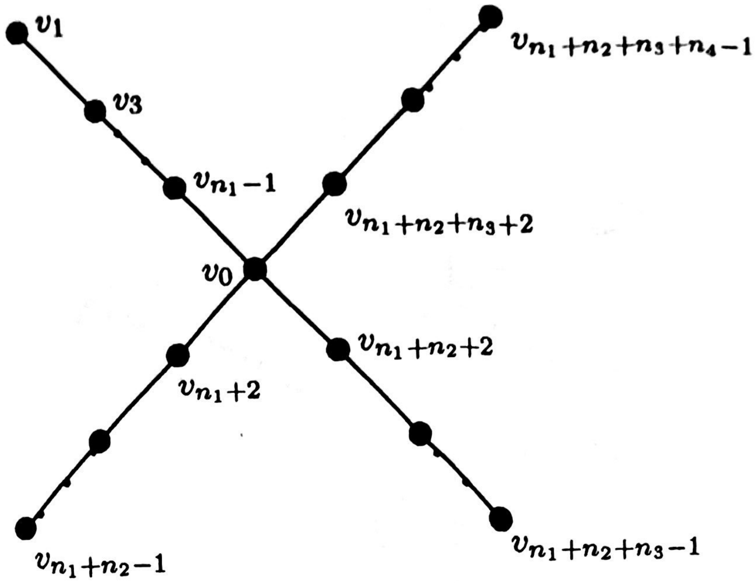


Figure 2(a): X-tree $G_3 = (P_{m_1}; P_{m_2}; P_{m_3}; P_{m_4} : K_1)$

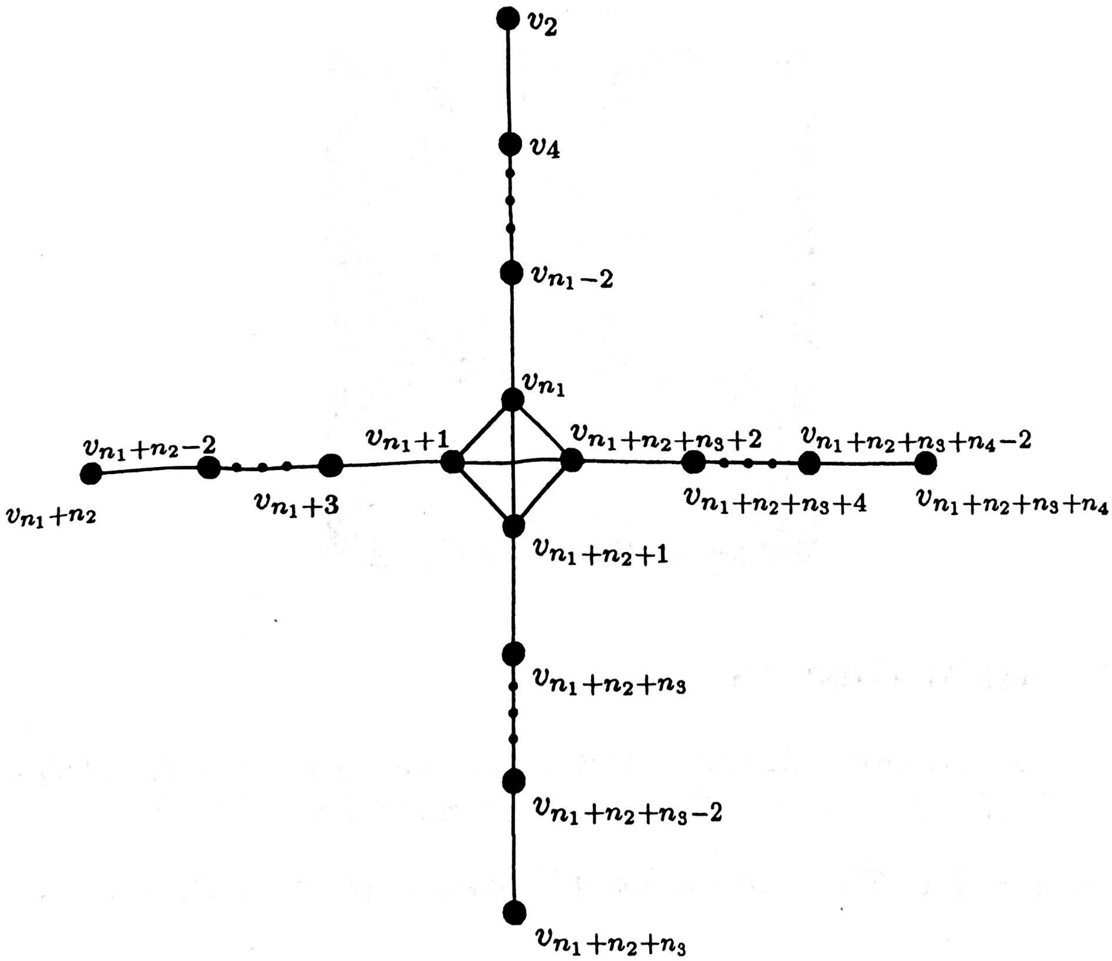
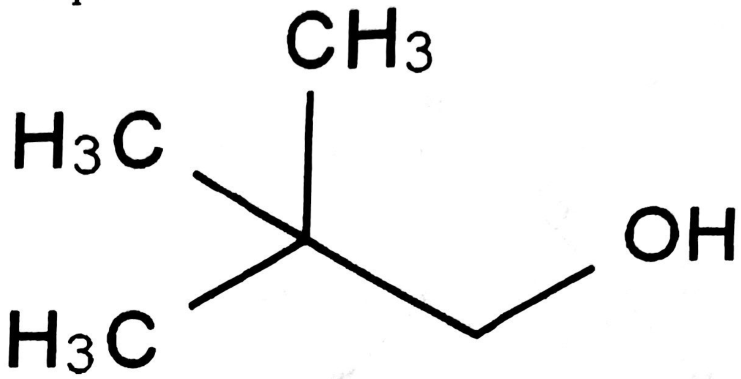


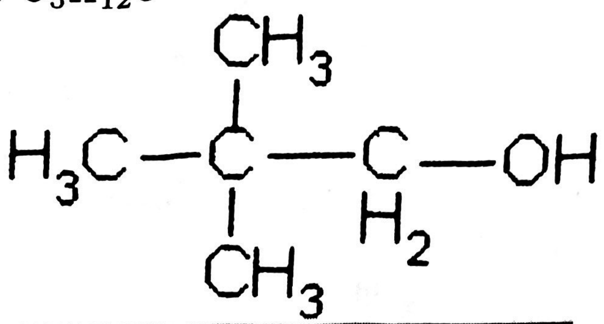
Figure 2(b): Graph $G_4 = (K_4 \odot P_{r_1}; P_{r_2}; P_{r_3}; P_{r_4})$

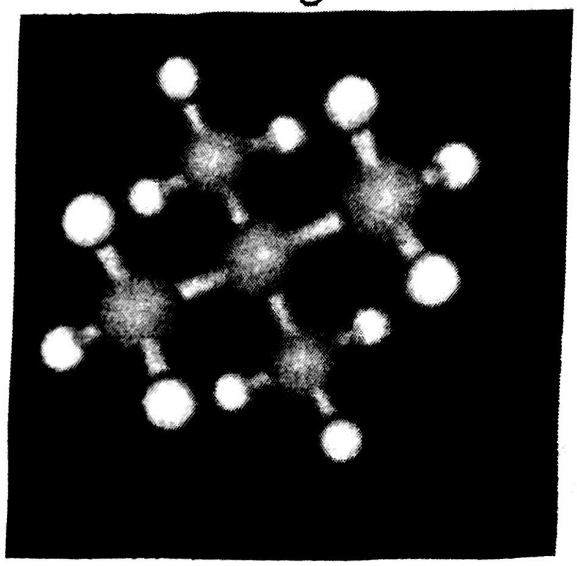
Example of Chemical Molecule of X-tree

2,2-Dimethyl-1-Propanol



Molecular Formula: $C_5H_{12}O$





3 Main Results

In this section we calculate the Wiener index of $(C_3 \odot P_{n_1}; P_{n_2}; P_{n_3})$, $(K_4 \odot P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4})$, Y-tree, X-tree, con(Y-tree) and con(X-tree).

Theorem 3.1. *The Wiener index of Y-tree $G = (P_{n_1}; P_{n_2}; P_{n_3} : K_1)$ is*

$$\begin{aligned} W(G) = {} & \frac{1}{6}[n_1^3 + n_2^3 + n_3^3 + 2(n_1 + n_2 + n_3)] \\ & + \frac{1}{2}\{[((n_1 + n_2)^2 + n_3^2 + n_1^2(n_2 + n_3) + n_2^2(n_1 + n_3) \\ & + n_3^2(n_1 + n_2) + 2(n_1 n_3 + n_2 n_3)]\} \end{aligned}$$

Proof. Let $G = (P_{n_1}; P_{n_1}; P_{n_1} : K_1)$
$W(G) = S_0 + S_1 + S_2 + S_3$ where
S_0 is the sum of the distances from the central vertex v_0 to all other vertices of the Y-tree,
S_1 is the sum of the distances of all the vertices from the path P_{n_1} to itself and the vertices of P_{n_2} and P_{n_3},
S_2 is the sum of the distances from the path P_{n_2} to itself and the vertices of P_{n_3},
S_3 is the sum of the distances from the path P_{n_3} to itself.

$$S_0 = \sum_{i=1}^{3} \frac{n_i(n_i+1)}{2}$$

$$\begin{aligned}
S_1 = {} & \frac{n_1(n_1-1)}{6} + [(n_1+1) + (n_1+1+1) + (n_1+1+2) + \cdots + (n_1+1) \\
& + (n_2-1)] + [n_1 + (n_1+1) + (n_1+2) + \cdots + n_1 + (n_2-1)] \\
& + [(n_1-1) + (n_1-1) + 1 + (n_1-1) + 2 + \cdots + (n_1-1) + (n_2-1)] \\
& + [(n_1-1) + (n_1-1) + 1 + (n_1-1) + 2 + \cdots + (n_1-1) + (n_2-1)] \\
& + [2 + (2+1) + (2+2) + \cdots + (2 + (n_2-1))] \\
& + [(n_1+1) + (n_1+1+1) + (n_1+1+2) + \cdots + (n_1+1) + (n_3-1)] \\
& + [n_1 + (n_1+1) + (n_1+2) + \cdots + n_1 + (n_3-1)] \\
& + \cdots + [2 + (2+1) + (2+2) + \cdots + (2 + (n-3-1))] \\
= {} & \frac{n_1(n_1^2-1)}{6} + n_2[(n_1+1) + n_1 + (n_1-1) + (n_1-2) + \cdot + 2] \\
& + n_1[1 + 2 + \cdots + (n_2-1)] \\
& + n_3[(n_1+1) + n_1 + (n_1-1) + \cdots + 2] \\
& + n_1[1 + 2 + \cdots + (n_3-1)]
\end{aligned}$$

$$\begin{aligned}
S_1 = {} & \frac{n_1(n_1^2-1)}{6} + n_2\left[\frac{(n_1+1)(n_1+2)}{2} - 1\right] \\
& + \frac{n_2(n_2-1)n_1}{2} + n_3\left[\frac{(n_1+1)(n_1+2)}{2} - 1\right] \\
& + \frac{n_3(n_3-1)n_1}{2}
\end{aligned}$$

$$S_1 = \frac{n_1(n_1^2-1)}{6} + \frac{n_1}{2}[(n_2+n_3)(n_1+3) + n_2(n_2-1) + n_3(n_3-1)]$$

$$S_2 = \frac{n_2(n_2^2-1)}{6} + n_3\left[\frac{(n_2+1)(n_2+2)}{2} - 1\right] + \frac{n_3(n_3-1)n_2}{2}$$

$$S_2 = \frac{n_2(n_2^2-1)}{6} + \frac{n_3}{2}[n_2^2 + 3n_2 + n_2(n_3-1)]$$

$$S_3 = \frac{n_3(n_3^2-1)}{6}$$

Hence $W(G) = S_0 + S_1 + S_2 + S_3$

$$\begin{aligned} W(G) = &\sum_{i=1}^{3} W(P_{n_i}) + \frac{1}{2}\sum_{i=1}^{3} n_i(n_i+1) \\ &+ \frac{n_1}{2}[(n_2+n_3)(n_1+3) + n_2(n_2-1) + n_3(n_3-1)] \\ &+ \frac{n_3}{2}[n_2(n_2+3) + n_2(n_3-1)] \end{aligned}$$

where $W(P_{n_i}) = \frac{(n_i^3-n_i)}{6}$, $i = 1, 2, 3$.

$$\begin{aligned} &= \frac{n_1^3-n_1}{6} + \frac{n_2^3-n_2}{6} + \frac{n_3^3-n_3}{6} \\ &+ \frac{n_1(n_1+1)}{2} + \frac{n_2(n_2+1)}{2} + \frac{n_3(n_3+1)}{2} \\ &+ \frac{n_1}{2}[n_1n_2 + 3n_2 + n_1n_3 + 3n_3 + n_2^2 - n_2 + n_3^2 - n_3] \\ &+ \frac{n_3}{2}[n_2^2 + 3n_2 + n_2n_3 - n_2] \\ &= \frac{n_1^3+n_2^3+n_3^3}{6} - \frac{1}{6}(n_1+n_2+n_3) \\ &+ \frac{1}{2}[n_1^2 + n_2^2 + n_3^2 + n_1 + n_2 + n_3] \\ &+ \frac{n_1}{2}[n_1n_2 + n_1n_3 + 2n_2 + 2n_3 + n_2^2 + n_3^2] \\ &+ \frac{n_3}{2}[n_2^2 + 2n_2 + n_2n_3] \\ &= \frac{n_1^3+n_2^3+n_3^3}{6} - \frac{1}{6}(n_1+n_2+n_3) \end{aligned}$$

$$+\frac{1}{2}[n_1+n_2+n_3+n_1^2+n_2^2+n_3^2+n_1^2(n_2+n_3)+n_2^2(n_1+n_3)$$
$$+n_3^2(n_1+n_2)+2n_1n_2+2n_1n_3+2n_2n_3]$$
$$=\frac{n_1^3+n_2^3+n_3^3}{6}-\frac{n_1-n_2-n_3}{2}+\frac{3n_1+3n_2+3n_3}{6}$$
$$+\frac{1}{2}\{n_1^2+n_2^2+n_3^2+n_1^2(n_2+n_3)+n_2^2(n_1+n_3)$$
$$+n_3^2(n_1+n_2)+2n_1n_2+2n_1n_3+2n_2n_3\}$$
$$=\frac{1}{6}[n_1^3+n_2^3+n_3^3+2(n_1+n_2+n_3)]$$
$$+\frac{1}{2}\{(n_1+n_2)^2+n_3^2+n_1^2(n_2+n_3)+n_2^2(n_1+n_3)$$
$$+n_3^2(n_1+n_2)+2(n_1n_3+n_2n_3)\}$$

□

Theorem 3.2. *The Wiener index of $G=C_3\odot(P_{n_1};P_{n_2};P_{n_3})$ is*

$$W(G)=\frac{1}{6}[n_1^3+n_2^3+n_3^3-(n_1+n_2+n_3)]$$
$$+\frac{1}{2}\{n_1^2(n_2+n_3)+n_2^2(n_1+n_3)+n_3^2(n_1+n_2)\}$$

Proof. Let $G=C_3\odot(P_{n_1};P_{n_2};P_{n_3})$
$W(G)=S_1+S_2+S_3$ where
S_1 is the sum of all the vertices from the path P_{n_1} to itself and the vertices of P_{n_2} and P_{n_3},
S_2 is the sum of the distances from the path P_{n_2} to itself and the vertices of P_{n_3},
S_3 is the sum of the distances from the path P_{n_3} to itself.

$$S_1=W(P_{n_1})+\frac{1}{2}[n_1n_2(n_1+1)+n_1n_2(n_2-1)]$$
$$+\frac{1}{2}[n_1n_3(n_1+1)+n_1n_3(n_3-1)]$$

$$S_2=W(P_{n_2})+\frac{1}{2}[n_2n_3(n_2+1)+n_2n_3(n_3-1)]$$

$$S_3=W(P_{n_3})$$

Hence $W(G) = S_1 + S_2 + S_3$

$$W(G) = \sum_{i=1}^{3} W(P_{n_i}) + \frac{1}{2}\{n_1n_2(n_1+1) + n_2n_3(n_2+1)]$$
$$+ n_1n_3(n_1+1) + n_1n_2(n_2-1) + n_1n_3(n_3-1)$$
$$+ n_2n_3(n_3-1)\}$$

$$W(G) = \frac{n_1^3 - n_1}{6} + \frac{n_2^3 - n_2}{6} + \frac{n_3^3 - n_3}{6}$$
$$+ \frac{1}{2}\{n_1^2n_2 + n_1n_2 + n_2^2n_3 + n_2n_3 + n_1^2n_3 + n_1n_3 + n_1n_2^2 - n_1n_2$$
$$+ n_1n_3^2 - n_1n_3 + n_2n_3^2 - n_2n_3\}$$
$$= \frac{1}{6}[n_1^3 + n_2^3 + n_3^3 - (n_1 + n_2 + n_3)]$$
$$+ \frac{1}{2}\{n_1^2n_2 + n_2^2n_3 + n_1^2n_3 + n_1n_2^2 + n_1n_3^2 + n_2n_3^2\}$$
$$= \frac{1}{6}[n_1^3 + n_2^3 + n_3^3 - (n_1 + n_2 + n_3)]$$
$$+ \frac{1}{2}\{n_1^2(n_2 + n_3) + n_2^2(n_1 + n_3) + n_3^2(n_1 + n_2)\}$$

□

Theorem 3.3. *The Wiener index of the X-tree, $G = (P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4} : K_1)$ is*

$$W(G) = \frac{1}{6}[n_1^3 + n_2^3 + n_3^3 + n_4^3 + 2n_1 + 2n_2 + 2n_3 + 2n_4]$$
$$+ \frac{1}{2}\{n_1^3 + n_2^3 + n_3^3 + n_4^3 + n_1^2n_2 + n_2^2n_3 + n_3^2n_4 + n_4^2n_1 - n_1n_2$$
$$- n_1n_3 - n_1n_4 + (2n_1 + 2n_2 + 3)n_3^2 + (n_2^2 + n_4^2 + 3)n_3$$
$$+ 3n_2n_3 + n_4 + n_1n_2n_3 + n_2n_3n_4\}$$

Proof. Let $G = (P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4} : K_1)$
$W(G) = S_0 + S_1 + S_2 + S_3 + S_4$ where
S_0 is sum of the distance from the central vertex v_0 to all other vertices of the X-tree,

S_1 is the sum of all the vertices from the path P_{n_1} to itself and the vertices of P_{n_2}, P_{n_3} and P_{n_4},
S_2 is the sum of the distances from the path P_{n_2} to itself and the vertices of P_{n_3} and P_{n_4},
S_3 is the sum of the distances from the path P_{n_3} to itself and the vertices of P_{n_4},
S_4 is the sum of the distances from the path P_{n_4} to itself.

$$S_0 = \sum_{i=1}^{4} \frac{n_i(n_i+1)}{2}$$

$$\begin{aligned} S_1 = \frac{n_1(n_1^2-1)}{6} &+ [(n_1+1)+(n_1+1+1)+(n_1+1+2) \\ &+\cdots+(n_1+1)+(n_2-1)] \\ &+[n_1+(n_1+1)+(n_1+2)+\cdots+n_1+(n_2-1)] \\ &+[(n_1-1)+(n_1-1)+1+(n_1-1)+2+\cdots+(n_1-1)+(n_2-1) \\ &+\cdots+[2+(2+1)+(2+2)+\cdots+(2+(n_2-1))] \\ &+[(n_1+1)+(n_1+1+1)+(n_1+1+2)+\cdots+(n_1+1)+(n_3-1)] \\ &+[n_1+(n_1+1)+(n_1+2)+\cdots+(n_1+(n_3-1))] \\ &+\cdots+[2+(2+1)+(2+2)+\cdots+(2+(n_3-1))] \\ &+\cdots+[(n_1+1)+(n_1+1+1)+(n_1+1+2)+\cdots+(n_1+1)+(n_4-1)] \\ &+[n_1+(n_1+1)+(n_1+2)+(n_1+(n_4-1))] \\ &\cdots+[2+(2+1)+(2+2)+\cdots+(2+(n_4-1))] \end{aligned}$$

$$\begin{aligned} S_1 = \frac{n_1(n_1^2-1)}{6} &+ n_2[(n_1+1)+n_1+(n_1-1)+(n_1-2)+\cdot+2] \\ &+n_1[1+2+\cdots+(n_2-1)] \\ &+n_3[(n_1+1)+n_1+(n_1-1)+\cdots+2] \\ &+n_1[1+2+\cdots+(n_3-1)] \\ &+n_4[(n_1+1)+n_1+(n_1-1)+\cdots+2] \\ &+n_1[1+2+\cdots+(n_4-1)] \end{aligned}$$

$$S_1 = \frac{n_1(n_1^2-1)}{6} + n_2\left[\frac{(n_1+1)(n_1+2)}{2} - 1\right]$$
$$+ \frac{n_2(n_2-1)n_1}{2} + n_3\left[\frac{(n_1+1)(n_1+2)}{2} - 1\right]$$
$$+ \frac{n_3(n_3-1)n_1}{2} + n_4\left[\frac{(n_1+1)(n_1+2)}{2} - 1\right]$$
$$+ \frac{n_4(n_4-1)n_1}{2}$$

$$S_1 = \frac{n_1(n_1^2-1)}{6} + \frac{n_1}{2}$$
$$[(n_2+n_3)(n_1+3) + n_2(n_2-1) + n_3(n_3-1) + n_4(n_4-1)]$$

$$S_2 = \frac{n_1(n_1^2-1)}{6} + n_3\left[\frac{(n_2+1)(n_2+2)}{2} - 1\right]$$
$$= \frac{n_3(n_3-1)n_2}{2} + n_4\left[\frac{(n_2+1)(n_2+2)}{2} - 1\right]$$
$$+ \frac{n_4(n_4-1)n_2}{2}$$

$$S_2 = \frac{n_3}{2}[n_2(n_3+1) + n_2(n_3-1) + n_2(n_4-1)]$$

$$S_3 = \frac{n_1(n_1^2-1)}{6} + n_4\left[\frac{(n_3+1)(n_3+2)}{2} - 1\right]$$
$$+ \frac{n_4(n_4-1)n_3}{2}$$

$$S_3 = \frac{n_4}{2}[n_3(n_3+3) + n_3(n_4-1)]$$

$$S_4 = \frac{n_4(n_4^2-1)}{6}$$

Hence $W(G) = S_0 + S_1 + S_2 + S_3 + S_4$

$$W(G)=\sum_{i=1}^{4}W(P_{n_i})+\sum_{i=1}^{4}\frac{n_i(n_i+1)}{2}$$
$$+\frac{1}{2}\{n_3(n_2+n_3)(n_1+3)+n_1n_2(n_2-1)$$
$$+n_1n_3(n_3+1)+n_1n_4(n_4-1)$$
$$+n_3n_2(n_2+3)+n_3n_2(n_3+1)$$
$$+n_3n_2(n_4-1)+n_4n_3(n_3+3)$$
$$+n_4n_3(n_4-1)\}$$
$$=\sum_{i=1}^{4}\left(W(P_{n_i})+\frac{n_i(n_i+1)}{2}\right)$$
$$+\frac{1}{2}\{(n_3-1)[(n_1n_3+n_3n_2]$$
$$+(n_4-1)[n_3n_2+n_1n_4+n_4n_3]$$
$$+(n_1+3)n_3(n_2+n_3)+(n_2+n_3)n_3n_2$$
$$+(n_3+3)n_3n_4+n_1n_2(n_2-1)\}$$

$$W(G)=\sum_{i=1}^{4}\left(W(P_{n_i})+\frac{n_i(n_i+1)}{2}\right)$$
$$+\frac{1}{2}\{n_1n_2(n_1-1)+(n_3-1)(n_1+n_2)n_3$$
$$+(n_4-1)[n_3n_2+n_4(n_1+n_3)]$$
$$+(n_1+3)n_3(n_2+n_3)+(n_2+3)n_2n_3$$
$$+(n_3+3)n_3n_4\}$$

$$W(G)=\frac{n_1^3-n_1}{6}+\frac{n_2^3-n_2}{6}+\frac{n_3^3-n_3}{6}+\frac{n_4^3-n_4}{6}$$
$$+\frac{n_1(n_1+1)}{6}+\frac{n_2(n_2+1)}{6}+\frac{n_3(n_3+1)}{6}+\frac{n_4(n_4+1)}{6}$$
$$+\frac{1}{2}\{n_1^2n_2-n_1n_2+n_1n_3^2-n_1n_3+2n_2n_3^2-n_2n_3+n_2n_3n_4-n_2n_3$$

$$+ n_1n_4^2 - n_1n_4 + n_3n_4^2 - n_3n_4 + n_1n_2n_3 + n_1n_3^2 + 3n_2n_3 + 3n_3^2$$
$$+ n_2^2n_3 + 3n_2n_3 + n_3^2n_4 + 3n_3n_4\}$$
$$= \frac{1}{6}[n_1^3 + n_2^3 + n_3^3 + n_4^3 + 2(n_1 + n_2 + n_3 + n_4)]$$
$$+ \frac{1}{2}\{n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_1^2n_2 + n_2^2n_3 + n_3^2n_4 + n_4^2n_1$$
$$- n_1(n_2 + n_3 + n_4) + (2n_1 + 2n_2 + 3)n_3^2 + (n_2^2 + n_4^2 + 3)n_3$$
$$+ 3n_2n_3 + n_4 + n_1n_2n_3 + n_2n_3n_4\}$$

□

Theorem 3.4. *The Wiener index of a graph* $G = K_4 \odot (P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4})$ *is*

$$W(G) = \frac{1}{6}[n_1^3 + n_2^3 + n_3^3 + n_4^3 - (n_1 + n_2 + n_3 + n_4)]$$
$$+ \frac{1}{2}\{n_1^2(n_2 + n_3 + n_4) + n_2^2(n_1 + n_3 + n_4) + n_3^2(n_1 + n_2 + n_4)$$
$$+ n_4^2(n_1 + n_2 + n_3)\}$$

Proof. Let $G = K_4 \odot (P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4})$
$W(G) = S_1 + S_2 + S_3 + S_4$ where
S_1 is the sum of all the vertices from the path P_{n_1} to itself and the vertices of P_{n_2}, P_{n_3} and P_{n_4},
S_2 is the sum of the distances from the path P_{n_2} to itself and the vertices of P_{n_3} and P_{n_4},
S_3 is the sum of the distances from the path P_{n_3} to itself and P_{n_4},
S_4 is the sum of the distances from the path P_{n_4} to itself.

$$S_1 = W(P_{n_1}) + \frac{1}{2}[n_1n_2(n_1 + 1) + n_1n_2(n_2 - 1)]$$
$$+ n_1n_3(n_1 + 1) + n_1n_3(n_3 - 1)$$
$$+ n_1n_4(n_1 + 1) + n_1n_4(n_4 - 1)\}$$

$$S_2 = W(P_{n_2}) + \frac{1}{2}\{n_2n_3(n_2 + 1) + n_2n_3(n_3 - 1)$$
$$+ n_2n_4(n_2 + 1) + n_2n_4(n_4 - 1)\}$$

$$S_3 = W(P_{n_3}) + \frac{1}{2}\{n_3n_4(n_3 + 1) + n_3n_4(n_4 - 1)]$$

$$S_4 = W(P_{n_4})$$

Hence

$$W(G) = \sum_{i=1}^{4} S_i$$

$$= \sum_{i=1}^{4} W(P_{n_i}) + \frac{1}{2}\{n_1n_2(n_1+n_2)$$

$$+ n_2n_3(n_2+n_3) + n_1n_3(n_1+n_3)$$

$$+ n_1n_4(n_1+n_4) + n_2n_4(n_2+n_4)$$

$$+ n_3n_4(n_3+n_4)\}$$

$$W(G) = \sum_{i=1}^{4} W(P_{n_i}) + \frac{1}{2}\prod_{\substack{i,j=1\\ i\neq j}}^{4} n_in_j(n_i+n_j)$$

$$W(G) = \frac{n_1^3-n_1}{6} + \frac{n_2^3-n_2}{6} + \frac{n_3^3-n_3}{6} + \frac{n_4^3-n_4}{6}$$

$$+ \frac{1}{2}\{n_1n_2(n_1+n_2) + n_2n_3(n_2+n_3) + n_1n_3(n_1+n_3)$$

$$+ n_1n_4(n_1+n_4) + n_2n_4(n_2+n_4) + n_3n_4(n_3+n_4)\}$$

$$= \frac{1}{6}[n_1^3+n_2^3+n_3^3+n_4^3 - (n_1+n_2+n_3+n_4)]$$

$$+ \frac{1}{2}\{n_1^2n_2 + n_1n_2^2 + n_2^2n_3 + n_2n_3^2 + n_1^2n_3 + n_1n_3^2 + n_1^2n_4 + n_1n_4^2$$

$$+ n_2^2n_4 + n_2n_4^2 + n_3^2n_4 + n_3n_4^2\}$$

$$= \frac{1}{6}[n_1^3+n_2^3+n_3^3+n_4^3 - (n_1+n_2+n_3+n_4)]$$

$$+ \frac{1}{2}\{n_1^2(n_2+n_3+n_4) + n_2^2(n_1+n_3+n_4) + n_3^2(n_1+n_2+n_4)$$

$$+ n_4^2(n_1+n_2+n_3)\}$$

□

From Theorems 3.1-3.4 we have the following results.

Theorem 3.5. *Let* $G = (P_{n_1}; P_{n_2}; P_{n_3} : K_1)$. *Then*

$$W(con(G)) = W(P_{m_1}; P_{m_2}; P_{m_3} : K_1) + W(C_3 \odot (P_{r_1}; P_{r_2}; P_{r_3}))$$

Where,

$$m_i = \begin{cases} \frac{n_i-1}{2} & \text{if } n_i \text{ is odd} \\ \frac{n_i}{2} & \text{if } n_i \text{ is even} \end{cases} \quad i = 1,2,3$$

$$r_i = \begin{cases} \frac{n_i+1}{2} & \text{if } n_i \text{ is odd} \\ \frac{n_i}{2} & \text{if } n_i \text{ is even} \end{cases} \quad i = 1,2,3$$

Proof. By Theorem 1.1 and 2.1, $W(con(G)) = W(G_1) + W(G_2)$. Where $G_1 = (P_{m_1}; P_{m_2}; P_{m_3} : K_1)$ and $G_2 = C_3 \odot (P_{r_1}; P_{r_2}; P_{r_3})$

$$\begin{aligned} W(con(G)) = &\frac{1}{6}[m_1^3 + m_2^3 + m_3^3 + 2(m_1 + m_2 + m_3)] \\ &+ \frac{1}{2}\{(m_1 + m_2)^2 + m_3^2 + m_1^2(m_2 + m_3) + m_2^2(m_1 + m_3) \\ &+ m_3^2(m_1 + m_2) + 2(m_1 m_3 + m_2 m_3)\} \\ &+ \frac{1}{6}[r_1^3 + r_2^3 + r_3^3 - (r_1 + r_2 + r_3)] \\ &+ \frac{1}{2}\{r_1^2(r_2 + r_3) + r_2^2(r_1 + r_3) + r_3^2(r_1 + r_2)\} \\ = &\frac{1}{6}[m_1^3 + m_2^3 + m_3^3 + r_1^3 + r_2^3 + r_3^3 + 2(m_1 + m_2 + m_3) \\ &- (r_1 + r_2 + r_3)] + \frac{1}{2}\{(m_1 + m_2)^2 + m_3^2 + m_1^2(m_2 + m_3) \\ &+ m_2^2(m_1 + m_3) + m_3^2(m_1 + m_2) + 2(m_1 m_3 + m_2 m_3) \\ &+ r_1^2(r_2 + r_3) + r_2^2(r_1 + r_3) + r_3^2(r_1 + r_2)\} \end{aligned}$$

□

Theorem 3.6. *Let $G = (P_{n_1}; P_{n_2}; P_{n_3}; P_{n_4} : K_1)$. Then*

$$W(con(G)) = W(P_{m_1}; P_{m_2}; P_{m_3}; P_{m_4} : K_1) + W(K_4 \odot (P_{r_1}; P_{r_2}; P_{r_3}; P_{r_4}))$$

Where,

$$m_i = \begin{cases} \frac{n_i-1}{2} & \text{if } n_i \text{ is odd} \\ \frac{n_i}{2} & \text{if } n_i \text{ is even} \end{cases} \quad i = 1,2,3,4$$

$$r_i = \begin{cases} \frac{n_i+1}{2} & \text{if } n_i \text{ is odd} \\ \frac{n_i}{2} & \text{if } n_i \text{ is even} \end{cases} \quad i = 1,2,3,4$$

Proof. By Theorem 1.1 and 2.2, $W(con(G)) = W(G_1) + W(G_2)$.
Where $G_1 = (P_{m_1}; P_{m_2}; P_{m_3}; P_{m_4} : K_1)$ and
$G_2 = (K_4 \odot (P_{r_1}; P_{r_2}; P_{r_3}; P_{r_4}))$

$$\begin{aligned}
W(con(G)) = &\frac{1}{6}[m_1^3 + m_2^3 + m_3^3 + m_4^3 + 2(m_1 + m_2 + m_3 + m_4)] \\
&+ \frac{1}{2}\{m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_1^2 m_2 + m_2^2 m_3 + m_3^2 m_4 + m_4^2 m_1 \\
&- m_1(m_2 + m_3 + m_4) + (2m_1 + 2m_2 + 3)m_3^2 + (m_2^2 + m_4^2 + 3)m_3 \\
&+ 3m_2 m_3 + m_4 + m_1 m_2 m_3 + m_2 m_3 m_4\} \\
&+ \frac{1}{6}[r_1^3 + r_2^3 + r_3^3 + r_4^3 - (r_1 + r_2 + r_3 + r_4)] \\
&+ \frac{1}{2}\{r_1^2(r_2 + r_3 + r_4) + r_2^2(r_1 + r_3 + r_4) + r_3^2(r_1 + r_2 + r_4) \\
&+ r_4^2(r_1 + r_2 + r_3)\} \\
= &\frac{1}{6}\left[\sum_{i=1}^{4}(m_i^3 + r_i^3) + 2\sum_{i=1}^{4} m_i - \sum_{i=1}^{4} r_i\right] \\
&+ \frac{1}{2}\left\{\sum_{i=1}^{4} m_i^2 + r_1^2(r_2 + r_3 + r_4) + r_2^2(r_1 + r_3 + r_4)\right. \\
&+ r_3^2(r_1 + r_2 + r_4) + r_4^2(r_1 + r_2 + r_3) + m_1^2 m_2 + m_2^2 m_3 + m_3^2 m_4 \\
&+ m_4^2 m_1 - m_1(m_2 + m_3 + m_4) + (2m_1 + 2m_2 + 3)m_3^2 \\
&\left. + (m_2^2 + m_4^2 + 3)m_3 + 3m_2 m_3 + m_4 + m_1 m_2 m_3 + m_2 m_3 m_4\right\}.
\end{aligned}$$

□

4 Conclusion

Finding Wiener index for congraph is a challenging work. We have done for Y-tree and X-tree. In future we would like to work on complicated graph structures.

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