CONNECTIVITY OF SINGLE-ELEMENT COEXTENSIONS OF A BINARY MATROID

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Abstract. Given an n-connected binary matroid, we obtain a necessary and sufficient condition for its single-element coextensions to be n-connected.

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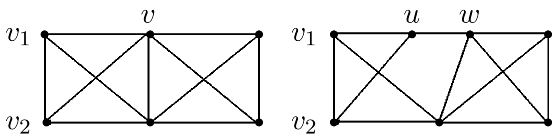
1. Introduction

For undefined terminologies, we refer to Oxley [6]. The point-splitting operation is a fundamental operation in respect of connectivity of graphs. It is used to characterize 3-connected graphs in the classical Tutte's Wheel Theorem [9] and also to characterize 4-connected graphs by Slater [8]. This operation is defined as follows.

Definition 1.1 ([8]). *Let G be a graph with a vertex v of degree at least $2n-2$ and let $T = \{vv_1, vv_2, \ldots, vv_{n-1}\}$ be a set of $n-1$ edges of G incident to v. Let G'_T be the graph obtained from G by replacing v by two adjacent vertices u and w such that u is adjacent to $v_1, v_2, \ldots, v_{n-1}$, and w is adjacent to the vertices which are adjacent to v except $v_1, v_2, \ldots, v_{n-1}$. We say G'_T arises from G by n-point splitting (see the following figure).*



G $\qquad$ G'_T

Slater [8] obtained the following result to characterize 4-connected graphs.

Theorem 1.2 ([8]). *Let G be an n-connected graph and let T be a set of $n-1$ edges incident to a vertex of degree at least $2n-2$. Then the graph G'_T is n-connected.*

In this paper, we extend the above theorem to binary matroids.

Azadi [1] extended the n-point splitting operation on graphs to binary matroids as follows.

Definition 1.3 ([1]). *Let M be a binary matroid with standard matrix representation A over the field $GF(2)$ and let T be a subset of the ground set $E(M)$ of M. Let A'_T be the matrix obtained from A by adjoining one extra row to matrix A whose entries are 1 in the columns labeled by the elements of T and 0 otherwise and also having one extra column labeled by a with 1 in the last row and 0 elsewhere. Denote the vector matroid of A'_T by M'_T. We say that M'_T is obtained from M by element splitting with respect to the set T.*

For example, the following matrices A and A'_T represent the Fano matroid F_7 and its element splitting matroid with respect to the set $T = \{1, 2, 3\} \subset E(F_7)$.

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}, \quad A'_T = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & a \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Given a graph H, let $M(H)$ denote the circuit matroid of H. A matroid N is a *single-element coextension* of a matroid M if $N/e = M$ for some element e of N.

Definition 1.3 is an extension of Definition 1.1 as $M(G)'_T = M(G'_T)$ for a set T of edges incident to a vertex of a graph G. Note that if M is a binary matroid, then the element splitting matroid M'_T is also binary and it is a coextension of M by the element a as $M'_T/a = M$. In fact, we prove in Lemma 2.1 that every coextension of a binary matroid M by a non-loop and non-coloop element is the element splitting matroid M'_T for some $T \subset E(M)$.

Dalvi et al. [4, 5] characterized the graphic (cographic) matroids M whose single-element coextensions M'_T are again graphic (cographic). Let M be an n-connected binary matroid. Borse and Mundhe [3] obtained sufficient conditions for the matroid $M'_T \backslash a$ to be n-connected. In this paper, we obtain a necessary and sufficient condition for M'_T to be n-connected. The following is the main theorem of the paper.

Main Theorem 1.4. *Let $n \geq 2$ be an integer and M be an n-connected binary matroid with $|E(M)| \geq 2n-2$. Suppose $T \subset E(M)$ with $|T| = n-1$. Then M'_T is n-connected if and only if $|Q| \geq 2|Q \cap T|$ for every cocircuit Q of M intersecting T.*

We also prove that Theorem 1.2 follows from Main Theorem 1.4 under a mild restriction.

Azadi [1] obtained the following result for M'_T to be n-connected, in terms of the circuits of M containing an odd number of elements of T.

Theorem 1.5 ([1])**.** *Let $n \geq 2$ be an integer and M be an n-connected binary matroid with $|E(M)| \geq 2n-2$. Suppose $T \subset E(M)$ with $|T| = n-1$. Then M'_T is n-connected if and only if for any set $A \subset E(M)$ with $|A| = n-2$, there exists a circuit C of M containing an odd number of elements of T and is contained in $E(M) - A$.*

We provide an alternate shorter proof of Theorem 1.5 in the third section.

In Section 2, we provide some properties of M'_T. Main Theorem 1.4 is proved in Section 3. In the last section, we discuss consequences of Main Theorem 1.4 to the graphs.

2. Preliminaries

We prove below that the single-element coextension of a binary matroid M by a non-loop and non-coloop element is nothing but an element splitting matroid M'_T for some $T \subset E(M)$.

Lemma 2.1. *Let M and N be binary matroids. Then N is a coextension of M by a non-loop and non-coloop element if and only if $N = M'_T$ for some $T \subset E(M)$.*

Proof. Suppose $N = M'_T$ for some $T \subset E(M)$. Then the ground set of N is $E(M) \cup \{a\}$ and $N/a = M$. Hence N is a coextension of M by the element a. Let A be the standard matrix representation of M over $GF(2)$. By Definition 1.3, in the matrix A'_T of M'_T, the column labeled by a has 1 in the last row and 0 elsewhere, and the columns labeled by the elements of T have 1 in the last row. This shows that a is neither a loop nor a coloop of N.

Conversely, suppose N is a coextension of M by a non-loop and non-coloop element a. Let T_1 be a cocircuit of N containing a and let $T = T_1 - \{a\}$. Then T is a non-empty subset of $E(M)$. We can write the standard matrix representation B of N such that the column of B labeled by a has entry 1 in the last row and 0 elsewhere. Since T_1 is a cocircuit of N, the last row of B contains 1 in the columns corresponding to T_1 and 0 elsewhere. Let C be the matrix obtained from B by deleting the last row and the column corresponding to a. Then $M[C] = N/a = M$. Thus B can be obtained from C by adding one extra row which has entries 1 below the elements corresponding to T and then adding a column labeled by a which has entry 1 in the last row and 0 elsewhere. Therefore, by Definition 1.3, $B = C'_T$. Hence $N = M[B] = M[C'_T] = M'_T$. □

Henceforth, we use the notation M'_T for a single-element coextension of a binary matroid M.

We need the following results.

Lemma 2.2 ([1])**.** *Let M be a binary matroid and $T \subseteq E(M)$. If $\mathcal{C}$ is the collection of circuits of M, then every circuit of M'_T belongs to one of the following type.*

(i). $\mathcal{C}_1 = \{C \in \mathcal{C} \colon |C \cap T| \text{ is even}\}$

(ii). $\mathcal{C}_2 = \{C \cup \{a\} \colon C \in \mathcal{C} \text{ and contains an odd number of elements of } T\}$

(iii). $\mathcal{C}_3 =$ *set of minimal members of* $\{C_1 \cup C_2 \colon C_1, C_2 \in \mathcal{C},\ C_1 \cap C_2 = \emptyset$ *and C_1 and C_2 each contains an odd number of elements of T such that $C_1 \cup C_2$ does not contain any member of $\mathcal{C}_1$.*$\}$

Lemma 2.3 ([2]). *Let M be a binary matroid. Suppose r and r' are the rank functions of M and M'_T, respectively. If $A \subset E(M) \cup \{a\}$, then rank of A is given by*

(i). $r'(A) = r(A - \{a\}) + 1$ if $a \in A$.
(ii). $r'(A) = r(A) + 1$ if $a \notin A$ and A contains a circuit C of M with $|C \cap T|$ odd.
(iii). $r'(A) = r(A)$ if $a \notin A$ and A does not contain any circuit C of M with $|C \cap T|$ odd.

Corollary 2.4. *Let M be a binary matroid and $T \subseteq E(M)$. Then $r'(M'_T) = r(M) + 1$.*

Lemma 2.5 ([7]). *Let M be a binary matroid and $\mathcal{C}^*$ be the collection of cocircuits of M. Suppose $T \subseteq E(M)$ does not contain a cocircuit of M. Then every cocircuit of M'_T belongs to one of the following type.*

(i). $\mathcal{Q}_1^* = \{(C^* - T) \cup \{a\} : C^* \in \mathcal{C}^*$ and T is a proper subset of $C^*\}$,
(ii). $\mathcal{Q}_2^* = \{C^* : C^* \in \mathcal{C}^*\}$,
(iii). $\mathcal{Q}_3^* = \{(C^* \Delta T) \cup \{a\} : C^* \in \mathcal{C}^*, 1 \le |C^* \cap T| < |T|$ and C^* does not contain $D^* - T$ for any $D^* \in \mathcal{C}^*$ and $T \subset D^*\}$,
(iv). $\mathcal{Q}_4^* = \{((C_1^* \cup C_2^* \cup \cdots \cup C_k^*) - T) \cup \{a\} : k \ge 2,\ C_i^* \in \mathcal{C}^*, C_i^* \cap T \ne \emptyset, C_i^*$ are mutually disjoint and $(C_1^* \cup C_2^* \cup \cdots \cup C_k^*) - T$ does not contain $D^* - T$ for any $D^* \in \mathcal{C}^*$ and $T \subset D^*\}$.
(v). $\mathcal{Q}_5^* = \{T \cup \{a\}\}$.

3. Proofs

In this section, we prove Main Theorem 1.4 and also provide an alternate shorter proof of Theorem 1.5.

We need the following result.

Lemma 3.1 ([6], pp 296). *If $n \ge 2$ and M is an n-connected matroid with $|E(M)| \ge 2(n-1)$, then all circuits and all cocircuits of M have at least n elements.*

Suppose M is an n-connected binary matroid with $|E(M)| \ge 2(n-1)$ and $T \subset E(M)$. By Definition 1.3, there is a cocircuit of M'_T contained in $T \cup \{a\}$. Therefore, if $|T| < n-1$, then M'_T contains a cocircuit of size less than n by Lemma 2.5 and hence M'_T is not n-connected by Lemma 3.1. Hence we assume that $|T| \ge n-1$.

We obtain below an obvious necessary condition for M'_T to be n-connected.

Lemma 3.2. *Let $n \ge 2$ be an integer and M be an n-connected binary matroid with $|E(M)| \ge 2n-2$. Suppose $T \subset E(M)$ with $|T| = n-1$. If M'_T is n-connected, then $|Q| \ge 2|Q \cap T|$ for every cocircuit Q of M intersecting T.*

Proof. Suppose M'_T is n-connected. Assume that there is a cocircuit Q of M intersecting T such that $|Q| < 2|Q \cap T|$. By Lemma 2.5 (iii), $Q \Delta T \cup \{a\}$ contains a cocircuit, say X, of M'_T. Then $|X| \le |Q \Delta T \cup \{a\}| = |Q| + |T| - 2|Q \cap T| + 1 < |T| + 1 = n$, a contradiction by Lemma 3.1. □

We now prove that the obvious necessary condition for M'_T to be n-connected stated in the above lemma is sufficient also.

Proposition 3.3. *Let $n \ge 2$ be an integer and M be an n-connected binary matroid with $|E(M)| \ge 2n-2$. Suppose $T \subset E(M)$ with $|T| = n-1$. If $|Q| \ge 2|Q \cap T|$ for every cocircuit Q of M intersecting T, then M'_T is n-connected.*

Proof. Assume that $|Q| \ge 2|Q \cap T|$ for every cocircuit Q of M intersecting T. We proceed by contradiction. Suppose M'_T is not n-connected. Then there exists an $(n-1)$-separation (A, B) of M'_T. Therefore

$$\min\{|A|, |B|\} \ge n-1 \text{ and } r'(A) + r'(B) - r'(M'_T) \le n-2. \ \ldots\ldots (*)$$

Suppose $|A| \ge n$ and $|B| \ge n$. Without loss of generality, we may assume that $a \in B$. By Lemma 2.3 and by $(*)$,

$$r(A) + r(B - \{a\}) - r(M) \le r'(A) + r'(B) - 1 - (r'(M'_T) - 1) \le n-2.$$

Therefore $(A, B - \{a\})$ forms an $(n-1)$-separation of M, a contradiction.

Therefore $|A| = n-1$ or $|B| = n-1$. We may assume that $|A| = n-1$. Then A is independent in M by Lemma 3.1. Hence, by Lemma 2.2, A is independent in M'_T also.

Claim: A is a coindependent in M'_T.

Assume that A is not coindependent in M'_T. Then A contains some cocircuit Q of M'_T. Therefore $|Q| \leq |A| = n-1$. By Lemma 3.1, Q is not a cocircuit of M. Further, by Lemma 2.5, Q does not belong to $\mathcal{Q}_2^*$. Hence Q belongs to one of the four classes $\mathcal{Q}_1^*$, $\mathcal{Q}_3^*$, $\mathcal{Q}_4^*$ and $\mathcal{Q}_5^*$.

(1). Suppose $Q \in \mathcal{Q}_1^*$. Then $Q = (C^* - T) \cup \{a\}$, where C^* is a cocircuit of M containing T. Then, by hypothesis, $|C^*| \geq 2|C^* \cap T| = 2|T| = 2n-2$. Therefore

$$n-1 \geq |Q| = |C^*| - |T| + 1 \geq (2n-2) - (n-1) + 1 = n,$$

a contradiction.

(2). Suppose $Q \in \mathcal{Q}_4^*$. Then $Q = ((C_1^* \cup C_2^* \cup \cdots \cup C_k^*) - T) \cup \{a\}$, where $k \geq 2$ and C_i^* are mutually disjoint cocircuits of M and each of them contains at least one element of T. Since M is n-connected, $|C_i^*| \geq n$ for each i by Lemma 3.1. Hence, we have

$$|Q| \geq |(C_1^* \cup C_2^*) - T| + 1 \geq |C_1^*| + |C_2^*| - |T| + 1 \geq 2n - (n-1) + 1 = n+2 > n-1 \geq |Q|,$$

again a contradiction.

(3). Suppose $Q \in \mathcal{Q}_3^*$. Then $Q = (C^* \Delta T) \cup \{a\}$, where C^* is a cocircuit of M intersecting T. Hence

$$|Q| = |C^* \Delta T| + 1 = |C^*| + |T| - 2|C^* \cap T| + 1 \geq |T| + 1 = n > n-1 \geq |Q|,$$

a contradiction.

(4). Suppose $Q \in \mathcal{Q}_5^*$. So $Q = T \cup \{a\}$. This gives $|Q| = n$, a contradiction.

Thus in all the four cases, we get a contradiction. This proves the claim.

Therefore A is independent and coindependent in the matroid M'_T. Hence $r'(A) = |A|$ and $r'(B) = r'(M'_T)$. This gives $n-1 = |A| = r'(A) = r'(A) + r'(B) - r'(M'_T) \leq n-2$, a contradiction. Thus we get a contradiction in each case. Therefore M'_T is n-connected. □

Main Theorem 1.4 follows obviously from Lemma 3.2 and Proposition 3.3.

For $2 \leq n \leq 4$, we get the following weaker sufficient conditions for M'_T to be n-connected.

Corollary 3.4. *Let $n \in \{2,3,4\}$ and let M be n-connected binary matroid. Suppose $T \subset E(M)$ with $|T| = n-1$. If $|Q| \geq 2n-2$ for every cocircuit Q containing T, then M'_T is n-connected.*

Proof. Let Q be a cocircuit of M intersecting T. By Proposition 3.3, it is sufficient to prove that $|Q| \geq 2|Q \cap T|$. If $T \subseteq Q$, then $|Q| \geq 2n-2 = 2|T| = 2|Q \cap T|$. Suppose $T \nsubseteq Q$. Then $|Q \cap T| < |T| = n-1$ and hence $|Q \cap T| \leq n-2$. Since $2 \leq n \leq 4$, we have $2|Q \cap T| \leq 2(n-2) = 2n-4 \leq n$. By Lemma 3.1, $|Q| \geq n$ and so $|Q| \geq 2|Q \cap T|$. □

We combine Main Theorem 1.4 and Theorem 1.5 and provide a shorter proof of Theorem 1.5.

Theorem 3.5. *Let $n \geq 2$ be an integer and M be an n-connected binary matroid with $|E(M)| \geq 2n-2$. Suppose $T \subset E(M)$ with $|T| = n-1$. Then the following statements are equivalent.*

(i). M'_T is n-connected.
(ii). $|Q| \geq 2|Q \cap T|$ for every cocircuit Q of M intersecting T.
(iii). For any subset $A \subset E(M)$ with $|A| = n-2$, there exists a circuit C of M containing an odd number of elements of T and is contained in $E(M) - A$.

Proof. (i) $\implies$ (ii) follows from Lemma 3.2 and (ii) $\implies$ (i) follows from Proposition 3.3.

(i) $\implies$ (iii). Suppose (i) holds but (iii) does not hold. Then there is a subset A of $E(M)$ with $|A| = n-2$ such that no circuit of M containing an odd number of elements of T is contained in $E(M) - A$. Let $A' = A \cup \{a\}$ and $B = E(M) - A$. Then $|A'| = n-1$ and $|B| \geq n-1$. Let r and r' be the rank function of M and M'_T, respectively. By Lemma 3.1, A contains neither a cocircuit of M nor a cocircuit of M'_T. Hence $r(B) = r(M)$ and $r'(B) = r'(M'_T)$. Also, by Lemma 2.3(iii), $r(B) = r'(B)$. This gives $r(M) = r'(M'_T)$, a contradiction by Corollary 2.4. Hence (i) implies (iii).

(iii) $\implies$ (i). Suppose (iii) holds but (i) does not hold. Then M'_T has an $(n-1)$-separation (A, B). Therefore

$$\min\{|A|, |B|\} \geq n-1 \text{ and } r'(A) - r'(B) - r'(M'_T) \leq n-2. \;\ldots\ldots (*)$$

Without loss of generality, assume that $a \in A$. By Lemma 2.3(i), $r'(A) = r(A - \{a\}) + 1$. If $|A| \geq n$, then, by $(*)$,

$$r(A - \{a\}) + r(B) - r(M) \leq r'(A) - 1 + r'(B) - (r'(M'_T) - 1) \leq n - 2.$$

Therefore $(A - \{a\}, B)$ is an $(n-1)$-separation of M, a contradiction. Hence $|A| = n - 1$. Then $|A - \{a\}| = n - 2$. By (iii) and Lemma 2.3(ii), $r'(B) = r(B) + 1$. Therefore

$$r(A - \{a\}) + r(B) - r(M) \leq r'(A) - 1 + r'(B) - 1 - (r'(M'_T) - 1) = n - 3.$$

This shows that (A, B) is an $(n-2)$-separation of M, a contradiction. Thus (iii) implies (i). □

4. Consequences to Graphs

In this section, we prove that Proposition 3.3 is a matroid extension of Theorem 1.2.

We need the following result.

Theorem 4.1 ([6], pp. 328). *For $n \geq 2$, let G be a graph without isolated vertices and with at least $n + 1$ vertices. Then the circuit matroid $M(G)$ is n-connected if and only if G is n-connected and has no cycle with fewer than n edges.*

By Theorem 4.1, the circuit matroid $M(G)$ of an n-connected graph G is not n-connected if G contain a cycle of length less than n. Therefore we derive Theorem 1.2 from Proposition 3.3 by assuming that G has girth at least n.

Theorem 4.2. *Suppose G is an n-connected graph of girth at least n, where $n \geq 2$. Let T be a set of $n - 1$ edges incident to a vertex of degree at least $2n - 2$ in G. Then the n-point splitting graph G'_T is n-connected.*

Proof. Let $M = M(G)$. Then $M'_T = M(G'_T)$. We prove that M'_T is n-connected. By Theorem 4.1, M is n-connected. Let Q be a cocircuit of M intersecting T. By Proposition 3.3, it is sufficient to prove that $|Q| \geq 2|Q \cap T|$. On the contrary, assume that $|Q| < 2|Q \cap T|$. As $Q = (Q - T) \cup (Q \cap T)$, $|Q| = \dfrac{|Q|}{2} + \dfrac{|Q|}{2} = |Q - T| + |Q \cap T|$ and hence $|Q - T| < \dfrac{|Q|}{2} < |Q \cap T|$. Let u be the vertex of G of degree at least $2n - 2$ such that the edges of G belonging to T are incident to u. Since Q is a cocircuit of $M(G)$, the graph $G - Q$ is disconnected and it has two components, say C_1 and C_2. We may assume that C_2 contains the vertex u. Let $Q \cap T = \{uu_1, uu_2, \ldots, uu_k\}$. Then $u_1, u_2, \ldots, u_k$ are vertices of C_1. Let $v_1, v_2, \ldots, v_r$ be the end vertices of the edges belonging to $Q - T$ in C_1. Then $r \leq |Q - T| < |Q \cap T|$. Since $|Q| < 2|Q \cap T| \leq 2|T| = 2n - 2$ and degree of u is at least $2n - 2$, there is at least one edge uw incident to u in $G - Q$. Then the edge uw is in C_2. Let $A = \{v_1, v_2, \ldots, v_r, u\}$. Then $G - A$ is disconnected, leaving u_i for some $i \in \{1, 2, \ldots, k\}$ in one component and the vertex w is in an another component. However, $|A| = r + 1 \leq |Q \cap T| \leq |T| = n - 1$, a contradiction to the fact that G is n-connected. Thus M'_T is n-connected. By Theorem 4.1, G'_T is n-connected. □

Corollary 4.3. *Let G be a 3-connected simple graph and T be a set of two edges incident to a vertex of G of degree at least four. Then the graph G'_T is 3-connected.*

We now prove that one can obtain a 3-regular, 3-connected graph from the given 3-connected simple graph by repeated applications of 3-point splitting operation.

Corollary 4.4. *A 3-regular, 3-connected simple graph can be obtained from the given 3-connected simple graph by a finite sequence of the 3-point splitting operation.*

Proof. Let G be a 3-connected simple graph. Then degree of every vertex of G is at least three. Suppose G contains a vertex v of degree $k > 3$. Let $T = \{x, y\}$ be a set of two edges incident at v. By Corollary 4.3, G'_T is 3-connected. The vertex of v of G is replaced by two vertices v' and v'' with degrees 3 and $k - 1$, respectively in G'_T. Thus one application of 3-point splitting on a vertex of degree $t > 3$ results into a 3-connected graph with one additional vertex of degree less than t. By a finite sequence of 3-point splitting operation we can get a 3-connected graph with no vertex of degree greater than three. Clearly, this graph will be 3-regular. □

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