Computing cardinalities of subsets of S_n with k adjacencies

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Abstract

Given a permutation $\pi = (\pi_1, \pi_2, \pi_3, \ldots, \pi_n)$ over the alphabet $\Sigma = \{0, 1, \ldots, n-1\}$, π_i and π_{i+1} are said to form an *adjacency* if $\pi_{i+1} = \pi_i + 1$ where $1 \leq i \leq n-1$. The set of permutations over Σ is a symmetric group denoted by S_n. $S_n(k)$ denotes the subset of permutations with exactly k adjacencies. We study four adjacency types and efficiently compute the cardinalities of $S_n(k)$. That is, we compute for all k $\mid S_n(k) \mid$ for each type of adjacency in $O(n^2)$ time. We define reduction and show that $S_n(n-k)$ is a multiset consisting exclusively of $\mu \in Z^+$ copies of $S_k(0)$ where μ depends on n, k and the type of adjacency. We derive an expression for μ for all types of adjacency.
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1 Introduction

A permutation, on an alphabet Σ, is a sequence where every object in Σ occurs precisely once. The set of permutations with n symbols is denoted by S_n. Given a permutation π in R_n where π= (π_1, π_2, π_3,....,π_n) over the alphabet $\Sigma = \{0, 1, ..., n-1\}$, π_i and π_{i+1} form an *adjacency* if $\pi_{i+1} = \pi_i + 1$; we call this as a *regular* or Type 1 *adjacency* [4]. In contrast, π_i and π_j form an *inversion* if $\pi_i > \pi_j$ and $i < j$. Given π in S_n, (π_i, π_{i+1}, π_{i+2},....,π_j) is a *sublist* of π where $1 \leq i < j \leq n$. $I_n = (0, 1, 2, ..., n-1)$ is the *identity* or *sorted* permutation of S_n; it has exactly $n-1$ Type 1 adjacencies. Likewise, the reverse order permutation denoted by R_n is $(n-1, n-2, n-3, n-4, ..., 0)$; it has no adjacencies.

A natural way of sorting a permutation by comparisons is to increase the number of adjacencies and reduce the number of inversions. Sorting permutations with various operations has applications in genetics and computer network architectures. In genetics, a genome is modelled by a permutation and a mutation is modelled by the corresponding operation, say a transposition. Given a permutation π, a transposition $\delta(i, j, k)$ moves the sublist $(\pi_i, ..., \pi_{j-1})$ to the position just after π_k. A single application of an operation that corresponds to a particular *generator* is called as a *move*. For transposition, a generator is specified by a valid triplet (i, j, k) and all such triplets form a generator set of $O(n^3)$ size.

The minimum number of moves that are required to sort a given permutation π is called its distance. For example, the minimum number of transpositions that are required to sort a given permutation π is called its transposition distance, $d_t(\pi)$. The problem of determining $d_t(\pi)$ is known to be computationally intractable [3]. Prefix transposition and suffix transposition are restricted versions of transposition where a moved sublist is either a prefix or a suffix respectively. Transposition, prefix transposition, and sufix transposition are called as *block-moves* [1].

$S_n(k)$ denotes the subset of S_n with exactly k adjacencies. A maximal sequence of zero or more adjacencies between consecutive symbols of a permutation is called as a *block*. If $B = (\pi_i, \pi_{i+1} ..., \pi_j)$ is a block then every pair of consecutive symbols form an adjacency and π_i and π_j do not form adjacencies with π_{i-1} and π_{j+1} respectively. If $\pi = (4, 5, 6, 0, 1, 2, 3)$ then π has two blocks (4,5,6) and (0,1,2,3) of sizes three and four respectively. Further, (4,5,6) is the leftmost or *leading* block and (0,1,2,3) is the rightmost or *trailing* block.

A permutation with no adjacencies is *irreducibe* or *reduced*. Otherwise, it is *reducibe*. A permutation $\pi \in S_n(k)$ *reduces* to $\sigma \in S_{n-k}(0)$ if σ is obtained by eliminating all adjacencies in π. For example, (3, 4, 1, 2, 0) in S_5 reduces to an *irreducible* permutation (2, 1, 0) in S_3. Let B^i denote i^{th} block from the left out of total p blocks of π and let $B^i = (B^i_1, B^i_2, \ldots, B^i_{i_{last}})$. The rank of B^i is the rank of B^i_1 when $(B^1_1, B^2_1, B^3_1, \ldots, B^p_1)$ are considered in increasing order. When the reduction process replaces a block B with its first symbol f then we say that B is *collapsed* into f. In the following Lemmas we show that reduction yields a unique permutation and then we specify the resultant permutation. The following procedure reduces π into a unique irreducible permutation. Chitturi and Das [4] design a linear time algorithm to realize the same.

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procedure REDUCE
    for all i do
        replace B^i with B^i_1.
        from every remaining symbol greater than B^i_1 subtract i_last − 1.
    end for
end procedure
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Lemma 1 Procedure REDUCE yields a unique permutation.

Proof: Let B^i be the i^{th} block that will be processed in iteration i $(1, 2, \ldots)$. Let n_{i-1} be the number of symbols in the resultant permutation at the end of iteration $i-1$. We prove the Lemma by showing that the following loop invariant P holds after every iteration. P: Iteration i yields a unique permutation with alphabet $(0, 1, 2, \ldots, n_{i-1} - (|B^i| - 1))$ where all adjacencies

in B^i are eliminated.
The default values at the beginning of the first iteration are: the size of imaginary block $|B^0| = 1$ and $n_0 = n - 1$ so that the alphabet is $(0, 1, 2, \ldots, n-1)$. P trivially holds at the beginning of the first iteration. Clearly, all adjacencies in B^i are eliminated in iteration i because B^i collapses into B^i_1. Consider the first iteration. We have following two cases. Case(i): $B^i_{i_{last}} = n - 1$ and Case(ii): $B^i_{i_{last}} < n - 1$. Recall that the alphabet is $(0, 1, 2, 3, \ldots, n-1)$.
Case(i): $1_{last} - 1$ symbols with largest values are removed from their respective positions of the current permutation and the symbols to their right are moved to the left by $1_{last} - 1$ positions. This clearly results in another unique permutation over $(0, 1, 2, \ldots, n - (|B^1| - 1))$.
Case(ii): $1_{last} - 1$ symbols are removed from the current permutation from their respective positions and the symbols to their right are moved to the left by $1_{last} - 1$ positions. Clearly this yields a permutation of $(0, 1, 2, 3, \ldots, B^i_1, B^i_1 + 1_{last}, B^i_1 + 1_{last} + 1, \ldots, n-1)$. However, all the symbols with value greater than B^i_1 are decremented by $1_{last} - 1$. Thus, we obtain a unique permutation over $(0, 1, 2, \ldots, n - (|B^1| - 1))$.
This argument can be extended to the subsequent iterations. Thus, after the last iteration we obtain a unique permutation in $S_{n-k}(0)$ where k is the total number of adjacencies in π.■

In the above procedure we considered blocks from the left end. However, the result holds even if we collapse the blocks in an arbitrary order.

Lemma 2 Let π be a member of S_n constituted by exactly k blocks and let σ be its reduced form. The following assertions hold: (a) $\sigma \in S_k(0)$ and (b) a block with rank i in π reduces to i in σ.

Proof: Let the blocks be $B_1, B_2, \ldots, B_k$ with corresponding first symbols $f_1, f_2, \ldots, f_k$ where $f_1 < f_2 <, \ldots, < f_{k-1} < f_k$. Because π is a permutation and from our definition of a block it follows that $f_i = 1 + \Sigma_{j=1 \ldots i-1}|B_j|$. Collapsing any B_j where $j > i$ does not alter B_i. When all blocks B_j where $j < i$ are collapsed, each block reduces in size by $|B_j| - 1$ and all symbols in B_i are decremented by $|B_j| - 1$. So, the total decrement for f_i is $\Sigma_{j=1 \ldots i-1}(|B_j| - 1)$. Thus, in σ, $f_i = 1 + \Sigma_{j=1 \ldots i-1}|B_j| - \Sigma_{j=1 \ldots i-1}(|B_j| - 1) = 1 + (i - 1) = i$. However, when B_i is collapsed then corresponding to B_i only $f_i = i$ remains.■

We extend Type 1 adjacency to yield three variations: Types 2 through 4. Type 2 has an adjacency in addition to Type 1 where if $\pi_n = n - 1$ then it forms a *b-adjacency* with (an imagined) $\pi_{n+1} = n$. Type 3 has an adjacency in addition to Type 1 where if $\pi_1 = 0$ then it forms an *f-adjacency* with (imagined) $\pi_0 = -1$. Type 4 is the union of Type 1 adjacency, b-adjacency and f-adjacency. Christie [2] showed that if π is reduced to δ then (i $d_t(\pi) = d_t(\delta)$ and (ii) an optimum sorting sequence exists that does no break any existing adjacencies in π. This result applies to all block-move and yields a correspondence between adjacency types and types of block moves. For example, Type 2 adjacency is directly applicable to a prefi transposition where a moved sublist is always a prefix. When we sort a permutation with prefix transpositions then we need not move: (i) $n-1$ i it is already in the last position and (ii) a trailing block with highest ranl if it exists [10]. Thus, the size of the permutation that we need to sor effectively reduces. In fact, the permutation that we need to sort is shorte than the given permutation by the number of adjacencies. Likewise, Typ 3 adjacency is directly applicable to a suffix transposition where a movec sublist is always a suffix. Type 4 adjacency is directly applicable to a transposition where any sublist can be moved. Here, if 0 is at the first index o a leading block with lowest rank exists; or $n-1$ is at the final index or a trailing block of highest rank exists then they need not be moved again. In fact, the permutation can be reduced and one can sort the resultant permutation. This concept is employed to design a more efficient algorithm to sort permutations with transpositions [4]. Sorting permutations with an operation O having a generator set G has applications in computation of genetic dissimilarity (under O) and latency in computer interconnection networks. Cayley graph with $n!$ vertices and $O(n!|G|)$ edges models the interconnection network corresponding to O [4, 10].

The integer sequences that we generated in this article were also generated by other equations in other contexts. Tanny [8] calls Type 1 adjacency a *succession* and Roselle [5] states that a *rise* in a permutation exists at a position i if $\pi_i < \pi_{i+1}$. Roselle determined the cardinality of $S(n,r,s)$, the number of permutations in S_n with r rises and s successions: $S(n,r,s) = \binom{n-1}{s} S(n-s, r-s, 0)$. Tanny studied the cardinalities of the sets of permutations with n symbols and k successions. He gave the expression for $f(n,k)$ as follows where D_i is a derangement number for size i. $f(n,k) = \binom{n-1}{k}(D_{n-k} + D_{n-1-k})$. Tanny also studied circular successions where π_i and $\pi_{(i+1)}$ form an adjacency if $\pi_{i+1} = (1+\pi_i)$ where $i+1$ is computed mod n. Tanny showed that $lim_{n\to\infty}(Q^*(n,k)/n!) = e^{-1}/k!$ where $Q^*(n,k)$ denotes the number of permutations with k circular successions. The cardinalities of $S_0(n)$ and $S_1(n)$ etc. for Type 1 adjacencies occur in OEIS [6] with sequence numbers A000255, A000166 etc.. The cardinalities of $S_0(n), S_1(n), S_2(n), \ldots$ of Type 2 adjacency (applicable to Type 3 adjacency also) occur in OEIS with sequence numbers A000166 denoting subfactorial or rencontres numbers; derangements: number of permutations of n symbols with no fixed points; A000240: rencontres numbers: number of permutations of S_n with exactly one fixed point etc.. The cardinalities of $S_0(n)$, $S_1(n)$ and $S_2(n)$ of Type 4 adjacency occur in OEIS with the following sequence numbers. A000757: Number of cyclic permutations of n symbols with no $[i]$ immediately followed by $[i+1]$ where $[i]$ denotes i mod n; A135799: second column ($k = 1$) of triangle A134832 (circular succession numbers); A134515: third column ($k = 1$) of triangle A134832, etc..

We construct recurrence relations for the cardinality of $S_n(k)$, that is $f(n,k)$ for all types of adjacencies. We show that $f(n,k)$ is exclusively determined by $f(n-2,x)$ and $f(n-1,y)$ for appropriate values of x and y. Thereby, we show that $f(n,k)$ for all k can be computed in $O(n^2)$ time. Furthermore, we show that there are integral copies of $S_k(0)$ in $S_n(n-k)$.

2 Type 1 adjacencies

Recall that π_i and π_{i+1} form an adjacency if $\pi_{i+1} = \pi_i + 1$. The following Theorem establishes a recurrence relation to compute $|S_n(\text{k})|$ for the Type 1 adjacencies.

Theorem 3 Let $f(n,k)$ be the cardinality of $S_n(k)$. Then $f(n,k) = f(n-1,k-1) + (n-1-k) * f(n-1,k) + (k+1) * f(n-1,k+1)$ where $0 \leq k < n$.

Proof: We denote the number of adjacencies in a permutation π in S_n with $\alpha(\pi)$ and the number of permutations in S_n having $\alpha(\pi)$ adjacencies with $f(n, \alpha(\pi))$. Recall that the alphabet of S_{n-1} is $\{0, 1, 2, 3, \ldots, n-2\}$ whereas the alphabet of S_n has $n-1$ in addition. Let $\alpha(\pi^*) = q$ for π^* in S_{n-1}. When a $\pi \in S_n$ is formed from π^* by inserting $n-1$ we have the following three cases: (i) $\alpha(\pi) = \alpha(\pi^*)$, (ii) $\alpha(\pi) = \alpha(\pi^*) + 1$ and (iii) $\alpha(\pi) = \alpha(\pi^*) - 1$.

Case (i): If $n-1$ neither succeeds $n-2$ nor is inserted between $a, a+1$ for some a then $\alpha(\pi) = \alpha(\pi^*)$.

Case (ii): The symbol $n-1$ can create an adjacency only if it immediately succeeds $n-2$. However, in such a case it cannot destroy an existing adjacency; $\alpha(\pi) = \alpha(\pi^*) + 1$. Thus, it is not possible to insert $n-1$ in any position where it simultaneously creates one adjacency and destroys one adjacency.

Case (iii): If $n-1$ is inserted between $a, a+1$ for some a then $\alpha(\pi) = \alpha(\pi^*) - 1$. We determine the number of permutations π of S_n with $\alpha(\pi) = k$ that can be generated from some permutations in S_{n-1} corresponding to each of these cases.

In order to generate $\pi \in S_n$ from $\pi^* \in S_{n-1}$ one can insert $n-1$ in any of the n positions corresponding to $n-2$ internal and two external positions. Let $\alpha(\pi^*) = k$. We want to determine for a given π^* how many

$\pi \in S_n$ exist such that $\alpha(\pi^*) = \alpha(\pi)$. However, in order to maintai the same number of adjacencies $k+1$ positions are forbidden; k position corresponding to existing adjacencies and one corresponds to b-adjacency that is, placing $n-1$ in the last position. Thus, the contribution of $S_{n-1}(k$ to $f(n,k)$ is $(n-1-k)*f(n-1,k)$. Let $\alpha(\pi^*) = k-1$. In order to create $\pi \in S_n$ from π^* where $\alpha(\pi) = k$ the only possibility is that $n-1$ is inserte to the immediate right of $n-2$. Thus, contribution of $S_{n-1}(k-1)$ t $f(n,k)$ is exactly $f(n-1,k-1)$. Finally, we determine the contributio of $S_{n-1}(k+1)$ to $f(n,k)$. Here, any one of the $k+1$ adjacencies can b broken by inserting $n-1$ in between. Thus, the contribution of $S_{n-1}(k+1$ to $f(n,k)$ is $(k+1)*f(n-1,k+1)$. Note that $f(n,k)$ is restricted to th above cases. The Theorem follows.■

3 Adjacency Variations

3.1 Type 2 adjacency

Type 2 adjacency has b-adjacency in addition to the adjacencies of Type 1. If $\pi_n = n-1$ then π_n and (imagined) $\pi_{n+1} = n$ form an adjacency. I_n has n adjacencies and R_n has zero adjacencies. If $\pi = (4\ 6\ 3\ 5\ 0\ 2\ 1\ 7)$ then $(4\ 6\ 3\ 5\ 0\ 2\ 1)$ is the reduced form of it where π_n is deleted because $\pi_n = n-1$. Type 2 and Type 3 are symmetric.

Theorem 4 Let $f(n,k)$ denote the number of permutations in S_n with exactly k adjacencies. Then the recurrence relation for $f(n,k)$ is:
$f(n,k) = (f(n-1,k-1) - f(n-2,k-2)) * 2 + f(n-2,k-2) +$
$(f(n-1,k+1) - f(n-2,k)) * (k+1) + f(n-2,k) * (n-k-1) +$
$(f(n-1,k) - f(n-2,k-1)) * (n-k-2) + f(n-2,k+1) * (k+1); \quad 0 \leq k \leq n.$

Proofs for Theorems 4, and 6 are identical. Please refer to the proof of Theorem 6.

3.2 Type 3 adjacency

In Type 3 adjacency if $\pi_1 = 0$ then (imagined) $\pi_0 = -1$ and π_1 form an adjacency. I_n has n adjacencies and R_n has none. Type 2 and Type 3 adjacencies are symmetrical. The recurrences governing $|S_k(n)|$ for all k and their base values are identical for Type 2 and Type 3 adjacencies.

3.3 Type 4 adjacency

Type 4 adjacency has a b-adjacency and an f-adjacency in addition to Type 1 adjacency. I_n and R_n have the maximum and the minimum number of adjacencies; $n+1$ and zero respectively. If $\pi = (0\ 4\ 6\ 3\ 5\ 2\ 1\ 7)$ then $(4\ 6\ 3$

5 2 1) is the reduced form of it where π_n is deleted because $\pi_n = n-1$ and π_1 is deleted because $\pi_1 = 0$. Theorem 6 establishes a recurrence relation to compute $| S_k(n) |$. First, we prove the following lemma.

Lemma 5 The number of permutations in S_{n-1} with $k+1$ Type 2 or Type 4 adjacencies that do not end with $n-2$ is $f(n-1, k+1) - f(n-2, k)$.

Proof: Let $n-2$ be in the last position then it forms a b-adjacency. Thus, the remaining $n-2$ symbols that belong to S_{n-2} must form k adjacencies. Among these symbols if $n-3$ is placed in position $n-2$ it is considered as a b-adjacency (however, it forms a regular adjacency with $n-2$ that is already in position $n-1$). That is, we are looking at all ways of obtaining k adjacencies with $n-2$ symbols. The corresponding count is $f(n-2, k)$. So, the number of permutations with $k+1$ adjacencies that do not end with $n-2$ is $f(n-1, k+1) - f(n-2, k)$.

Theorem 6 Let $f(n, k)$ denote the number of permutations in S_n with exactly k adjacencies. Then the recurrence relation for $f(n, k)$ is:
$f(n, k) = (f(n-1, k) - f(n-2, k-1)) * (n-k-2) +$
$(f(n-1, k-1) - f(n-2, k-2)) * 2 +$
$f(n-2, k-2) + (f(n-1, k+1) - f(n-2, k)) * (k+1) + f(n-2, k) * (n-k-1) +$
$f(n-2, k+1) * (k+1); \quad 0 \leq k \leq n+1.$

Proof: If $\pi_n = n-1$ then π_n and imaginary π_{n+1} form an adjacency. Likewise, if $\pi_1 = 0$ then imaginary $\pi_0 = -1$ and π_1 form an adjacency. The recurrences are formed by studying the composition of $\pi \in S_n$ from $\pi^* \in S_{n-1}$ by inserting $n-1$. Such composition can be partitioned into the following cases where $\pi \in S_n(k)$ can be obtained only from permutations in $S_{n-1}(x)$ where $x \in \{k-1, k, k+1, k+2\}$. First we list the cases and then we show the respective contribution of each case.
Case(i) $x = k$, that is $\alpha(\pi) = \alpha(\pi^*)$. The insertion of $n-1$ does not alter the number of adjacencies.
Case(ii) $x = k-1$, that is $\alpha(\pi) = \alpha(\pi^*) + 1$. $n-1$ immediately succeeds $n-2$, $\pi_n = n-1$ or both. The last scenario occurs when $\pi^*_{n-1} = n-2$. Then if $\pi_n = n-1$ then the b-adjacency of $n-2$ is broken. However, a regular adjacency (between $n-2$ and $n-1$) and one b-adjacency ($\pi_n = n-1$) are created.
Case(iii) $x = k+1$, that is $\alpha(\pi) = \alpha(\pi^*) - 1$. If $\pi^*_{n-1} \neq n-2$ and $n-1$ is inserted between $x, x+1$ for some x then $\alpha(\pi) = \alpha(\pi^*) - 1$. Also, if $\pi^*_{n-1} = n-2$ then $n-1$ can be inserted in a position where it neither breaks or creates an adjacency. That is, position n where it creates an adjacency (Case (ii) covers this scenario) and $k+1$ positions where adjacencies exist are forbidden.
Case(iv) $x = k+2$, that is $\alpha(\pi) = \alpha(\pi^*) - 2$. Consider π^* where

$\pi^*_{n-1} = n-2$; here π^*_{n-1} and (imagined) π^*_n form an adjacency; thus, if $n-1$ is inserted into π^* in a position other than n then $\alpha(\pi) = \alpha(\pi^*) - 1$ further if $n-1$ breaks an existing adjacency in π^* then $\alpha(\pi) = \alpha(\pi^*) - 2$.

Contribution of Case(i): $x = k$. The number of permutations in S_n that π^* generates such that $\alpha(\pi^*) = \alpha(\pi)$ is to be determined. To generate $\pi \in S_n$ from $\pi^* \in S_{n-1}$ one can insert $n-1$ in any of the n positions ($n-2$ internal positions and the two extremes).

Case(i-a): $\pi^*_{n-1} = n-2$. If $n-1$ is inserted in the last position then the number of adjacencies increases by one. That is, $n-1$ forms two new adjacencies by being at the last position and by succeeding $n-2$. However, the existing adjacency of $n-2$ by being at the last position of its respective permutation is broken yielding $\alpha(\pi) = k+1$.

If $\pi_n \neq n-1$ then the existing adjacency of the last symbol of π^*_{n-1} i.e. $n-2$ is automatically broken because after inserting $n-1$, $n-2$ is not the largest symbol. If $n-1$ does not break an existing adjacency then one obtains $\alpha(\pi) = k-1$. Further, if $n-1$ breaks an existing adjacency of Type 1 then $\alpha(\pi) = k-2$. Thus, for this sub-case $\alpha(\pi) \in \{k+1, k-1, k-2\}$. That is, if $x = k$ and $\pi^*_{n-1} = n-2$ then no permutations in $S_n(k)$ can be generated.

Case(i-b): $\pi^*_{n-1} \neq n-2$. If $n-1$ is inserted in a position where it does not create or break an adjacency then $\alpha(\pi^*) = \alpha(\pi)$. Out of n positions $n-k-2$ positions exist where $n-1$ can be inserted. k of the excluded positions correspond to existing adjacencies in π^* that must not be broken. The remaining two excluded positions correspond to n and the position succeeding $n-2$ where the insertion would create a new adjacency.

Per Lemma 5, $f(n-1,k) - f(n-2,k-1)$ denotes the number of permutations where $\pi^*_{n-1} \neq n-2$ and $\alpha(\pi^*) = k$. Thus, the contribution of $S_{n-1}(k)$ to $f(n,k)$ is $(f(n-1,k) - f(n-2,k-1)) * (n-k-2)$.

Contribution of Case(ii): $x = k-1$. If $\pi^*_{n-1} = n-2$ then $\pi_n = n-1$; the corresponding contribution is $f(n-2,k-2)$. If $\pi^*_{n-1} \neq n-2$ then $n-1$ can be inserted at the last position or immediately after $n-2$ with a contribution of $(f(n-1,k-1) - f(n-2,k-2)) * 2$. Thus, the contribution of $S_{n-1}(k-1)$ to $f(n,k)$ is $(f(n-1,k-1) - f(n-2,k-2)) * 2 + f(n-2,k-2)$.

Contribution of Case(iii): $x = k+1$. If $\pi^*_{n-1} \neq n-2$ then any of the existing $k+1$ adjacencies can be broken. Otherwise, $\pi_n \neq n-1$ and $n-1$ does not break any of the existing k adjacencies; hence the last position and k other positions, i.e. a total of $k+1$ positions are forbidden. Recall that $(f(n-1,k+1) - f(n-2,k))$ is the number of permutations in S_{n-1} with $k+1$ adjacencies that do not end with $n-2$ (Lemma 5). Thus, the contribution of $S_{n-1}(k+1)$ to $f(n,k)$ is $(f(n-1,k+1) - f(n-2,k)) * (k+1) +$

$f(n-2,k)*(n-k-1)$.

Contribution of Case(iv): $x = k+2$. Two of the existing adjacencies must be broken. Note that by inserting $n-1$ between a and $a+1$ for some a only one adjacency can be broken. Thus, the only feasibility is that $\pi^*_{n-1} = n-2$ and $n-1$ breaks one of the existing Type 1 adjacencies. That is, by not being in the last position of π, $n-1$ breaks the existing b-adjacency of $n-2$ in π^*. Additionally, it breaks a Type 1 adjacency. Thus, the contribution of $S_{n-1}(k+2)$ to $f(n,k)$ is $f(n-2,k+1)*(k+1)$. The Theorem follows. ■

Consider Type 2 and Type 4 adjacencies. When a permutation π starts with a 0 then the number of Type 4 adjacencies of π is one greater than that of Type 2 adjacencies. In general, the cardinalities of $S_n(k)$ differ for Type 2 and Type 4 adjacencies. However, the cases shown in the proof of Theorem 6 are identical to the cases one obtains for Theorem 4. Thus, the recurrence relations are identical. By symmetry, Type 2 and Type 3 have the same recurrence relation. Therefore, the proof of Theorem 6 suffices for Type 2, Type 3 and Type 4.

Theorem 7 (a) There are integral copies of $S_k(0)$ in $S_n(n-k)$. Let the multiplicative factor be $\mu \in Z^+$. (b) For Type 1, $\mu = \binom{n-1}{k-1}$; for Type 2, $\mu = \Sigma_{i=0}^{n-k}\binom{n-i-1}{k-1}$; for Type 3, $\mu = \Sigma_{i=0}^{n-k}\binom{n-i-1}{k-1}$ and for Type 4, $\mu = \binom{n-1}{k-1} + 2(\Sigma_{i=1}^{n-k}\binom{n-i-1}{k-1}) + \Sigma_{i=1}^{n-k-1}\Sigma_{j=1}^{n-k-i}\binom{n-i-j-1}{k-1}$.

Proof: When permutations are denoted by their reduced forms, for any type of adjacency, we seek to show that $S_n(n-k)$ is a multiset composed exclusively of some $\mu \in Z^+$ copies of $S_k(0)$. π in $S_n(n-k)$ yields a particular γ of $S_k(0)$ upon reduction (Lemma 1). Let $\oplus$ denote concatenation operation where $(\pi_i, \pi_{i+1}, \dots, \pi_j) \oplus (\pi_{j+1}, \pi_{j+2}, \dots, \pi_k) = (\pi_i, \pi_{i+1}, \dots, \pi_j, \pi_{j+1}, \dots, \pi_k)$. $(A^1, A^2, A^3, \dots, A^k)$ is a k-cut of I_n if $A^1 \oplus A^2 \oplus A^3 \oplus, \dots, \oplus A^k = I_n$. Let $rank(\gamma_i)$ denote the *rank* of γ_i in γ and the rank of A^i be the rank of A^i_1 in $\{A^1_1, A^2_1, \dots, A^k_1\}$. Let $A = (A^1, A^2, A^3, \dots, A^k)$ be a k-cut of I_n. Consider a permutation of A^* of A where $A^* = (A^{i_1}, A^{i_2}, A^{i_3}, \dots, A^{i_k})$ and rank of A^{i_j} equals $rank(\gamma_j)$. Due to Lemmas 1 and 2 we have the following result.

The permutation $A^* = A^{i_1} \oplus A^{i_2} \oplus A^{i_3} \oplus, \dots, \oplus A^{i_k}$ reduces to γ. (I)

We call the leftmost and the rightmost blocks as the *leading* and *trailing* blocks respectively. For each type of adjacency we consider an arbitrary permutation γ in $S_k(0)$ and show that a particular number of permutations from $S_n(n-k)$ reduce to it. Because this applies to every permutation in $S_k(0)$ it follows that there are integral copies of $S_k(0)$ in $S_n(n-k)$ proving part (a) of Theorem.

Type 1 adjacency We seek to count the number of permutations π in S_n that yield γ in $S_k(0)$ upon reduction. Due to (I), A^* is the only permutation of A (where A is a k-cut of I_n) that reduces to γ. So, we have the following result.
The number of valid k-cuts of I_n determine the number of permutations in S_n that yield γ. (II)

Any k-cut A is specified by assigning size to each of A^i (that is specifying $|A^i|$ for all i); the contents of A^i are automatically determined. For example, consider a 3-cut of I_5 where $|A^1| = 1$, $|A^2| = 2$ and $|A^3| = 2$. Clearly, the 3-cut is $((0), (1, 2), (3, 4))$. That is, any valid k-cut can be specified by assigning $|A^i|$ for all i such that $|A^i| > 0$ and $\Sigma|A^i| = n$. The solution to this problem is given by the number of solutions to the following integer solutions problem: $x_1 + x_2 +, \dots, x_k = n$ where for all i $x_i > 0$. The number of solutions is $\binom{n-1}{k-1}$. The next statement follows.
There are exactly $\binom{n-1}{k-1}$ copies of $S_k(0)$ in $S_n(n-k)$ for Type 1 adjacency. (III)

Type 2 adjacency includes b-adjacency. Thus, a trailing block with highest rank vanishes. That is $(2, 3, 1, 4, 5, 6)$ in its reduced form is $(2, 3, 1)$ where $(4, 5, 6)$ vanishes due to b-adjacency. Due to this property, for Type 2 adjacency, there are additional permutations in $S_n(n-k)$ that reduce to a particular permutation γ in $S_k(0)$ when compared to Type 1 adjacency. For example, let $(3, 2, 1)$ of $S_3(0)$ be obtained by reducing a permutation π in $S_4(1)$. For Type 1 adjacency, the possibilities for π are $(3, 4, 2, 1), (4, 2, 3, 1)$ and $(4, 3, 1, 2)$. However, for Type 2, π can also be $(3, 2, 1, 4)$. That is, a trailing block of highest rank can exist that will vanish upon reduction. Likewise, $(3, 2, 1, 4, 5)$ and $(3, 4, 2, 1, 5)$ of $S_5(2)$ also yield $(3, 2, 1)$ of $S_3(0)$ where the trailing blocks with highest rank are of sizes two and one respectively. The reduced permutation has k symbols. Thus, the size of the trailing block with highest rank is at most $n - k$. The statement below follows.
A given trailing block with highest rank yields additional permutations in $S_n(n-k)$ for Type 2 adjacency. (IV)

Similar to Type 2 adjacency, in Type 4 adjacency a trailing block with highest rank vanishes. Moreover, due to the presence of additional f-adjacency, a leading block with lowest rank also vanishes. Thus, the next observation follows.
Every combination of a leading block with lowest rank and a trailing block

with highest rank yields additional permutations in $S_n(n-k)$ for Type 4 adjacency. (V)

Type 2 adjacency From (IV) there are $\binom{n-1}{k-1}+b$ copies of $S_n(n-k)$ in S_n where b is the number of permutations due to a trailing block with highest rank. Clearly, this block can have sizes $1, 2, \ldots, n-k$. In fact, $\binom{n-1}{k-1}$ corresponds to the case when trailing block does not exist. Let the size of the this trailing block be i then the remaining $n-i$ symbols must form k blocks. The corresponding count due to (III) is $\binom{n-i-1}{k-1}$. The next result follows. Note that for Type 3 adjacency, leading block with lowest rank takes place of the trailing block with highest rank. Thus, due to symmetry, this result holds for Type 3 adjacency also.
The total count for Type 2 adjacency is $\Sigma_{i=0}^{n-k}\binom{n-i-1}{k-1}$. (VI)

Type 4 adjacency Similar to Type 2 adjacency, we have up to $n-k$ adjacencies that can exist in the trailing block with highest rank; however, we can also have up to $n-k$ adjacencies that can exist in the leading block with lowest rank (V). Thus, we have the following cases: (i) Neither a leading nor a trailing block that will vanish upon reduction exists. (ii) Trailing block that will vanish upon reduction exists. (iii) Leading block that will vanish upon reduction exists. (iv) Both a leading block and a trailing block that will vanish upon reduction exist. (III) yields the corresponding solution for Case(i). When the lower index starts from one instead of zero then (VI) yields the corresponding solution for Case(ii). Recall that $i=0$ in (VI) corresponds to a lack of either leading or a trailing block and this scenario is already covered in Case(i). Case(iii) and Case(ii) are symmetric.
Case(iv): First we observe that the lengths of leading and trailing blocks uniquely specify them. A leading block of length i is $(1, 2, 3, \ldots, i)$ and a trailing block of length i is $(n-i, n-i+1, \ldots, n-2, n-1)$. Thus, the lengths vary from 1 through $n-k-1$. Note that if the length equals $n-k$ then either the leading or the trailing block is present and these scenarios are covered by Cases (ii) and (iii). The subcases for the trailing block are analyzed here. When the trailing block has size i then the leading block has size j where $j \in \{1, 2, \ldots, n-k-i\}$. For each valid value of j, the remaining $n-i-j$ symbols that do not belong to either the leading or the trailing block form k Type 1 adjacencies. Thus, the total number of such permutations is given by $\Sigma_{i=1}^{n-k-1}\Sigma_{j=1}^{n-k-i}\binom{n-i-j-1}{k-1}$. Let the size of the leading block be i. Then the possible sizes of trailing block are $\{1, 2, \ldots, n-k-i\}$. However, when each of these sizes is considered for the trailing block then size i for the leading block is also considered. Thus, all cases are covered and the above equation yields the final count for Case(iv). Thus, for all cases the total count is $\binom{n-1}{k-1}+2(\Sigma_{i=1}^{n-k}\binom{n-i-1}{k-1})+\Sigma_{i=1}^{n-k-1}\Sigma_{j=1}^{n-k-i}\binom{n-i-j-1}{k-1}$. ■

Let R be the subset of S_n containing all reducible permutations of S_n in reduced form. Then $S_n(0) \bigcup R$ is called a *vector alphabet* of S_n and it is denoted by $\nu(S_n)$. Note that vector alphabet has permutations from $\{S_1, S_2, S_3, \ldots, S_n\}$. Let $n + \delta$ be the maximum possible adjacencies for a given Type (1, 2, 3, or 4) of adjacency. That is, $\delta = -1, 0, 0, 1$ respectively for adjacency Types 1 through 4. Let $(S_k(0))^{c_k}$ denote c_k copies of the set $S_k(0)$ where $c_k \in Z^+$. The corollaries given below follow. All permutations are presumed to be represented by their corresponding reduced permutations.

Corollary 7.1 $\nu(S_n) = \cup_{k=1}^{n+\delta} S_k(0)$ ♦

Corollary 7.2 $S_n = \cup_{k=1}^{n+\delta} (S_k(0))^{c_k}$ ♦

Corollary 7.3 For sufficiently large n, $|\nu(S_n)| \leq (1+\epsilon)|S_n(0)|$ for some $\epsilon \to 0^+$.

Proof We denote $|S_i(0)|$ by s_i. $|S_n(0)| \approxeq n!/e$ [5] [9].
$|\nu(S_n)| = \sum_{i=1}^{n} s_i$
$= s_n + s_{n-1} + s_{n-2} + \ldots + s_1 \approxeq$
$n!/e + (n-1)!/e + (n-2)!/e + (1)!/e =$
$n!/e(1 + 1/n + 1/(n(n-1)) + \ldots + 1/n!)$.
For sufficiently large n and $\omega \to 0^+$, the above expression is bounded above by $n!/e(1 + (1+\omega)/n) \leq (1+\epsilon)|S_n(0)|$. ϵ can be made arbitrarily small by choosing correspondingly large value for n.

Theorem 8 Cardinalities of $S_n(k)$ can be computed for all k in $O(n^2)$ time.

Proof: Consider Type 2 adjacency. For a particular i and j we have the following equation where $f(i,j)$ denotes the cardinality of $S_i(j)$. $f(i,j) =$
$(f(i-1,j-1) - f(i-2,j-2)) * 2 + f(i-2,j-2) +$
$(f(i-1,j+1) - f(i-2,j)) * (j+1) + f(i-2,j) * (i-j-1) +$
$(f(i-1,j) - f(i-2,j-1)) * (i-j-2) + f(i-2,j+1) * (j+1)$.

The computation of $f(i,j)$ is done with direct dynamic programming. Imagine a $n \times n$ table F where $F(i,j)$ is to hold the value of $f(i,j)$. We fill this matrix row wise starting from row 1. Clearly, $F(1,0)$, $F(1,1)$, $F(2,0)$, $F(2,1)$ and $F(2,2)$ can be easily computed by direct enumeration of permutations of sizes one and two. Consider the computation of $F(3,j)$ it depends only on $F(1,x)$ and $F(2,y)$ for some values of x and y. However, these values are already computed. Because there are fixed number of terms on the RHS of the equation, $F(3,j)$ can be computed in $O(1)$ time. This argument holds for any $F(i,j)$ for $i > 2$. Table F has n^2 entries. Thus, for all i and j, $F(i,j)$ can be computed in n^2 $O(1)$= $O(n^2)$ time. Recall that $F(n,k)$ equals cardinality of $S_n(k)$. So, the last row contains cardinalities of $S_n(k)$ for all k. This argument holds for all adjacency types.■

4 Conclusion

The cardinalities of $S_n(k)$ i.e. $\forall_k \mid S_n(k) \mid$ for a given type of adjacency are computed in $O(n^2)$ time. It is shown that an integral copies of $S_k(0)$ exist in $S_n(n-k)$ for all types of adjacency and the associated multiplicative factors are derived.

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