A class of bipartite and antipodal graphs and their uniform posets

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Abstract

Let Γ denote a bipartite and antipodal distance-regular graph with vertex set X, diameter D and valency k. Firstly, we determine such graphs Γ when $D \geq 8$, $k \geq 3$ and their corresponding quotient graphs are Q-polynomial: Γ is $2d$-cube if $D = 2d$; Γ is either $(2d+1)$-cube or the doubled Odd graph if $D = 2d + 1$. Secondly, by defining a partial order $\leq$ on X we obtain a grading poset $(X, \leq)$ with rank D. In [Š. Miklavič, P. Terwilliger, Bipartite Q-polynomial distance-regular graphs and uniform posets. J. Algebr. Combin. 225–242 (2013)], the authors determined precisely whether the poset $(X, \leq)$ for D-cube is uniform. In this paper we prove that the poset $(X, \leq)$ for doubled Odd graph is not uniform.

Key words: Distance-regular graph; Bipartite and Antipodal; Uniform poset

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1 Introduction

Let Γ denote a bipartite and antipodal distance-regular graph with vertex set X, diameter D and valency k. In this paper we determine such graphs Γ when $D \geq 8$, $k \geq 3$ and their corresponding quotient graphs are Q-polynomial: Γ is $2d$-cube if $D = 2d$; Γ is either $(2d+1)$-cube or the doubled Odd graph if $D = 2d + 1$.

Fix a vertex $x \in X$ and define a partial order $\leq$ on X as follows: for $y, z \in X$

$$y \leq z \quad \text{if and only if} \quad \partial(x, y) + \partial(y, z) = \partial(x, z),$$

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where ∂ denotes the path-length distance function for Γ. Denote this pa tially ordered set by $(X, \leq)$. For $0 \leq i \leq D$, let $\Gamma_i(x) = \{y \in X | \partial(x, y)$ $i\}$. Then the partition $\{\Gamma_i(x)\}_{i=0}^{D}$ of X forms a grading of the poset $(X, \leq$ in sense of [9, Section 1].

Terwilliger [9] introduced the uniform property for posets. In that wo he described the algebraic structure of the uniform posets and displaye eleven infinite families of examples. Worawannotai [12] found another fan ily of uniform posets using the polar spaces. Kang and Chen [4] obtaine a family of uniform posets using the nonisotropic subspaces of a unitar polar space. Liu [7] discussed the incidence algebra of the attenuated spac poset. Miklavič and Terwilliger [8] considered a uniform poset for bipartit distance-regular graphs based on their Q-polynomial properties. Hou e *al.* [3] studied the uniform poset for the folded $(2n + 1)$-cube by using it Q-polynomial property.

Motivated by the above connection between the Q-polynomial propert of distance-regular graphs and uniform posets, it is natural to consider th relation between the distance-regular graphs whose quotient graphs are Q polynomial and the uniform posets. To simplify this investigation, in th present paper we will determine whether the corresponding poset $(X, \leq)$ i uniform for our determined graphs: D-cube and doubled Odd graph. W remark that the work for D-cube was completed by Miklavič and Terwillige [8]. Therefore we discuss the case of the doubled Odd graph.

This paper is organized as follows. In Section 2 we recall some def initions and basic facts concerning distance-regular graphs and uniforn posets. In Section 3 we discuss a class of bipartite and antipodal graph whose quotients are Q-polynomial. In section 4 we show that the pose $(X, \leq)$ for the doubled Odd graph is not uniform. Our main results are Theorem 3.3 and Theorem 4.2.

2 Preliminaries

In this section we recall some basic facts concerning distance-regular graphs and uniform posets.

2.1 Distance-regular graphs

Let X denote a nonempty finite set. Let $V = \mathbb{R}^X$ denote the $\mathbb{R}$-vector space of column vectors with coordinates indexed by X, and let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra of matrices with rows and columns indexed by X. We observe that $\mathrm{Mat}_X(\mathbb{R})$ acts on V by left multiplication. For all $y \in X$, let $\hat{y}$ denote the element of V with a 1 in y coordinate and 0 in all other coordinates.

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set $\mathcal{R}$. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x,y) | x, y \in X\}$. We call D the *diameter* of Γ. For vertices $x, y \in X$ with $\partial(x,y) = h$, let $P^h_{ij}(x,y) = \{z \in X | \partial(x,z) = i, \partial(z,y) = j\}$. We say Γ is *regular with valency* k whenever $|P^0_{11}(x,x)| = k$ for all vertex $x \in X$. We say Γ is *distance-regular* whenever for all integers h, i, j $(0 \le h, i, j \le D)$ and for all vertices $x, y \in X$ with $\partial(x,y) = h$, the number

$$p^h_{ij} = |P^h_{ij}(x,y)|$$

is independent of x and y. The constants p^h_{ij} are called the *intersection numbers* of Γ. We abbreviate $c_i := p^i_{1i-1}$ $(1 \le i \le D)$, $a_i := p^i_{1i}$ $(0 \le i \le D)$, $b_i := p^i_{1i+1}$ $(0 \le i \le D-1)$. For the rest of this paper we assume Γ is distance-regular graph with diameter $D \ge 3$. By the triangle inequality, for $0 \le h, i, j \le D$ we have $p^h_{ij} = 0$ (resp. $p^h_{ij} \ne 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. In particular $c_i \ne 0$ for $1 \le i \le D$ and $b_i \ne 0$ for $0 \le i \le D-1$.

We now recall the Bose-Mesner algebra of Γ. For $0 \le i \le D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (x,y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \ne i \end{cases} \qquad (x, y \in X). \tag{1}$$

We call A_i the ith *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. Let M be the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ spanned by $A_0, A_1, \ldots, A_D$. We call M the *Bose-Mesner algebra* of Γ. By [2, p. 45] M has a second basis $E_0, E_1, \ldots, E_D$ such that (i) $E_0 = |X|^{-1}J$; (ii) $\sum_{i=0}^{D} E_i = I$; (iii) $E^t_i = E_i$ $(0 \le i \le D)$; (iv) $E_iE_j = \delta_{ij}E_i$ $(0 \le i, j \le D)$, where J (resp. I) denotes all 1's matrix (resp. identity matrix). We call $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of Γ.

We say Γ is Q-polynomial (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of primitive idempotents) whenever for $0 \le i \le D$, E_i is an entry-wise polynomial in E_1 with degree exactly i.

We now recall the dual Bose-Mesner algebra of Γ. Fix a vertex $x \in X$. For $0 \le i \le D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ with (y,y)-entry

$$(E^*_i)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \ne i \end{cases} \qquad (y \in X). \tag{2}$$

We call E^*_i the ith *dual idempotent* of Γ with respect to x [10, p. 378]. For convenience set $E^*_i = 0$ for $i < 0$ or $i > D$. Let $M^* = M^*(x)$ be the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ spanned by $E_0^*, E_1^*, \ldots, E_D^*$. We call M^* the *du* *Bose-Mesner algebra* of Γ with respect to x [10, p. 378]. Observe

$$V = E_0^*V + E_1^*V + \cdots + E_D^*V \quad \text{(orthogonal direct sum).}$$

For $0 \leq i \leq D$ let $\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}$. Then the subspac E_i^*V $(0 \leq i \leq D)$ has a basis $\{\hat{y} | y \in \Gamma_i(x)\}$.

2.2 Uniform posets

In this subsection we continue to assume Γ is distance-regular with diamete $D \geq 3$.

Fix a vertex $x \in X$. Define a partial order $\leq$ on X such that for a $y, z \in X$,

$$y \leq z \quad \text{if and only if} \quad \partial(x, y) + \partial(y, z) = \partial(x, z).$$

For $y, z \in X$ define $y < z$ whenever $y \leq z$ and $y \neq z$. We say tha z *covers* y whenever $y < z$ and there does not exist a vertex $w \in X$ sucl that $y < w < z$. For $0 \leq i \leq D$ each vertex in $\Gamma_i(x)$ covers exactl c_i vertices in $\Gamma_{i-1}(x)$, and is covered by exactly b_i vertices in $\Gamma_{i+1}(x)$ Therefore the partition $\{\Gamma_i(x)\}_{i=0}^D$ of X forms a *grading of the poset* $(X, \leq$ [9].

Definition 2.1. Let A be the adjacency matrix of Γ and let E_i^* be the *it* dual idempotent of Γ with respect to $x \in X$. Define matrices $R = R(x$ and $L = L(x)$ by

$$R = \sum_{i=0}^{D-1} E_{i+1}^* A E_i^*, \quad L = \sum_{i=1}^{D} E_{i-1}^* A E_i^*.$$

Note that $R = L^t$. We call R, L *raising matrix* and *lowering matrix*, respectively.

By (2) and Definition 2.1, it is direct to obtain the following results.

Lemma 2.2. *The following* (i), (ii) *hold.*

(i) *For* $0 \leq i \leq D - 1$

$$(RE_i^*)_{yz} = \begin{cases} 1 & \text{if } y \in \Gamma_{i+1}(x), \ z \in \Gamma_i(x), \ z < y, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *For* $1 \leq i \leq D$

$$(LE_i^*)_{yz} = \begin{cases} 1 & \text{if } y \in \Gamma_{i-1}(x), \ z \in \Gamma_i(x), \ y < z, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.3. *The following* (i), (ii) *hold.*

(i) *For* $0 \leq i \leq D-1$,

$$RE_i^*\hat{y} = \begin{cases} \sum_{\substack{z\in\Gamma_{i+1}(x)\\ y<z}} \hat{z} & \text{if } y \in \Gamma_i(x), \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *For* $1 \leq i \leq D$,

$$LE_i^*\hat{y} = \begin{cases} \sum_{\substack{z\in\Gamma_{i-1}(x)\\ z<y}} \hat{z} & \text{if } y \in \Gamma_i(x), \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, we recall the uniform structure for a partially ordered set [9]. The structure of a uniform poset involves the notion about a parameter matrix. By a *parameter matrix* we mean a tridiagonal matrix $U = (e_{ij})_{1\leq i,j\leq D}$ with entries in $\mathbb{R}$ satisfying

(i) $e_{ii} = 1$ for $1 \leq i \leq D$;

(ii) $e_{i,i-1} \neq 0$ for $2 \leq i \leq D$ or $e_{i-1,i} \neq 0$ for $2 \leq i \leq D$;

(iii) the principal submatrix $(e_{ij})_{r\leq i,j\leq p}$ is nonsingular for $1 \leq r \leq p \leq D$.

For convenience we abbreviate $e_i^- := e_{i,i-1}$ for $2 \leq i \leq D$, $e_i^+ := e_{i,i+1}$ for $1 \leq i \leq D-1$, and define $e_1^- := 0$, $e_D^+ := 0$.

By a *uniform structure* we mean a pair (U, f) where $U = (e_{ij})_{1\leq i,j\leq D}$ is a parameter matrix and $f = (f_1, f_2, \ldots, f_D)^t$ is a column vector in $\mathbb{R}^D$ such that the following equation

$$e_i^- RL^2 + LRL + e_i^+ L^2 R = f_i L$$

holds on $E_i^* V$ for $1 \leq i \leq D$ [8]. In this case, we also say the poset $(X, \leq)$ is *uniform*.

Note that the definition of uniform structure for a poset in [9] is the same as that in [8].

3 Bipartite and antipodal distance-regular graph

In this section we consider a class of bipartite and antipodal graphs whose quotients are Q-polynomial.

Let Γ denote a distance-regular graph with vertex set X and diameter D. Recall that Γ is *bipartite* whenever $a_i = 0$ for $0 \leq i \leq D$, and is *almost-bipartite* whenever $a_i = 0$ for $0 \leq i \leq D-1$ and $a_D \neq 0$.

For a given graph Γ of diameter D, we define the distance-D graph $\Gamma^{(D)}$ to be the graph with the same vertex set as Γ, and two vertices are adjacent whenever they are at distance D in the graph Γ. Γ is called *antipodal* if the distance-D graph $\Gamma^{(D)}$ is a disjoint union of cliques. In this case, we define the *folded graph* of Γ as the graph $\overline{\Gamma}$ with vertices being maximal cliques of $\Gamma^{(D)}$, and two maximal cliques are adjacent if there is an edge between them in Γ. The graph $\overline{\Gamma}$ is known as *antipodal quotient* of Γ. Moreover, if all maximal cliques in $\Gamma^{(D)}$ have the same size r then Γ is also called an *antipodal r-cover* of $\overline{\Gamma}$.

The following are two examples of bipartite and antipodal distance-regular graphs.

• *Hamming graph $H(D,2)$ (D-cube)*: Let X be the Cartesian product of D copies of $\{0,1\}$. Two vertices $x = (x_1, x_2, \ldots, x_D)$, $y = (y_1, y_2, \ldots, y_D) \in X$ are adjacent whenever $|\{i | x_i \neq y_i,\ 1 \leq i \leq D\}| = 1$. It is easy to check that for $x, y \in X$, $\partial(x,y) = l$ if and only if $|\{i | x_i \neq y_i,\ 1 \leq i \leq D\}| = l$. By [2, p. 261] $H(D,2)$ is a distance-regular graph with diameter D and intersection numbers

$$b_i = D - i, \quad c_i = i \quad (0 \leq i \leq D).$$

It is known that $H(D,2)$ is bipartite and antipodal, whose antipodal quotient graph is called *folded D-cube* [2, p. 264]. Moreover, $H(D,2)$ is Q-polynomial.

• *Doubled Odd graph*: Let S be a set of cardinality $2d+1$. The *doubled Odd graph* on S, often denoted by $2.O_{d+1}$, is the graph whose vertices are the d-subsets and $(d+1)$-subsets of S, and two vertices x, y are adjacent whenever $x \subset y$ or $y \subset x$. It is easy to check that for vertices x, y, $\partial(x,y) = l$ if and only if $|x \cup y - x \cap y| = l$. By [2, p. 260] $2.O_{d+1}$ is a distance-regular graph with diameter $D = 2d+1$ and intersection numbers

$$b_i = d + 1 - \lfloor \frac{1}{2}(i+1) \rfloor, \quad c_i = \lfloor \frac{1}{2}(i+1) \rfloor \quad (0 \leq i \leq D),$$

where $\lfloor a \rfloor$ denotes the maximal integer less than or equal to a. It is known that $2.O_{d+1}$ is bipartite and antipodal, whose antipodal quotient graph is called *odd graph* ([2, Proposition 9.18]). Note that $2.O_{d+1}$ is not Q-polynomial.

For later use we introduce the following lemmas.

Lemma 3.1. ([2, p. 141]) *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Assume that Γ is an antipodal r-cover of its quotient graph $\overline{\Gamma}$, then $\overline{\Gamma}$ is bipartite if D is even, and $\overline{\Gamma}$ is almost-bipartite and $r = 2$ if D is odd.*

Lemma 3.2. ([6, Theorem 1.1]) *Let Γ denote an almost-bipartite distance-regular graph with diameter $d \geq 4$. Then Γ is Q-polynomial if and only if one of* (i)–(iii) *holds.*

(i) Γ *is the* $(2d+1)$*-gon.*

(ii) Γ *is the folded* $(2d+1)$*-cube.*

(iii) Γ *is the Odd graph on a set of cardinality* $2d+1$.

We now give our first main result of this paper.

Theorem 3.3. *Let Γ denote a bipartite and antipodal distance-regular graph with diameter $D \geq 8$ and valency $k \geq 3$. Assume the antipodal quotient of Γ is Q-polynomial, then the following* (i), (ii) *hold.*

(i) *if $D = 2d$, then Γ is the $2d$-cube.*

(ii) *if $D = 2d+1$, then Γ is either the $(2d+1)$-cube or the doubled Odd graph on a set of cardinality $2d+1$.*

Proof. (i) This result can be easily obtained by [5, Theorems 10.2, 12.2].
(ii) Let $\overline{\Gamma}$ denote the antipodal quotient of Γ. By Lemma 3.1, $\overline{\Gamma}$ is an almost-bipartite Q-polynomial distance-regular graph with diameter $d \geq 4$ and valency $k \geq 3$, and Γ is an antipodal double cover of $\overline{\Gamma}$. Then by Lemma 3.2, $\overline{\Gamma}$ is the folded $(2d+1)$-cube or the Odd graph on a set of cardinality $2d+1$. If $\overline{\Gamma}$ is the folded $(2d+1)$-cube, by [2, Proposition 9.2.8(ii)] we know that Γ is the $(2d+1)$-cube; if $\overline{\Gamma}$ is the Odd graph on a set of cardinality $2d+1$, by [2, Propositions 9.1.8, 9.1.9], we know that Γ is the $2.O_{d+1}$ since $d \geq 4$. □

4 The poset $(X, \leq)$ for $2.O_{d+1}$

Recall the poset $(X, \leq)$ from Subsection 2.2, and the graph $2.O_{d+1}$ from Section 3. In this section our aim is to show that the poset $(X, \leq)$ for $2.O_{d+1}$ is not uniform. To do this, we need the following lemma.

Lemma 4.1. *Let Γ denote $2.O_{d+1}$ with vertex set X and diameter $2d+1 \geq 9$. Fix a vertex $x \in X$. For $3 \leq i \leq 2d-1$ and i odd, there exist six vertices, say y, z, w_1, w_2, p_1, p_2, satisfying*

$$\begin{aligned} &y \in \Gamma_{i-1}(x), \ z \in \Gamma_i(x), \ \partial(y,z) = 3, \\ &w_j \in \Gamma_i(x), \ \partial(w_j, z) = 2, \quad (j = 1,2), \\ &p_j \in \Gamma_{i-1}(x), \ \partial(p_j, y) = 2, \\ &y < w_j, \ p_j < w_j, \ p_j < z. \end{aligned} \tag{3}$$

Proof. For convenience let the graph $2.O_{d+1}$ be defined on set $S := \{1, 2, \ldots, 2d+1\}$. Since $2.O_{d+1}$ is distance-transitive, without loss of generality, le $x = \{1, 2, \ldots, d\}$. Then put

$$
\begin{aligned}
y' &= \{1, 2, \ldots, t\} \cup \{d+1, d+2, \ldots, l\}, \\
z' &= \{1, 2, \ldots, t\} \cup \{d+2, d+3, \ldots, l+2\}, \\
w_1' &= \{1, 2, \ldots, t\} \cup \{d+1, d+2, \ldots, l+1\}, \\
w_2' &= \{1, 2, \ldots, t\} \cup \{d+1, d+2, \ldots, l, l+2\}, \\
p_1' &= \{1, 2, \ldots, t\} \cup \{d+2, d+3, \ldots, l+1\}, \\
p_2' &= \{1, 2, \ldots, t\} \cup \{d+2, d+3, \ldots, l, l+2\},
\end{aligned}
$$

where $t + l = 2d$, $l - t = i - 1$ $(d+1 \leq l \leq 2d-1)$. By simple calculation for any odd i with $3 \leq i \leq 2d-1$ and for any $j = 1, 2$ we have

$$
\begin{aligned}
&|y'| = d,\ y' \in \Gamma_{i-1}(x),\ |z'| = d+1,\ z' \in \Gamma_i(x),\ \partial(y', z') = 3, \\
&|w_j'| = d+1,\ w_j' \in \Gamma_i(x),\ |p_j'| = d,\ p_j' \in \Gamma_{i-1}(x).
\end{aligned}
$$

Moreover, it is not difficult to verify that the above vertices satisfying relation (3).

Denote by $\mathrm{Aut}_x(\Gamma)$ the stabilizer subgroup of x in the automorphism group of $2.O_{d+1}$. Since $2.O_{d+1}$ is distance-transitive, $\mathrm{Aut}_x(\Gamma)$ is transitive on $\Gamma_i(x)$ $(0 \leq i \leq 2d+1)$. Pick any $\sigma \in \mathrm{Aut}_x(\Gamma)$ and assume $y = \sigma(y'), z = \sigma(z'), w_1 = \sigma(w_1'), w_2 = \sigma(w_2'), p_1 = \sigma(p_1'), p_2 = \sigma(p_2')$. It follows that these vertices $x, y, z, w_1, w_2, p_1, p_2$ also satisfy the statement of our lemma. □

Theorem 4.2. *The poset $(X, \leq)$ for $2.O_{d+1}$ is not uniform.*

Proof. Suppose on the contrary that there exists a tridiagonal matrix $U = (e_{ij})_{1 \leq i,j \leq 2d+1}$ and a column vector $f = (f_1, f_2, \ldots, f_{2d+1})^t$ such that the following equation

$$e_i^- RL^2 + LRL + e_i^+ L^2 R = f_i L$$

holds on $E_i^* V$ for $1 \leq i \leq 2d+1$. Then for $1 \leq i \leq 2d+1$ and any given vertices $y, z \in X$, we have

$$e_i^- (RL^2 E_i^*)_{yz} + (LRLE_i^*)_{yz} + e_i^+ (L^2 RE_i^*)_{yz} = f_i (LE_i^*)_{yz}. \tag{4}$$

We now calculate (y, z)-entry of both sides of (4).

By simple calculation, it is direct that

$$(RL^2E_i^*)_{yz} = \sum_{u,v\in X} (E_{i-1}^* RE_{i-2}^*)_{yu}(E_{i-2}^* LE_{i-1}^*)_{uv}(E_{i-1}^* LE_i^*)_{vz}$$

$$= |\{(u,v)|u < y,\ u < v,\ v < z,\ u \in \Gamma_{i-2}(x),\ y,v \in \Gamma_{i-1}(x),\ z \in \Gamma_i(x)\}|. \quad (5)$$

$$(LRLE_i^*)_{yz} = \sum_{u,v\in X} (E_{i-1}^* LE_i^*)_{yu}(E_i^* RE_{i-1}^*)_{uv}(E_{i-1}^* LE_i^*)_{vz}$$

$$= |\{(u,v)|y < u,\ v < u,\ v < z,\ z,u \in \Gamma_i(x),\ y,v \in \Gamma_{i-1}(x)\}|. \quad (6)$$

$$(L^2RE_i^*)_{yz} = \sum_{u,v\in X} (E_{i-1}^* LE_i^*)_{yu}(E_i^* LE_{i+1}^*)_{uv}(E_{i+1}^* RE_i^*)_{vz}$$

$$= |\{(u,v)|y < u,\ u < v,\ z < v,\ y \in \Gamma_{i-1}(x),\ z,u \in \Gamma_i(x),\ v \in \Gamma_{i+1}(x)\}|. \quad (7)$$

$$(LE_i^*)_{yz} = 1 \text{ if } y \in \Gamma_{i-1}(x),\ z \in \Gamma_i(x),\ y < z, \text{ and } 0 \text{ otherwise.} \quad (8)$$

In particular, we consider the concrete value of (y,z)-entry for (5)–(8) in the case of y, z in Lemma 4.1: $y \in \Gamma_{i-1}(x), z \in \Gamma_i(x)$ ($3 \leq i \leq 2d-1$, i odd) with $\partial(y,z) = 3$. Combining Lemma 4.1 with the fact that $c_3 = 2$, we get

$$P_{12}^3(y,z) = \{w_1, w_2\}, \quad \text{where } w_1, w_2 \in \Gamma_i(x), \quad (9)$$

$$P_{21}^3(y,z) = \{p_1, p_2\}, \quad \text{where } p_1, p_2 \in \Gamma_{i-1}(x). \quad (10)$$

From (5)–(10), it is easy to verify

$$(RL^2E_i^*)_{yz} = 0, \quad (\text{by } \{(u,v)|(u,v) \text{ in } (5)\} = \emptyset) \quad (11)$$

$$(LRLE_i^*)_{yz} = 2, \quad (\text{by } \{(u,v)|(u,v) \text{ in } (14)\} = \{(w_1,p_1),(w_2,p_2)\}) \quad (12)$$

$$(L^2RE_i^*)_{yz} = 0, \quad (\text{by } \{(u,v)|(u,v) \text{ in } (7)\} = \emptyset) \quad (13)$$

$$(LE_i^*)_{yz} = 0. \quad (\text{by } (8)) \quad (14)$$

From (11)–(14), it follows that (4) does not hold for the vertices y, z in Lemma 4.1, a contradiction. □

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