3-EQUITABLE TOTAL LABELING OF GRAPHS

KM.Kathiresan[1] and R.Sivakumar [2]
Department of Mathematics,
Ayya Nadar Janaki Ammal College,
Sivakasi - 626124, Tamilnadu, India
email: [1]kathir2esan@yahoo.com, [2]sivamat15@gmail.com

ABSTRACT. Motivated by the existing 3-equitable labeling of graphs, in this paper we introduce a new graph labeling called 3-equitable total labeling and we investigate the 3-equitable total labeling of several families of graphs such as $C_n, W_n, SL_n, S(K_4, n)$ and $K_4^{(n)}$.

Keywords: 3-equitable labeling, 3-equitable total labeling.
AMS classification: 05C78.

1. Introduction

In general, a labeling of a graph is a map that carries some set of graph elements to numbers, most often to the positive or non negative integer. The most common choices of domain are the set of all vertices and edges (such labelings are called total labelings), the vertex-set alone (vertex-labelings) or the edge-set alone (edge-labelings). Other domains are possible. Several variations of graph labeling such as Graceful, Harmonious, Sequential, Magic, Antimagic, Consecutive, Strongly Consecutive etc., have been introduced by various authors (see Gallian [8]).

For graph theoretic terminology we follow [11]. Throughout this paper, all graphs are finite, simple and undirected. Let G be a graph with vertex set V and edge set E. A vertex labeling $f : V \to \{0, 1, 2, \ldots, k-1\}$, where $k \geq 1$, induces an edge labeling $\overline{f} : E \to \{0, 1, 2, \ldots, k-1\}$, defined by $\overline{f}(uv) = |f(u) - f(v)|$. By $v_f(i)$, we mean the number of vertices labeled i under f. Similarly, by $e_{\overline{f}}(i)$, we mean the number of edges labeled i under $\overline{f}$.

[2]Corresponding author

In 1990, Cahit [6] proposed the idea of distributing the vertex and edge labels among $\{0,\ 1,\ \ldots,\ k-1\}$ as evenly as possible to obtain a generalization of graceful labelings as follows. For any graph $G(V, E)$ and any positive integer k, assign vertex labels from $\{0,\ 1,\ \ldots,\ k-1\}$ so that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the number of edges labeled with i and the number of edges labeled with j differ by at most one. Cahit has called a graph with such an assignment of labels k-equitable. That is, a graph G is said to be *k-equitable*, if there exits a vertex labeling $f : V \to \{0, 1, 2, \ldots, k-1\}$ of G such that $|v_f(i) - v_f(j)| \leq 1$ and $\left|e_{\overline{f}}(i) - e_{\overline{f}}(j)\right| \leq 1$, for all $i \neq j$, $i, j = 0, 1, 2, \ldots k-1$. Note that $G(V, E)$ is graceful if and only if it is $|E|+1$-equitable and G is cordial if and only if it is 2-equitable. Cahit [4] has shown the following: C_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$; the triangular snake with n blocks is 3-equitable if and only if n is even; the friendship graph $\mathbf{C}_3^{(n)}$ is 3-equitable if and only if n is even; an Eulerian graph with $q \equiv 3 \pmod 6$ edges is not 3-equitable; and all caterpillars are 3-equitable [4]. Youssef [12] proved that if G is a k-equitable Eulerian graph with q edges and $k \equiv 2$ or $3 \pmod 4$ then $q \not\equiv k \pmod{2k}$. Cahit conjectures [4] that a triangular cactus with n blocks is 3-equitable if and only if n is even. In [6] Cahit proves that every tree with fewer than five end vertices has a 3-equitable labeling. He conjectures that all trees are k-equitable [5]. In 1999, Speyer and Szaniszlo [9] proved Cahit's conjecture for $k = 3$. Coles, Huszar, Miller, and Szaniszlo [7] proved that caterpillars, symmetric generalized n-stars (or symmetric spiders), and complete n-ary trees are 4-equitable.

Vaidya and Vihol [10] prove that any graph G can be embedded as an induced subgraph of a 3-equitable graph thereby ruling out any possibility of obtaining any forbidden subgraph characterization for 3-equitable graphs.

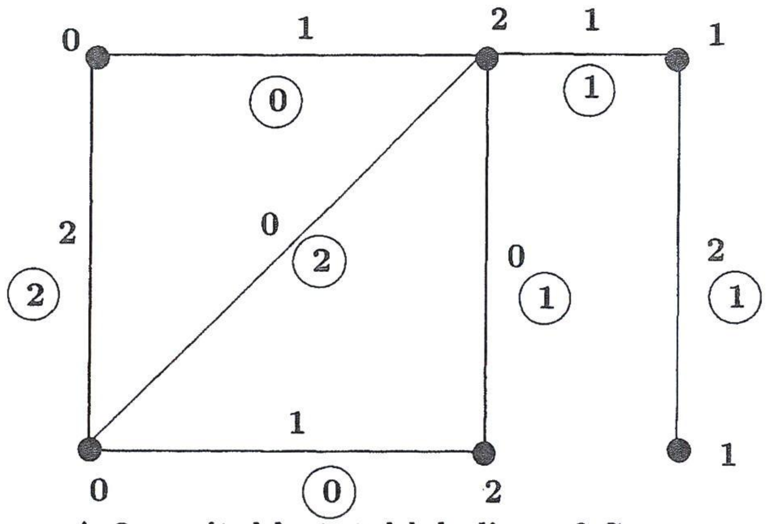
Bhut-Nayak and Telang have shown that crowns $C_n \odot K_1$ are k-equitable for $k = n, \ldots, 2n-1$ [2] and $C_n \odot K_1$ is k-equitable for all n when $k = 2, 3, 4, 5$ and 6 [1]. For further result on k-equitable labeling of a graph, the readers are advised to refer the survey article on graph labeling by Gallian [8].

Motivated by the 3-equitable labeling of graphs we introduce the following 3-equitable total labeling of graphs.

Definition 1.1. *Let G be a graph with vertex set V and edge set E. Define a total labeling $f : V \cup E \to \{0, 1, 2\}$ in such a way that $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$, for all $i \neq j$, $i, j = 0, 1, 2$. This induces an edge labeling $\overline{f}$ defined by $\overline{f}(uv) = (f(u) + f(v) + f(uv)) \pmod 3$, for all $uv \in E(G)$. A graph G is said to be 3-equitable total labeled if there is a total labeling f such that $\left|e_{\overline{f}}(i) - e_{\overline{f}}(j)\right| \leq 1$, for all $i \neq j$, $i, j = 0, 1, 2$. In this case f is called a 3-equitable total labeling of the graph G.*

The following example shows a 3-equitable total labeling of the graph G.

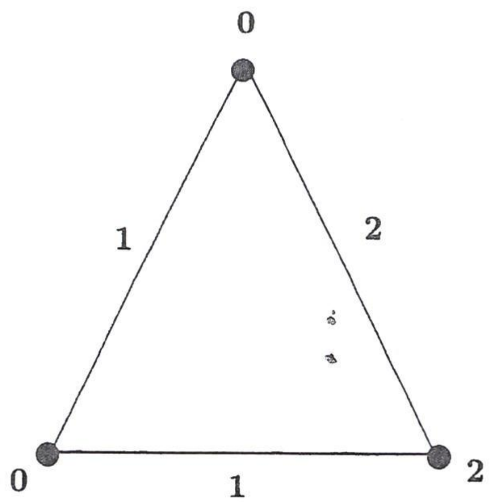
Example 1.2.



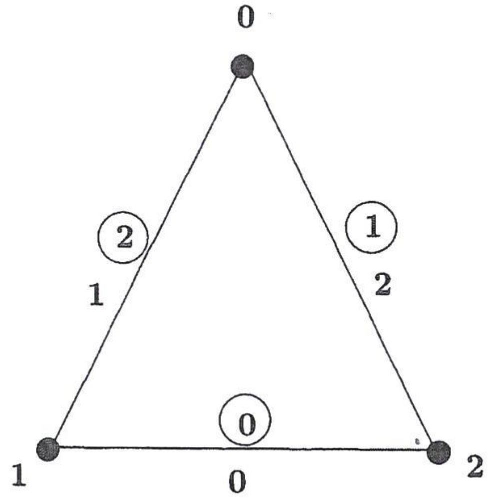
A 3-equitable total labeling of G

In this example $v_f(0) = v_f(1) = v_f(2) = 2$; $e_f(0) = e_f(2) = 2, e_f(1) = 3$; $e_{\overline{f}}(0) = e_{\overline{f}}(2) = 2, e_{\overline{f}}(1) = 3$ and hence G is 3-equitable total labeled.

Remark 1.3. Not every assignment of numbers from the set $\{0, 1, 2\}$ to a graph is a 3-equitable total labeling. For example the following assignment is not a 3-equitable total labeling of C_3



However C_3 has the following 3-equitable total labeling.



2. Main results

The next theorem shows that all cycles have 3-equitable total labeling.

Theorem 2.1. *The cycle C_n on n vertices, $n \geq 3$, is a 3-equitable total labeled graph.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of C_n. Define $f : V \cup E \rightarrow \{0, 1, 2\}$ by

$$f(v_i) = \begin{cases} 0, & if \ \ i \equiv 1 \ (mod\ 3) \\ 1, & if \ \ i \equiv 2 \ (mod\ 3) \\ 2, & if \ \ i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 0, & if \ \ i \equiv 0 \ (mod\ 3) \\ 1, & if \ \ i \equiv 2 \ (mod\ 3) \\ 2, & if \ \ i \equiv 1 \ (mod\ 3) \end{cases}$$

where $v_{n+1} = v_1$.

The following table shows that C_n is a 3-equitable total labeled graph

n	$v_f(0)$	$v_f(1)$	$v_f(2)$	$e_f(0)$	$e_f(1)$	$e_f(2)$	$e_{\bar{f}}(0)$	$e_{\bar{f}}(1)$	$e_{\bar{f}}(2)$
3m	m	m	m	m	m	m	m	m	m
3m+1	m+1	m	m	m	m	m+1	m	m	m+1
3m+2	m+1	m+1	m	m	m+1	m+1	m+1	m	m+1

□

Theorem 2.2. *The wheel W_n , $n \geq 3$, on n spokes is a 3-equitable total labeled graph.*

Proof. Let $V(W_n) = \{u\} \cup \{ v_i \mid 1 \leq i \leq n\}$, where u is the centre of the wheel.

Define $f : V \cup E \to \{0, 1, 2\}$ by

$$f(v_i) = \begin{cases} 0, & if \ \ i \equiv 1 \ (mod\ 3) \\ 1, & if \ \ i \equiv 2 \ (mod\ 3) \\ 2, & if \ \ i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 0, & if \ \ i \equiv 0 \ (mod\ 3) \\ 1, & if \ \ i \equiv 2 \ (mod\ 3) \\ 2, & if \ \ i \equiv 1 \ (mod\ 3) \end{cases}$$

where $v_{n+1} = v_1$,

$$f(uv_i) = \begin{cases} 0, & if \ \ i \equiv 1 \ (mod\ 3) \\ 1, & if \ \ i \equiv 2 \ (mod\ 3) \\ 2, & if \ \ i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(u) = \begin{cases} 0, & if \ \ n \equiv 0 \ (mod\ 3) \\ 1, & if \ \ n \equiv 1 \ (mod\ 3) \\ 2, & if \ \ n \equiv 2 \ (mod\ 3) \end{cases}$$

The following table shows that W_n is a 3-equitable total labeled graph

n	$v_f(0)$	$v_f(1)$	$v_f(2)$	$e_f(0)$	$e_f(1)$	$e_f(2)$	$e_{\bar{f}}(0)$	$e_{\bar{f}}(1)$	$e_{\bar{f}}(2)$
3m	m+1	m	m	2m	2m	2m	2m	2m	2m
3m+1	m+1	m+1	m	2m+1	2m	2m+1	2m	2m+1	2m+1
3m+2	m+1	m+1	m+1	2m+1	2m+2	2m+1	2m+1	2m+1	2m+2

□

Definition 2.3. *A Shell graph SL_n on n vertices $v_1, v_2, \cdots, v_n$ is a graph with edge set $E(SL_n) = \{v_1v_i \mid 3 \le i \le n-1\} \bigcup \{v_iv_{i+1} \mid 1 \le i \le n\}$, where $v_{n+1} = v_1$. Thus SL_n has n vertices and $2n-3$ edges. The graph defined as Shell graph SL_n is commonly known in the literature as the fan F_{n-1}.*

In the next theorem we shall show that the shell graph SL_n, $n \ge 5$, is a 3-equitable total labeled graph.

Theorem 2.4. *The shell graph SL_n , $n \ge 5$, is a 3-equitable total labeled graph.*

Proof. Define $f : V \cup E \to \{0,1,2\}$ by

$$f(v_i) = \begin{cases} 0, & if \ \ i \equiv 1 \ (mod\ 3) \\ 1, & if \ \ i \equiv 2 \ (mod\ 3) \\ 2, & if \ \ i \equiv 0 \ (mod\ 3) \end{cases}$$

for all $i = 1, 2, \cdots, n$

$$f(v_iv_{i+1}) = \begin{cases} 2, & if \ \ i \equiv 1 \ (mod\ 3) \\ 1, & if \ \ i \equiv 2 \ (mod\ 3) \\ 0, & if \ \ i \equiv 0 \ (mod\ 3) \end{cases}$$

where $v_{n+1} = v_1$, for all $i = 1, 2, \cdots, n$

$$f(v_1v_i) = \begin{cases} 0, & if \ \ i \equiv 0 \ (mod\ 3) \\ 1, & if \ \ i \equiv 1 \ (mod\ 3) \\ 2, & if \ \ i \equiv 2 \ (mod\ 3) \end{cases}$$

for all $i = 3, 4, \cdots, n-1$.

The following table shows that SL_n is a 3-equitable total labeled graph

n	$v_f(0)$	$v_f(1)$	$v_f(2)$	$e_f(0)$	$e_f(1)$	$e_f(2)$	$e_{\overline{f}}(0)$	$e_{\overline{f}}(1)$	$e_{\overline{f}}(2)$
3m	m	m	m	2m-1	2m-1	2m-1	2m-1	2m-1	2m-1
3m+1	m+1	m	m	2m	2m	2m-1	2m-1	2m	2m
3m+2	m+1	m+1	m	2m	2m+1	2m	2m	2m	2m+1

□

Definition 2.5. *A K_4-star, denoted by $K_4^{(n)}$, is a graph whose vertex set is $V(K_4^{(n)}) = \{v_0\} \bigcup \{v_{i,k} \mid 1 \leq i \leq 3,\ 1 \leq k \leq n\}$ and edge set is $E(K_4^{(n)}) = \{v_0 v_{i,k} \mid 1 \leq i \leq 3,\ 1 \leq k \leq n\} \bigcup \{v_{i,k} v_{i+1,k} \mid 1 \leq i \leq 3,\ 1 \leq k \leq n\}$, where $v_{4,k} = v_{1,k}$.*

Theorem 2.6. *K_4-star is a 3-equitable total labeled graph.*

Proof. Define $f : V \cup E\left(K_4^{(n)}\right) \to \{0,1,2\}$ as follows $f(v_0) = 0,\ f(v_{i,k}) = i-1,\ i = 1,2,3;\quad k = 1,2,\ldots,n$ and

$$f(v_0 v_{i,k}) = \begin{cases} 0, & if\ \ i \equiv 1\ (mod\ 3) \\ 1, & if\ \ i \equiv 2\ (mod\ 3) \\ 2, & if\ \ i \equiv 0\ (mod\ 3) \end{cases}$$

$$f(v_{i,k} v_{i+1,k}) = \begin{cases} 1, & if\ \ i \equiv 1\ (mod\ 3) \\ 0, & if\ \ i \equiv 2\ (mod\ 3) \\ 2, & if\ \ i \equiv 0\ (mod\ 3) \end{cases}$$

for $i = 1,2,3$; for $k = 1,2,\ldots,n$.

It is easy to find that $v_f(0) = n+1$, $v_f(1) = v_f(2) = n$, $e_f(0) = e_f(1) = e_f(2) = 2n$.

The induced labeling $\overline{f}$ is given by

$$\overline{f}(v_0 v_{i,k}) = \begin{cases} 0, & if\ \ i \equiv 1\ (mod\ 3) \\ 2, & if\ \ i \equiv 2\ (mod\ 3) \\ 1, & if\ \ i \equiv 0\ (mod\ 3) \end{cases}$$

$$f(v_{i,k} v_{i+1,k}) = \begin{cases} 2, & if\ \ i \equiv 1\ (mod\ 3) \\ 0, & if\ \ i \equiv 2\ (mod\ 3) \\ 1, & if\ \ i \equiv 0\ (mod\ 3) \end{cases}$$

Therefore, $e_{\overline{f}}(0) = e_{\overline{f}}(1) = e_{\overline{f}}(2) = 2n$.
Thus K_4-star $K_4^{(n)}$ is a 3-equitable total labeled graph for all n. □

Definition 2.7. *A K_4-snake on n-blocks, denoted by $S(K_4, n)$, is a graph whose vertex set is $\{v_1, v_2, \ldots, v_{n+1}\} \bigcup \{a_i, b_i \mid 1 \leq i \leq n\}$ and edge set is $\{v_iv_{i+1} \mid 1 \leq i \leq n\} \bigcup \{a_iv_i, a_iv_{i+1}, a_ib_i, b_iv_i, b_iv_{i+1} \mid 1 \leq i \leq n\}$.*

Theorem 2.8. *K_4 – snake on n-blocks is a 3-equitable total labeled graph, $n \geq 2$.*

Proof. Define $f : V \cup E(S(K_4, n)) \to \{0, 1, 2\}$ as follows $f(v_i) = 0$, $f(a_i) = 1$, $f(b_i) = 2$, $f(a_ib_i) = f(v_iv_{i+1}) = 0$, $f(a_iv_i) = f(a_iv_{i+1}) = 1$, $f(b_iv_i) = f(b_iv_{i+1}) = 2$, $1 \leq i \leq n$. Then $v_f(0) = n + 1$, $v_f(1) = v_f(2) = n$, $e_f(0) = e_f(1) = e_f(2) = 2n$. We can show that $e_{\overline{f}}(0) = e_{\overline{f}}(1) = e_{\overline{f}}(2) = 2n$, where $\overline{f}$ is the induced labeling. Then $S(K_4, n)$ is a 3-equitable total labeled graph. □

We feel that it is very hard to find an example of a graph which is not 3-equitable total labeled. So we conclude this paper with the following conjecture.

Conjecture: All graphs are 3-equitable total labeled.

ACKNOWLEDGMENTS

We are grateful to the reviewers for their valuable comments which improve the presentation of our paper.

REFERENCES

[1] V. N. Bhat-Nayak and S. Telang, Cahit-equitability of coronas, *Ars Combinatoria,* vol. 71, pp. 3-32, 2004.

[2] V. N. Bhat-Nayak and S. Telang, Cahit-k-equitability of $C_n \circ K_1$, $k = n$ to $2n - 1$, $n \geq 3$, *Congressus Numerantium,* vol. 155, pp. 131-213, 2002.

[3] I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, *Ars Combinatoria,* vol. 23, pp. 201-230, 1987.

[4] I. Cahit, On cordial and 3-equitable labellings of graphs, *Utilitas Mathematica,* vol. 37, pp. 189-198, 1990.

[5] I. Cahit, Recent results and open problems on cordial graphs, *Contemporary Methods in Graph Theory,* R. Bodendiek (ed.), Wissenschaftsverlag, Mannheim, pp. 209-230, 1990.

[6] I. Cahit, Status of graceful tree conjecture in 1989, in Topics in Combinatorics and Graph Theory, R.Bodendiek and R.Henn (eds), Physica-Verlag, Heidelberg, 1990.

[7] Z. Coles, A. Huszar, J. Miller and Z. Szaniszlo, 4-equitable tree labelings, preprint.

[8] J. A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics,* DS6, pp. 1-432, 2017.

[9] D. Speyer and Z. Szaniszlo, Every tree is 3-equitable, *Discrete Mathematics,* vol. 220, pp. 283-289, 2000.

[10] S. K. Vaidya and P. L. Vihol, Embedding and NP-complete problems for 3-equitable graphs, *Inter. J. Contemporary Advanced Math.*, vol. 2(No.1), pp. 1-7, 2011.

[11] D. B. West, Introduction to Graph Theory, *Pearson Education Pvt. Ltd., New Delhi, India,* 2002.

[12] M. Z. Youssef, New families of graceful graphs, *Ars Combinatoria,* vol. 67, pp. 303-311, 2003.