

# $C_6$ -DECOMPOSITION OF THE TENSOR PRODUCT OF COMPLETE GRAPHS

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ABSTRACT. Let  $G$  be a simple and finite graph. A graph is said to be decomposed into subgraphs  $H_1$  and  $H_2$  which is denoted by  $G = H_1 \oplus H_2$ , if  $G$  is the edge disjoint union of  $H_1$  and  $H_2$ . If  $G = H_1 \oplus H_2 \oplus \dots \oplus H_k$ , where  $H_1, H_2, \dots, H_k$  are all isomorphic to  $H$ , then  $G$  is said to be  $H$ -decomposable. Furthermore, if  $H$  is a cycle of length  $m$  then we say that  $G$  is  $C_m$ -decomposable and this can be written as  $C_m|G$ . Where  $G \times H$  denotes the tensor product of graphs  $G$  and  $H$ , in this paper, we prove that the necessary conditions for the existence of  $C_6$ -decomposition of  $K_m \times K_n$  are sufficient. Using these conditions it can be shown that every even regular complete multipartite graph  $G$  is  $C_6$ -decomposable if the number of edges of  $G$  is divisible by 6.

## 1. Introduction

Let  $C_m$ ,  $K_m$  and  $K_m - I$  denote cycle of length  $m$ , complete graph on  $m$  vertices and complete graph on  $m$  vertices minus a 1-factor respectively. By an  $m$ -cycle we mean a cycle of length  $m$ . All graphs considered in this paper are simple and finite. A graph is said to be *decomposed* into subgraphs  $H_1$  and  $H_2$  which is denoted by  $G = H_1 \oplus H_2$ , if  $G$  is the edge disjoint union of  $H_1$  and  $H_2$ . If  $G = H_1 \oplus H_2 \oplus \dots \oplus H_k$ , where  $H_1, H_2, \dots, H_k$  are all isomorphic to  $H$ , then  $G$  is said to be  $H$ -decomposable. Furthermore, if  $H$  is a cycle of length  $m$  then we say that  $G$  is  $C_m$ -decomposable and this can be written as  $C_m|G$ . A  $k$ -factor of  $G$  is a  $k$ -regular spanning subgraph. A  $k$ -factorization of a graph  $G$  is a partition of the edge set of  $G$  into  $k$ -factors. A  $C_k$ -factor of a graph is a 2-factor in which each component is a cycle of length  $k$ . A *resolvable  $k$ -cycle decomposition* (for short  $k$ -RCD) of  $G$  denoted by  $C_k||G$ , is a 2-factorization of  $G$  in which each 2-factor is a  $C_k$ -factor.

For two graphs  $G$  and  $H$  their tensor product  $G \times H$  has vertex set  $V(G) \times V(H)$  in which two vertices  $(g_1, h_1)$  and  $(g_2, h_2)$  are adjacent whenever

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$g_1g_2 \in E(G)$  and  $h_1h_2 \in E(H)$ . From this, note that the tensor product of graphs is distributive over edge disjoint union of graphs, that is if  $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$ , then  $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H)$ . Now, for  $h \in V(H)$ ,  $V(G) \times h = \{(v, h) | v \in V(G)\}$  is called a column of vertices of  $G \times H$  corresponding to  $h$ . Further, for  $y \in V(G)$ ,  $y \times V(H) = \{(y, v) | v \in V(H)\}$  is called a layer of vertices of  $G \times H$  corresponding to  $y$ . The problem of finding  $C_k$ -decomposition of  $K_{2n+1}$  or  $K_{2n} - I$  where  $I$  is a 1-factor of  $K_{2n}$ , is completely settled by Alspach, Gavlas and Sagna in two different papers (see [1, 13]). A generalization to the above complete graph decomposition problem is to find a  $C_k$ -decomposition of  $K_m * \overline{K}_n$ , which is the complete  $m$ -partite graph in which each partite set has  $n$  vertices. The study of cycle decompositions of  $K_m * \overline{K}_n$  was initiated by Hoffman et al. [5]. In the case when  $p$  is a prime, the necessary and sufficient conditions for the existence of  $C_p$ -decomposition of  $K_m * \overline{K}_n$ ,  $p \geq 5$  is obtained by Manikandan and Paulraja in [7, 8, 10]. Billington [2] has studied the decomposition of complete tripartite graphs into cycles of length 3 and 4. Furthermore, Cavenagh and Billington [4] have studied 4-cycle, 6-cycle and 8-cycle decomposition of complete multipartite graphs. Billington et al. [3] have solved the problem of decomposing  $(K_m * \overline{K}_n)$  into 5-cycles. Similarly, when  $p \geq 3$  is a prime, the necessary and sufficient conditions for the existence of  $C_{2p}$ -decomposition of  $K_m * \overline{K}_n$  is obtained by Smith (see [14]). For a prime  $p \geq 3$ , it was proved in [15] that  $C_{3p}$ -decomposition of  $K_m * \overline{K}_n$  exists if the obvious necessary conditions are satisfied. As the graph  $K_m \times K_n \cong K_m * \overline{K}_n - E(nK_m)$  is a proper regular spanning subgraph of  $K_m * \overline{K}_n$ . It is natural to think about the cycle decomposition of  $K_m \times K_n$ . The results in [7, 8, 10] also give necessary and sufficient conditions for the existence of a  $p$ -cycle decomposition, (where  $p \geq 5$  is a prime number) of the graph  $K_m \times K_n$ . In [9] it was shown that the tensor product of two regular complete multipartite graph is Hamilton cycle decomposable. Muthusamy and Paulraja in [11] proved the existence of  $C_{kn}$ -factorization of the graph  $C_k \times K_{mn}$ , where  $mn \not\equiv 2 \pmod{4}$  and  $k$  is odd. While Paulraja and Kumar [12] showed that the necessary conditions for the existence of a resolvable  $k$ -cycle decomposition of tensor product of complete graphs are sufficient when  $k$  is even.

In this paper, we prove that the obvious necessary conditions for  $K_m \times K_n$ ,  $2 \leq m, n$ , to have a  $C_6$ -decomposition are also sufficient. Among other results, here we prove the following main results.

It is not surprising that the conditions in Theorem 1.1 are "symmetric" with respect to  $m$  and  $n$  since  $K_m \times K_n \cong K_n \times K_m$ .

**Theorem 1.1.** *For  $2 \leq m, n$ ,  $C_6 | K_m \times K_n$  if and only if  $m \equiv 1$  or  $3 \pmod{6}$  or  $n \equiv 1$  or  $3 \pmod{6}$ .*

**Theorem 1.2.** *Let  $m$  be an even integer and  $m \geq 6$ , then  $C_6|K_m - I \times K_n$  if and only if  $m \equiv 0$  or  $2 \pmod{6}$*

## 2. $C_6$ Decomposition of $C_3 \times K_n$

**Theorem 2.1.** *For all  $n$ ,  $C_6|C_3 \times K_n$ .*

*Proof.* Following from the definition of tensor product of graphs, let  $U^1 = \{u_1, v_1, w_1\}$ ,  $U^2 = \{u_2, v_2, w_2\}, \dots, U^n = \{u_n, v_n, w_n\}$  form the partite set of vertices in  $C_3 \times K_n$ . Also,  $U^i$  and  $U^j$  has an edge in  $C_3 \times K_n$  for  $1 \leq i, j \leq n$  and  $i \neq j$  if the subgraph induce  $K_{3,3} - I$ , where  $I$  is a 1-factor of  $K_{3,3}$ . Now, each subgraph  $U^i \cup U^j$  is isomorphic to  $K_{3,3} - I$ . But  $K_{3,3} - I$  is a cycle of length six. Hence the proof.  $\square$

**Example 2.2.** *The graph  $C_3 \times K_7$  can be decomposed into cycles of length 6.*

**Solution.** *Let the partite sets (layers) of the tripartite graph  $C_3 \times K_7$  be  $U = \{u_1, u_2, \dots, u_7\}$ ,  $V = \{v_1, v_2, \dots, v_7\}$  and  $W = \{w_1, w_2, \dots, w_7\}$ . We assume that the vertices of  $U, V$  and  $W$  having same subscripts are the corresponding vertices of the partite sets. A 6-cycle decomposition of  $C_3 \times K_7$  is given below:*

$\{u_1, v_2, w_1, u_2, v_1, w_2\}, \{u_1, v_3, w_1, u_3, v_1, w_3\}, \{u_2, v_3, w_2, u_3, v_2, w_3\},$   
 $\{u_1, v_4, w_1, u_4, v_1, w_4\}, \{u_2, v_4, w_2, u_4, v_2, w_4\}, \{u_3, v_4, w_3, u_4, v_3, w_4\},$   
 $\{u_1, v_5, w_1, u_5, v_1, w_5\}, \{u_2, v_5, w_2, u_5, v_2, w_5\}, \{u_3, v_5, w_3, u_5, v_3, w_5\},$   
 $\{u_4, v_5, w_4, u_5, v_4, w_5\}, \{u_1, v_6, w_1, u_6, v_1, w_6\}, \{u_2, v_6, w_2, u_6, v_2, w_6\},$   
 $\{u_3, v_6, w_3, u_6, v_3, w_6\}, \{u_4, v_6, w_4, u_6, v_4, w_6\}, \{u_5, v_6, w_5, u_6, v_5, w_6\},$   
 $\{u_1, v_7, w_1, u_7, v_1, w_7\}, \{u_2, v_7, w_2, u_7, v_2, w_7\}, \{u_3, v_7, w_3, u_7, v_3, w_7\},$   
 $\{u_4, v_7, w_4, u_7, v_4, w_7\}, \{u_5, v_7, w_5, u_7, v_5, w_7\}, \{u_6, v_7, w_6, u_7, v_6, w_7\}.$

**Theorem 2.3.** [6] *Let  $m$  be an odd integer and  $m \geq 3$ . If  $m \equiv 1$  or  $3 \pmod{6}$  then  $C_3|K_m$ .*

**Theorem 2.4.** [13] *Let  $n$  be an even integer and  $m$  be an odd integer with  $3 \leq m \leq n$ . The graph  $K_n - I$  can be decomposed into cycles of length  $m$  whenever  $m$  divides the number of edges in  $K_n - I$ .*

## 3. $C_6$ Decomposition of $C_6 \times K_n$

**Theorem 3.1.** [13] *Let  $n$  be an odd integer and  $m$  be an even integer with  $3 \leq m \leq n$ . The graph  $K_n$  can be decomposed into cycles of length  $m$  whenever  $m$  divides the number of edges in  $K_n$ .*

**Lemma 3.2.**  $C_6|C_6 \times K_2$

*Proof.* Let the partite set of the bipartite graph  $C_6 \times K_2$  be  $\{u_1, u_2, \dots, u_6\}$ ,  $\{v_1, v_2, \dots, v_6\}$ . We assume that the vertices having the same subscripts are

the corresponding vertices of the partite sets. Now  $C_6 \times K_2$  can be decomposed into 6-cycles which are  $\{u_1, v_2, u_3, v_4, u_5, v_6\}$  and  $\{v_1, u_2, v_3, u_4, v_5, u_6\}$ .  $\square$

**Theorem 3.3.** For all  $n$ ,  $C_6 | C_6 \times K_n$ .

*Proof.* Let the partite set of the 6-partite graph  $C_6 \times K_n$  be  $U = \{u_1, u_2, \dots, u_n\}$ ,  $V = \{v_1, v_2, \dots, v_n\}$ ,  $W = \{w_1, w_2, \dots, w_n\}$ ,  $X = \{x_1, x_2, \dots, x_n\}$ ,  $Y = \{y_1, y_2, \dots, y_n\}$  and  $Z = \{z_1, z_2, \dots, z_n\}$ : we assume that the vertices of  $U, V, W, X, Y$  and  $Z$  having the same subscripts are the corresponding vertices of the partite sets. Let  $U^1 = \{u_1, v_1, w_1, x_1, y_1, z_1\}$ ,  $U^2 = \{u_2, v_2, w_2, x_2, y_2, z_2\}$ , ...,  $U^n = \{u_n, v_n, w_n, x_n, y_n, z_n\}$  be the sets of these vertices having the same subscripts. By the definition of the tensor product, each  $U^i$ ,  $1 \leq i \leq n$  is an independent set and the subgraph induced by each  $U^i \cup U^j$ ,  $1 \leq i, j \leq n$  and  $i \neq j$  is isomorphic to  $C_6 \times K_2$ . Now by Lemma 3.2 the graph  $C_6 \times K_2$  admits a 6-cycle decomposition. This completes the proof.  $\square$

#### 4. $C_6$ Decomposition of $K_m \times K_n$

**Proof of Theorem 1.1.** Assume that  $C_6 | K_m \times K_n$  for some  $m$  and  $n$  with  $2 \leq m, n$ . Then every vertex of  $K_m \times K_n$  has even degree and 6 divides in the number of edges of  $K_m \times K_n$ . These two conditions translate to  $(m-1)(n-1)$  being even and  $6 | m(m-1)n(n-1)$  respectively. Hence, by the first fact  $m$  or  $n$  has to be odd, i.e., has to be congruent to 1 or 3 or 5 (mod 6). The second fact can now be used to show that they cannot both be congruent to 5 (mod 6). It now follows that  $m \equiv 1$  or 3 (mod 6) or  $n \equiv 1$  or 3 (mod 6). Conversely, let  $m \equiv 1$  or 3 (mod 6). By Theorem 2.3,  $C_3 | K_m$  and hence  $K_m \times K_n = ((C_3 \times K_n) \oplus \dots \oplus (C_3 \times K_n))$ . Since  $C_6 | C_3 \times K_n$  by Theorem 2.1.

Finally, if  $n \equiv 1$  or 3 (mod 6), the above argument can be repeated with the roles of  $m$  and  $n$  interchanged to show again that  $C_6 | K_m \times K_n$ . This completes the proof.

**Proof of Theorem 1.2.** Assume that  $C_6 | K_m - I \times K_n$ ,  $m \geq 6$ . Certainly,  $6 | mn(m-2)(n-1)$ . But we know that if  $6 | m(m-2)$  then  $6 | mn(m-2)(n-1)$ . But  $m$  is even therefore  $m \equiv 0$  or 2 (mod 6).

Conversely, let  $m \equiv 0$  or 2 (mod 6). Notice that for each  $m$ ,  $\frac{m(m-2)}{2}$  is a multiple of 3. Thus by Theorem 2.4  $C_3 | K_m - I$  and hence  $K_m - I \times K_n = ((C_3 \times K_n) \oplus \dots \oplus (C_3 \times K_n))$ . From Theorem 2.1  $C_6 | C_3 \times K_n$ . The proof is complete.

#### 5. CONCLUSION

In view of the results obtained in this paper we draw our conclusion by the following corollary.

**Corollary 5.1.** *For any simple graph  $G$ . If*

- (1)  $C_3|G$  then  $C_6|G \times K_n$ , whenever  $n \geq 2$ .
- (2)  $C_6|G$  then  $C_6|G \times K_n$ , whenever  $n \geq 2$ .

*Proof.* We only need to show that  $C_3|G$ . Applying Theorem 2.1 gives the result.  $\square$

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