A note on putative $(101, 10)$-arcs in PG$(2, 11)$

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Abstract

An (n, r)-arc in PG$(2, q)$ is a set of n points such that each line contains at most r of the selected points. We show that in case of the existence of a $(101, 10)$-arc in PG$(2, 11)$ it only admits the trivial linear automorphism.

1 Introduction

Definition 1. An (n, r)-arc in PG$(2, q)$ is a set $\mathcal{B}$ of points such that each line contains at most r elements of $\mathcal{B}$.

The points of an (n, r)-arc in PG$(2, q)$ define the columns of a $3 \times n$ matrix generating a projective linear code over the field GF(q) with length n, dimension 3, and error correction capability $\lfloor (n-r-1)/2 \rfloor$ with respect to the Hamming metric.

Definition 2. Let $m_r(2, q)$ denote the maximum number n such that an (n, r)-arc in PG$(2, q)$ exists.

It is hard to determine the exact value for $m_r(2, q)$ and for $q \geq 11$ in most cases only lower and upper bounds are known (see [2]). An explicit construction of an (n, r)-arc in PG$(2, q)$ yields the bound $m_r(2, q) \geq n$.

For primes q it has been conjectured that $m_{q-1}(2, q) \leq q^2 - 2q + 1$ (see [1]) until in 2005 the first example of a $(q^2 - 2q + 2, q - 1)$-arc in PG$(2, q)$ for $q = 13$ has been found (see [3]). Recently, further examples have been constructed for $q = 19$, 31, 37, and 43 (see [4]).

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The smallest open case for which the existence of $(q^2-2q+2, q-1)$-arcs in $\mathrm{PG}(2,q)$ has not been solved yet is $q = 11$.

In this article we prove the following result.

Theorem 1. *In case of its existence a* $(101, 10)$*-arc in* $\mathrm{PG}(2, 11)$ *only admits the trivial linear automorphism.*

The complement of (n, r)-arcs in $\mathrm{PG}(2, q)$ are called blocking sets. A $(q^2 - 2q + 2, q - 1)$-arc in $\mathrm{PG}(2, q)$ corresponds to a double blocking set of size $3q - 1$ in $\mathrm{PG}(2, q)$.

Corollary 1. *In case of its existence a double blocking set in* $\mathrm{PG}(2, q)$ *of size* $3q - 1$ *for the prime field size* $q = 11$ *admits only the trivial linear automorphism.*

2 Construction of (n, r)-arcs in $\mathrm{PG}(2, q)$

If $\mathrm{G}(3, k, q)$ denotes the set of k-dimensional subspaces of the 3-dimensional canonical vector space $\mathrm{GF}(q)^3$ an (n, r)-arc in $\mathrm{PG}(2, q)$ corresponds to a set $\mathcal{B} \subseteq \mathrm{G}(3, 1, q)$ such that for all subspaces $H \in \mathrm{G}(3, 2, q)$ the inequality $|\{P \in \mathcal{B} \mid P \subseteq H\}| \leq r$ holds.

If $\mathrm{G}(3, 1, q) = \{P_1, \ldots, P_m\}$ and $\mathrm{G}(3, 2, q) = \{H_1, \ldots, H_m\}$, where $m := q^2 + q + 1$, we define the $m \times m$ incidence matrix $A = (a_{ij})$ where $a_{ij} := 1$ if $P_j \subseteq H_i$ and 0 otherwise.

If $j = (1, \ldots, 1)^T$ denotes the all-one column vector any binary column vector x satisfying $Ax \leq rj$ is equivalent to a $(j^T x, r)$-arc in $\mathrm{PG}(2, q)$.

Hence, the determination of $m_r(2, q)$ corresponds to the following integer linear programming problem:

$$m_r(2, q) = \max_{x \in \{0,1\}^m} \{j^T x \mid Ax \leq rj\}.$$

Definition 3. An (n, r)-arc $\mathcal{B}$ in $\mathrm{PG}(2, q)$ admits a subgroup $S \leq \mathrm{GL}(3, q)$ of the general linear group as a group of linear automorphisms if and only if $\mathcal{B}$ is a collection of orbits of S on $\mathrm{G}(3, 1, q)$.

If $S(P_1), \ldots, S(P_\ell)$ denote the orbits of $S \leq \mathrm{GL}(3, q)$ on the $\mathrm{G}(3, 1, q)$ and $S(H_1), \ldots, S(H_\ell)$ the orbits of S on $\mathrm{G}(3, 2, q)$ we define the incidence matrix $A^S = (a_{ij}^S)$ of orbits by $a_{ij}^S := |\{P \in S(P_j) \mid P \subseteq H_i\}|$.

If $w = (w_1, \ldots, w_\ell)^T$ is given by the orbit lenghts $w_i = |S(P_i)|$ any binary column vector x satisfying $A^S x \leq rj$ defines a $(w^T x, r)$-arc in $\mathrm{PG}(2, q)$ admitting $S \leq \mathrm{GL}(3, q)$ as a group of linear automorphisms.

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Considering groups $S \leq \mathrm{GL}(3,q)$ as groups of linear automorphisms we obtain the following integer linear programming problem to determine new lower bounds:

$$m_r(2,q) \geq m_r^S(2,q) := \max_{x\in\{0,1\}^\ell} \{w^T x \mid A^S x \leq rj\}.$$

The crucial part is that the square matrix A^S yields a reduction of the square matrix A from $m = q^2+q+1$ rows and columns to the number $\ell \leq m$ of orbits of S on the points and lines such that the integer linear programming solver can find solutions in reasonable time.

3 Putative (101, 10)-arcs in PG(2, 11)

In order to show that a putative $(101,10)$-arc in $\mathrm{PG}(2,11)$ only admits the trivial linear automorphism it is sufficient to show $m_{10}^S(2,11) < 101$ for all non-trivial subgroups $1 < S \leq \mathrm{GL}(3,11)$.

Since an (n,r)-arc in $\mathrm{PG}(2,11)$ admitting $1 < S \leq \mathrm{GL}(3,11)$ as a group of linear automorphisms also admits all cyclic subgroups $1 < T \leq S$ as groups of linear automorphisms we can restrict the list of groups to cyclic groups to show non-existence of the (n,r)-arc in $\mathrm{PG}(2,11)$.

Note that we can further restrict groups if we change our viewpoint from the general linear group to the projective linear group but since the number of groups to consider is small we stay with the matrix representation.

Furthermore, isomorphic (n,r)-arcs in $\mathrm{PG}(2,q)$ arise by conjugated subgroups. Therefore if C denotes the set of representatives of conjugacy classes of matrices in $\mathrm{GL}(3,11)$ we have to show $m_{10}^{\langle g\rangle}(2,11) < 101$ for all $g \in C \setminus \{aI \mid a \in \mathrm{GF}(11)\setminus\{0\}\}$, where I is the 3×3 identity matrix, in order to prove the non-existence of $(101,10)$-arcs in $\mathrm{PG}(2,q)$ admitting a non-trivial group of linear automorphisms.

Using GAP (see [5]) we computed the 1320 representatives g of conjugacy classes C of matrices of $\mathrm{GL}(3,11)$ which served as input for the integer linear programming $m_{10}^{\langle g\rangle}(2,11)$ we were solving with Gurobi (see [6]).

For each $g \in C \setminus \{aI \mid a \in \mathrm{GF}(11)\setminus\{0\}\}$ the exact value $m_{10}^{\langle g\rangle}(2,11)$ was smaller than 101 finally showing that either $(101,10)$-arcs in $\mathrm{PG}(2,11)$ do not exist or that they only admit the trivial linear automorphism. The runtime was less than 10 hours on a 1.2 GHz Intel Core m3 processor.

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