Co-secure Domination in Mycielski Graphs

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Abstract

A set $S \subseteq V(G)$ of a connected graph G is a co-secure dominating set, if S is a dominating set and for each $u \in S$, there exists a vertex $v \in V(G) - S$, such that $v \in N(u)$ and $(S - \{u\}) \cup \{v\}$ is a dominating set of G. The minimum cardinality of the co-secure dominating set in a graph G is the co-secure domination number, $\gamma_{cs}(G)$. In this paper, we characterise the Mycielski graphs with co-secure domination 2 and 3. We also obtained a sharp upper bound for $\gamma_{cs}(\mu(G))$.

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1 Introducton

Let $G = (V, E)$ be a simple graph with $|v(G)| = n$ and $|E(G)| = m$. The degree of a vertex v in G, $deg(v)$, is the total number of vertices adjacent to it. The open neighbourhood of a vertex v, $N(v) = \{u \in V(G) : uv \in E(G)\}$ and its closed neighbourhood is the set $N[v] = N(v) \cup \{v\}$. A set $S \subseteq V(G)$ is a dominating set if for every vertex $v \in V(G) - S$, there exists $u \in S$ such that v is adjacent to u. The minimum cardinality of a dominating set is the domination number of G, $\gamma(G)$. The concept of domination has been studied extensively by T.W. Haynes, et.al in [6]. William.F. Klostermeyer and C.M. Mynhardt studied about several types of domination parameters in [7]. It contains a detailed survey on the historical development and the strategies of protection of graphs using mobile guards.

Secured dominating sets can be considered as one of the strategies for protection of a graph by placing one or more guards at every vertex v of a subset S of $V(G)$, where a guard at a vertex v can protect any vertex in its closed neighbourhood. This minimizes the number of guards to secure a system so that it is cost effective [7].

Let $S \subseteq V(G)$ be a dominating set, if corresponding to each vertex $v \in V(G) - S$ there exists a vertex u in S such that v is adjacent to u and $(S - \{u\}) \cup \{v\}$ is a dominating set of G, then S is a **secure dominating set** of G. The minimum cardinality of a secure dominating set is the **secure domination number**, $\gamma_s(G)$. This concept was introduced by E. J. Cockayne, et.al in [4] and has been investigated by several authors in [1],[2] and [6].

J.M. Xu in [13] modelled the topology of an interconnection network by a simple graph, whose vertices represents components of a network and whose edges represents physical communication links between them. A detailed explanation of various network systems and the influence of graph parameters on the network system has been given in [9], [13].

The Mycielskian or Mycielski graph, $\mu(G)$ of an undirected graph G is a graph formed by a construction of Jan Mycielski with vertex set $V(\mu(G))$ $= \{u_1, u_2 ..., u_n\} \cup \{u_{1'}, u_{2'} ..., u_{n'}\} \cup \{w\}$ and $E(\mu(G)) = E(G) \cup \{u_i u_{j'} : u_i u_j \in E(G)\} \cup \{ u_{i'} u_j : u_i u_j \in E(G)\} \cup \{ u_{i'} w\}$, where $i, j \in \{1, 2, 3, ..., n\}$. Thus, $\mu(G)$ consists of $2n+1$ vertices and $3m + n$ edges, where n is the number of vertices and m is the number of edges of the given graph G, [10].

The Mycielskian of a graph produce large networks and keep some properties like fast multi-path communication, reliable resource sharing, high fault tolerance and diameter, which are essential for a good network [12], [11]. In this paper, we have characterised the Mycielski graph with co-secure domination number 2 and 3 and also obtained a sharp upper bound for the co-secure domination of Mycielski graphs.

2 Co-Secure Domination

The idea of **co-secure domination** has been motivated by a situation in which the set of guards in the dominating set S continue to protect the graph even after every guard in S is replaced by another guard from $V(G) - S$. S. Arumugam initiated this study in [1].

A dominating set S of a graph $G = (V, E)$ is called a **co-secure dominating set**(CSDS), if for each $u \in S$, there exists a vertex $v \in (V - S)$ such that $v \in N(u)$ and $(S - \{u\}) \cup \{v\}$ is a dominating set of G [1]. The minimum cardinality of a co-secure dominating set in G is the **co-secure domination number** $\gamma_{cs}(G)$. If G has isolated vertices, CSDS does not exists and hence the study of co-secure domination may be restricted to connected, non-trivial graphs.

We have characterized Mycielski graphs with co-secure domination number 2 and 3.

Theorem 2.1. *Let G be a connected graph with n vertices. Then $\gamma_{cs}(\mu(G)) = 2$ if and only if G has at least two vertices of degree $(n-1)$.*

Proof. Suppose G has at least two vertices of degree $(n-1)$, say u_1, u_2. Clearly $S = \{u_1, w\}$ is a dominating set of $\mu(G)$, since u_1 dominates all the vertices $u_i, u_{i'}$ of $\mu(G)$, except $u_{1'}$, and w dominates the vertex $u_{1'}$, where $i \in \{1, 2, 3, ..., n\}$.
Now, we have to check the co-security condition for S. For, consider the vertex $u_1 \in S$, there exists a vertex $u_2 \in V(\mu(G)) - S$ such that $u_2 \in N(u_1)$ and $(S - \{u_1\}) \cup \{u_2\}$ is a dominating set of $\mu(G)$. Since u_2 dominates all the vertices $u_i, u_{i'}$, where $i \in \{1, 2, 3, 4, .., n\}$ except $u_{2'} \in V(\mu(G))$ and the vertex w dominates $u_{2'}$. Now, consider the vertex $w \in S$, there exists a vertex $u_{1'} \in V(\mu(G)) - S$ such that $u_{1'} \in N(w)$ and $(S - \{w\}) \cup \{u_{1'}\}$ is a dominating set of $\mu(G)$, since u_1 dominates all the vertices $u_i, u_{i'}$, where $i \in \{1, 2, 3, 4, .., n\}$ except $u_{1'}$ and the vertex $u_{1'} \in S$. Thus $S = \{u_1, w\}$ is a co-secured dominating set of $\mu(G)$ and $\gamma_{cs}(\mu(G)) = 2$.

Conversely assume that $\gamma_{cs}(\mu(G)) = 2$.

For any graph G without isolates, we have $\gamma(G) \leq \gamma_{cs}(G)$, [1].

Thus, we have, $\gamma(\mu(G)) \leq \gamma_{cs}(\mu(G)) \Rightarrow \gamma(\mu(G)) \leq 2$.

Case(I) Let $\gamma(\mu(G)) = 1$

Clearly, a single vertex cannot dominate all the vertices of $\mu(G)$. Hence this case is not possible.

Case(II) Let $\gamma(\mu(G)) = 2$

II(i) Consider $S = \{u_i, u_j\}$ in $\mu(G)$

S will not dominate the vertex w and hence $\gamma(\mu(G)) > 2$.

II(ii) Consider $S = \{u_{i'}, w\}$ in $\mu(G)$

S cannot dominate the vertex u_i and hence S is not a dominating set of $\mu(G)$.

II(iii) Consider $S = \{u_i, u_{j'}\}$ in $\mu(G)$

- **When** $i \neq j$

The vertex $u_{i'}$ is not dominated by the set S. Hence S is not a dominating set of $\mu(G)$ and not a CSDS set of $\mu(G)$.

- **When** $i = j$

(a) $deg(u_i) \neq (n-1)$

The set S cannot dominates all the vertices of $\mu(G)$. Hence S is not a dominating set and not a CSDS set of $\mu(G)$.

(b) $deg(u_i) = (n-1)$

Clearly S is a dominating set of $\mu(G)$. But S is not a CSDS of $\mu(G)$. For, consider a vertex $u_j \in N(u_i)$, even if $deg(u_j) = (n-1)$, this vertex cannot replace $u_i \in S$, as the vertex $u_{j'}$ is not dominated by $(S - \{u_i\}) \cup \{u_j\}$. Hence S is not a CSDS of $\mu(G)$.

II(iv) Consider $S = \{u_i, w\}$ in $\mu(G)$

(a) $deg(u_i) \neq (n-1)$

The set S cannot dominates all the vertices of $\mu(G)$. Hence S is not a dominating set of $\mu(G)$.

(b) $deg(u_i) = (n-1)$

Since u_i is adjacent to all the vertices u_x, where $x \in \{1, 2, 3, ..., n\}$ of $\mu(G)$ and w dominates all the vertices $u_{x'}$, S is a dominating set of $\mu(G)$. Now, consider the co-security condition of the set S. Any vertex $u_j \in N(u_i)$, where $u_j \in V(\mu(G)) - S$ can replace u_i in S, only if $deg(u_j) = (n-1)$. Thus the set S will be a CSDS only when at least two vertices in G have degree $(n-1)$. Hence the proof. □

Theorem 2.2. *Let G be a connected graph of order n. Then $\gamma_{cs}(\mu(G)) = 3$ if and only if*
i) G has exactly one vertex of degree $(n-1)$

or

ii) $\gamma(G) = \gamma_{cs}(G) = 2$, where the vertices in the co-secured sets are adjacent.

Proof. **Case (i):** Assume that G has exactly one vertex u_i of degree $n-1$).

Consider S $= \{u_i, u_{i'}, w\}$ in $\mu(G)$. Clearly, S dominates all the vertices of $\mu(G)$. We have to prove the co-security condition of S. For, consider the vertex $u_i \in S$, there exists a $u_j \in V(\mu(G)) - S$ such that $u_j \in N(u_i)$ and set $(S - \{u_i\}) \cup \{u_j\}$ is a dominating set of $\mu(G)$. Since $u_{i'}$ dominates all the vertices u_x and w, except u_i, where $x \in \{1, 2, 3, ..., n\}$ and the vertex u_i is dominated by the vertex $u_j \in S$. Now, consider a vertex $u_k \in V(\mu(G)) - S$, where $u_k \in N(u_{i'})$. This vertex defends $u_{i'} \in S$, since the set $(S - \{u_{i'}\}) \cup \{u_k\}$ is a dominating set of $\mu(G)$. The vertex u_i dominates all the vertices $u_x, u_{x'}$ except $u_{i'}$ in $\mu(G)$ and u_k dominates $u_{i'}$ and the vertex $w \in S$, where $x \in \{1, 2, 3, ., i-1, i+1, ..., n\}$. Consider the vertex $w \in S$, there exists $u_{j'} \in V(\mu(G)) - S$ such that $u_{j'} \in N(w)$ and $(S - \{w\}) \cup \{u_{j'}\}$ is a dominating set of $\mu(G)$. Hence, S is a minimum co-secure dominating set of $\mu(G)$ and $\gamma_{cs}(\mu(G)) = 3$.

Case(ii): Assume that $\gamma(G) = \gamma_{cs}(G) = 2$, where vertices in the dominating set are adjacent.

Let $\{u_i, u_j\}$ be a dominating set of G. Since $\gamma_{cs}(G)= 2$, there exists vertices u_x and u_y in $V(G) - S$ such that $(S - \{u_i\}) \cup \{u_x\}$ and $(S - \{u_j\}) \cup \{u_y\}$ are dominating sets of G, where $u_x \in N(u_i)$ and $u_y \in N(u_j)$. Consider the set $S = \{u_i, u_j, w\}$ in $\mu(G)$. Clearly S dominates all the vertices of $\mu(G)$. The vertices u_i and u_j in S can be replaced by the vertices u_x, u_y respectively in $V(\mu(G)) - S$. Consider the vertex w, it is replaced by a vertex $u_{i'} \in V(\mu(G)) - S$ and the set $(S - \{w\}) \cup \{u_{i'}\}$ is a dominating set of $\mu(G)$, where $u_{i'} \in N(w)$. Hence $\gamma_{cs}(\mu(G)) = 3$.

Conversely assume that $\gamma_{cs}(\mu(G)) = 3$. We have $\gamma(\mu(G)) \leq \gamma_{cs}(\mu(G))$. When $\gamma(\mu(G)) \leq 3$, we consider the following cases.

Case (i) Consider $\gamma(\mu(G)) = 1$.

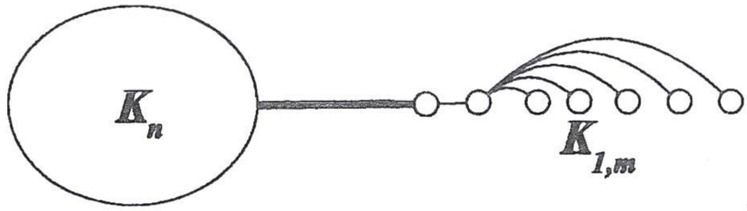
Clearly, a single vertex cannot dominate the graph $\mu(G)$. Hence this case is not possible.

Case(ii) Consider $\gamma(\mu(G)) = 2$.

ii(a) Let $S = \{u_i, u_j\}$ in $\mu(G)$.

Clearly S cannot dominate all the vertices of $\mu(G)$ and hence not a dominating set of $\mu(G)$.

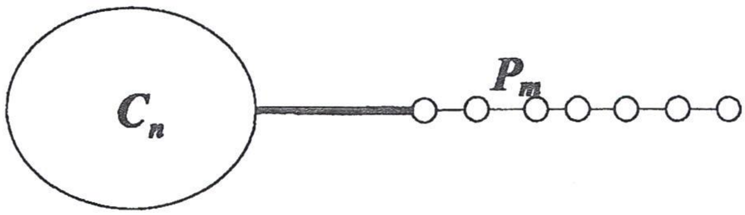
ii(b) Let $S = \{u_i, u_i\}$ in $\mu(G)$.



Here, $\gamma(G) = 2$ and $\gamma_{cs}(G) > 2$
$\gamma_{cs}(\mu(G)) = 2\ \gamma(G) + 1 = 5$.

Figure 1:

- If a cycle C_n is connected to a path P_m by an edge, where m,n ≥ 3, as in Figure 2, $\gamma_{cs}(\mu(G)) = \gamma_{cs}(G) + 1$.



Here, $\gamma_{cs}(G) = \gamma_{cs}(C_n) + \gamma_{cs}(P_m)$.
$\gamma_{cs}(\mu(G)) = \gamma_{cs}(G) + 1$.

Figure 2:

Conclusion

In this paper we have characterised Mycielski graphs for co-secure domination number 2 and 3. A sharp upper bound is obtained for the co-secure domination number of Mycielski graphs. The result can be used to obtain the co-secure domination number of generalised Mycielski graphs.

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