Chromatic Number and Hamiltonicity of Graphs

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Abstract

Let G be a k - connected ($k \geq 2$) graph of order n. If $\chi(G) \geq n-k$, then G is Hamiltonian or $K_k \vee (K_k^c \cup K_{n-2k})$ with $n \geq 2k+1$, where $\chi(G)$ is the chromatic number of the graph G.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. Let G be a graph. We use G^c to denote the complement of G. We also use $\chi(G)$, $\omega(G)$, and $\alpha(G)$ to denote the chromatic number, the clique number, and the independent (or stability) number of G, respectively. We use $G \vee H$ to denote the the join of two disjoint graphs G and H. If C is a cycle of G, we use $\overrightarrow{C}$ to denote the cycle C with a given direction. For two vertices x, y in C, we use $\overrightarrow{C}[x, y]$ to denote the consecutive vertices on C from x to y in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $\overleftarrow{C}[y, x]$. We use x^+ and x^- to denote respectively the successor and predecessor of a vertex x on C along the direction of C. We also use x^{++} to denote $(x^+)^+$. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if

G has a Hamiltonian cycle.

In this note, we will present a sufficient condition based on the chromatic number for the Hamiltonicity of graphs. The main result is as follows.

Theorem 1. Let G be a k - connected ($k \geq 2$) graph of order n. If $\chi(G) \geq n-k$, then G is Hamiltonian or $K_k \vee (K_k^c \cup K_{n-2k})$ with $n \geq 2k+1$.

2. The Lemmas

We will use the following results as our lemmas. The first one is an inequality established by Nordhaus and Gaddum in [3].

Lemma 1. Let G be a graph of order n. Then $\chi(G) + \chi(G^c) \leq n+1$.

The second one is the main result in [1].

Lemma 2. Let G be a k - connected ($k \geq 2$) graph with independent number $\alpha = k+1$. Let C be the longest cycle in G. Then $G[V(G) - V(C)]$ is complete.

3. Proofs

Proof of Theorem 1. Let G be a k - connected ($k \geq 2$) graph satisfying the conditions in Theorem 1. Assume that G is not Hamiltonian. Then $n \geq 2k+1$ (otherwise $\delta \geq k \geq \frac{n}{2}$ and G is Hamiltonian). Since $k \geq 2$, G contains a cycle. Choose a longest cycle C in G and give a direction on C. Since G is not Hamiltonian, there exists a vertex $x_0 \in V(G) - V(C)$. By Menger's theorem, we can find s ($s \geq k$) pairwise disjoint (except for x_0) paths P_1, P_2, ..., P_s between x_0 and $V(C)$. Let u_i be the end vertex of P_i on C, where $1 \leq i \leq s$. We assume that the appearance of u_1, u_2, ..., u_s agrees with the given direction on C. We use u_i^+ to denote the successor of u_i along the direction of C, where $1 \leq i \leq s$. Then a standard proof in Hamiltonian graph theory yields that $T := \{x_0, u_1^+, u_2^+, ..., u_s^+\}$ is independent (otherwise G would have cycles which are longer than C). Since $s \geq k$, we have an independent set $S := \{x_0, u_1^+, u_2^+, ..., u_k^+\}$ of size $k+1$ in G and a clique S of size $k+1$ in G^c. From Lemma 1, we have that

$$n+1 = n-k+k+1 \leq \chi(G) + \alpha(G)$$

$$= \chi(G) + \omega(G^c) \leq \chi(G) + \chi(G^c) \leq n+1.$$

Then $\chi(G) = n-k$ and $\alpha(G) = \omega(G^c) = \chi(G^c) = k+1$. Next we will present a claim and its proofs.

Claim 1. $G[V(G) - S]$ is a complete.

Proof of Claim 1. Suppose, to the contrary, that $G[V(G) - S]$ is not complete. Then there exist vertices x, $y \in V(G) - S$ such that $xy \notin E(G)$. Notice that S is independent in G. We can have a proper coloring for G in the following way. Use $n - |S| - 1 = n - k - 2$ different colors to color the vertices in $V(G) - S$ and another color to color all the vertices in S. Therefore $n - k = \chi(G) \leq |V(G) - S| - 1 + 1 = n - k - 1$, a contradiction. $\diamond$

Set $T_i := \overrightarrow{C}[u_i^{++}, u_{i+1}]$, where $1 \leq i \leq k$ and the index $k+1$ is regarded as 1. Obviously, $|T_i| \geq 1$ for each i with $1 \leq i \leq k$. Set $T := \{ i : |T_i| \geq 2 \}$. Next we, according to the different sizes of $|T|$, divide the remainder of the proofs into three cases.

Case 0 $|T| = 0$.

Since $|T| = 0$, we have $C = u_1 u_1^+ u_2 u_2^+ ... u_k u_k^+ u_1$. We first consider the case of $|V(G) - V(C)| \geq 2$. Since $\alpha(G) = k+1$, we have, by Lemma 2, that $G[V(G) - V(C)]$ is complete. Let z be a vertex in $V(G) - V(C) - \{ x_0 \}$. Then $x_0 z \in E(G)$. Since $z \in V(G) - S$ and $u_2 \in V(G) - S$, we, by Claim 1, have that $zu_2 \in E(G)$. Thus G has a cycle $x_0 z \overrightarrow{C}[u_2, u_1] P_1 x_0$ which is longer than C, a contradiction.

Next we consider the case $V(G) - V(C) = \{ x_0 \}$. Since S is independent and $d(w) \geq \delta \geq k$ for each vertex w in S, we must have that for any vertex $x \in S$ and any vertex $y \in V(G) - S$, $xy \in E(G)$. Thus G is $K_k \vee K_{k+1}^c$. Namely, G is $K_k \vee (K_k^c \cup K_{n-2k})$ with $n = 2k + 1$

Case 1 $|T| = 1$.

Without loss of generality, we assume that $|T_1| \geq 2$, $|T_r| = 1$ for each r with $2 \leq r \leq k$. We first consider the case of $|V(G) - V(C)| \geq 2$. Since $\alpha(G) = k+1$, we have, by Lemma 2, that $G[V(G) - V(C)]$ is complete. Let z be a vertex in $V(G) - V(C) - \{ x_0 \}$. Then $x_0 z \in E(G)$. Since $z \in V(G) - S$ and $u_3 \in V(G) - S$, we, by Lemma 1, have that $zu_3 \in E(G)$. Notice that u_3 is regarded as u_1 when $k = 2$. Thus G has a cycle $x_0 z \overrightarrow{C}[u_3, u_2] P_2 x_0$ which is longer than C, a contradiction.

Next we consider the case $V(G) - V(C) = \{ x_0 \}$. Obviously, we now have that $n \geq 2k + 2$. Let $T_1 = y_1 y_2 ... y_r u_2$, where $r \geq 1$. We first notice that $y_r x_0 \notin E(G)$ and $y_r u_s^+ \notin E(G)$, where $2 \le s \le k$ otherwise G would have cycles which are longer than C. We further notice that $u_1^+ y_r \in E(G)$ otherwise $\{x_0, u_1^+, u_2^+, ..., u_k^+, y_r\}$ would be an independent set of size $k+2$. We claim that $x_0 y_{r-1} \notin E(G)$ otherwise G would have a cycle $x_0 \overleftarrow{C}[y_{r-1}, u_1^+]\overrightarrow{C}[y_r, u_1]P_1 x_0$ which is longer than C. We further claim that $u_l^+ y_{r-1} \notin E(G)$ for each l with $2 \le l \le k$ otherwise G would have a cycle $x_0 P_l \overleftarrow{C}[u_l, y_r]\overrightarrow{C}[u_1^+, y_{r-1}]\overrightarrow{C}[u_l^+, u_1]P_1 x_0$ which is longer than C. Since $\{x_0, u_1^+, u_2^+, ..., u_k^+, y_{r-1}\}$ is not independent, we must have that $u_1^+ y_{r-1} \in E(G)$. Repeating this process, we can prove that $y_j u_1^+ \in E(G)$ for each j with $1 \le j \le r$, $x_0 y_j \notin E$ for each j with $1 \le j \le r$, and $y_j u_l^+ \notin E(G)$ for each j and l with $1 \le j \le r$ and $2 \le l \le k$.

Notice that $d(w) \ge \delta \ge k$ for each vertex $w \in S - \{u_1^+\} = \{x_0, u_2^+, ..., u_k^+\}$. We must have that $wu_s \in E(G)$ for each vertex $w \in S - \{u_1^+\}$ and each s with $1 \le s \le k$. Next we will prove that $u_1^+ u_t \in E(G)$ for each t with $1 \le t \le k$. Obviously, $u_1^+ u_1 \in E(G)$. Without loss of generality, we assume that $u_1^+ u_2 \notin E(G)$. In this case, we can have a proper coloring for G in the following way. Firstly, use $n - |S| = n - k - 1$ different colors to color the vertices in $V(G) - S = \{y_1, y_2, ..., y_r, u_1, u_2, ..., u_k\}$; secondly, use the color assigned to the vertex u_2 in the first step of coloring to color u_1^+; finally, use the color assigned to y_1 in the first step of coloring to color the vertices x_0, u_2^+, u_3^+, ... , u_k^+. Thus $n-k = \chi(G) \le n-k-1$, a contradiction.

Now $G[V(G)-S]$ is complete, $u_1^+ w \in E(G)$ for each $w \in V(G)-S$, $xy \in E(G)$ for each vertex $x \in \{x_0, u_2^+, ..., u_k^+\}$ and each $y \in \{u_1, u_2, ..., u_k\}$, and $xy \notin E(G)$ for each vertex $x \in \{x_0, u_2^+, ..., u_k^+\}$ and each $y \in \{y_1, y_2, ..., y_r\}$, we have that G is $K_k \vee (K_k^c \cup K_{n-2k})$ with $n \ge 2k+2$.

Case 2 $|T| \ge 2$.

Notice that $T_i \subseteq V(G) - S$ for each i with $1 \le i \le k$. We, by Claim 1, have that $G[T_1 \cup T_2 \cup \cdots \cup T_k]$ is complete. Since $|T| \ge 2$, there exist two different indexes i and j such that $u_i^- \in T_i$, $u_j^- \in T_j$ and therefore $u_i^- u_j^- \in E(G)$, where $1 \le i, j \le k$. Then we can easily find a cycle in G which is longer than C, a contradiction.

So the proof of Theorem 1 is completed. ◇

References

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