Crossing number bounds for mesh connected and 3-regular mesh of trees

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Abstract

In this paper we provide bounds for the crossing number of mesh connected trees and 3-regular mesh of trees.

Keywords: Crossing number, planar graph, mesh connected trees, mesh of trees.

1 Introduction

A drawing D of a graph G is a representation of G in the Euclidean plane R^2 where vertices are represented as distinct points and edges by simple polygonal arcs joining points that correspond to their end vertices. A drawing D is good or clean if it has the following properties: no edge crosses itself, no pair of adjacent edges crosses, two edges cross at most once, not more than two edges cross at one point.

The number of crossings in a drawing D is denoted by $Cr(D)$ and is called the crossing number of the drawing D. The crossing number $Cr(G)$ of a graph G is the minimum $Cr(D)$ taken over all good drawings D of G. If a graph G admits a drawing D with $Cr(D) = 0$ then G is said to be planar; otherwise it is non-planar. The study of crossing numbers has applications to VLSI design in theoretical computer science [6]. The VLSI (Very-large-scale integration) is the process of creating an integrated circuit (IC) by combining hundreds of thousands of transistors or devices into a single chip. VLSI began in the 1970s when complex semiconductor and communication technologies were being developed. The microprocessor is a VLSI device. The grid and the mesh of trees are among the best-known parallel architectures in the literature. Both of them enjoy efficient VLSI layouts. Most of the problems of VLSI design are modeled to some graph theoretic problem and most of the algorithms of VLSI physical design are based on graph structure.

For an arbitrary graph computing $Cr(G)$ is NP-hard [5]. Hence from a computational standpoint, it is infeasible to obtain exact solutions for graphs, in general, but more practical to explore bounds for the parameter values [3]. In this paper we consider the class of parallel architectures, called the mesh-connected trees [4] and 3-regular mesh of trees [8]. These networks are widely used in the area of broadcasting [7].

We obtain bounds for the crossing number of mesh connected trees, propose a new drawing and obtain a finer bound. We also consider the crossing number problem for the 3-regular mesh of trees. The problem of determining the VLSI layout area is open.

2 Mesh connected trees

A complete binary tree with h levels, denoted $T(h)$, has $2^h - 1$ vertices and $2^h - 2$ edges. The root of $T(h)$ which is assumed to be at level 1, has degree 2, and each internal vertex has degree 3. The diameter of $T(h)$ is $2(h-1)$. Throughout this paper the symbol N stands for $2^h - 1$.

The mesh connected trees or simply the MCT network is the multi-dimensional cross product of complete binary trees [4]. Informally, the N^r-node r-dimensional MCT, denoted as $MCT_r(N)$, is obtained from the N^r-node r-dimensional grid by replacing the liner connections along each grid dimension by the connections of an N-node complete binary tree.

The notation MCT [4] is used to refer generically to the class of networks that are called mesh connected trees, while the notation $MCT_r(N)$ is used to refer specifically to the N^r-node r-dimensional grid by replacing the linear connections along each grid dimension by the connections of an N-node complete binary tree. Figure 1 shows the 49-node 2-dimensional mesh connected trees $MCT_2(7)$.

The N^r node r-dimensional mesh-connected trees, $MCT_r(N)$, is the graph whose vertices comprise all the r-tuples $x = x_{r-1}, \cdots, x_1, x_0$, such that, for every i, x_i is a vertex of $T(h)$, and the pair (x, y) defines an edge in $MCT_r(N)$ if and only if x and y differ exactly in one index position i and (x_i, y_i) is an edge in $T(h)$. In what follows we let $r = 2$ and we estimate

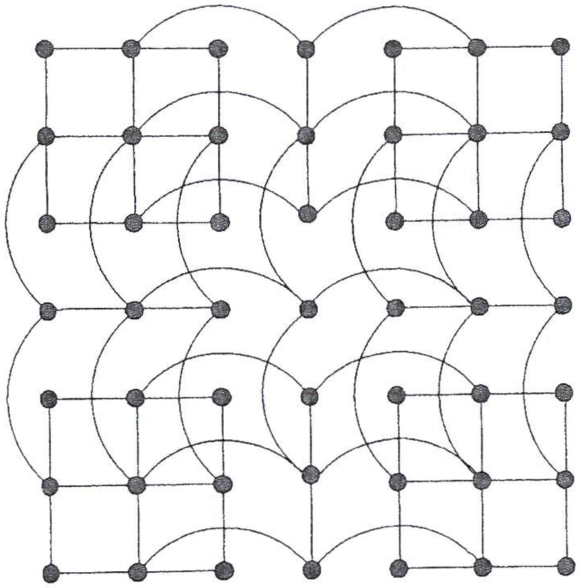

Figure 1: $MCT_2(7)$ or $M(3)$

the crossing number of $MCT_2(N), N = 2^h - 1$. For convenience of notation we write $MCT_2(N)$ as $MCT_2(h)$ or simply $M(h)$. There are three types of trees in $M(h)$; row trees, column trees and middle trees. Further there are four subnetworks $M(h-1)$ in $M(h)$. We call these subnetworks as top left (TL), top right (TR), bottom left (BL) and bottom right (BR) subnetworks.

In the construction of $M(h)$, the roots of the row trees, column trees and middle trees of the four copies of $M(h-1)$ are connected by paths of length two (trees on three vertices). The new vertices are connected so as to form two middle trees in $M(h)$. The edges of the trees on three vertices are termed as additional row edges (ARE), additional column edges (ACE) and additional middle edges (AME). These additional edges cross the four subnetworks $M(h-1)$ and contribute to the crossing number of $M(h)$. There are three types of crossings in $M(h)$.

1. Crossing of the additional row or column edges of $M(h)$ with the edges of $M(h-1)$.
2. Crossing of the additional row or column edges of $M(h)$ with the additional column or row edges of $M(h)$.
3. Crossing of the additional middle row or middle column edges of $M(h)$ with those of $M(h)$.

In the study of crossing numbers there is no specific method to obtain a lower bound for the crossing number. There is a simple inequality mentioned in [2] that provides a lower bound for the crossing number. This may or may not match with an upper bound. We use the notation $cr(A, B)$ to denote the number of crossings between any two edge subsets A and B [10].

Theorem 2.1. [2]*(Eüler's formula) In a connected plane graph G with ν vertices, ϵ edges and ϕ faces (regions), $\nu - \epsilon + \phi = 2$.*

Theorem 2.2. [2] *If G is a connected plane graph with girth g, then $g\phi \leq 2\epsilon$.*

Let D be a good drawing of $M(3)$ as in Fig.1. Then we have the following result.

Lemma 2.3. *Let G be $M(3)$. Then $Cr(G) \leq 28$.*

Theorem 2.4. *Let G be $M(h), h \geq 3$. Then*

$$Cr(G) \leq \sum_{i=0}^{h-3} 4^{h-2-i}[(i+1)2^{i+2}(2^{i+1}-1)+(i^3+i)+2^{i+1}(2^{i+1}-1)+(i\cdot 2^{i+1}+1)]$$

Proof. We use induction on h. By Lemma 2.3, the theorem is true for $h = 3$. We first prove that it is true for $h = 4$. Let D be a good drawing of $M(4)$ as in Fig.2. There are four copies of $M(3)$ in $M(4)$ along with additional edges. To estimate $Cr_D(M(4))$ we use the bound for $Cr_D(M(3))$ from lemma 2.3 and using symmetry of the network we begin counting the number of crossings of the additional row edges with the edges of top left (TL) subnetwork.

We observe that the top left subnetwork is $M(3)$ and it contains four copies of $M(2)$. The additional row edges in $M(4)$ cross the edges of two copies of $M(2)$. This accounts for $2^2(3)$ number of crossings. Further the additional row edges cross the $M(3)$ edges other than those in $M(2)$. This accounts for $2^2(3)$ number of crossings. Hence the number of crossings of the additional row edges with the edges of top left (TL) subnetwork is $2^3(3)$.

In the case of the number of crossings of additional row edges with the edges of top right subnetwork (TR) it is the same count as above together with two extra crossings. Hence the number of crossings of additional row edges with top left and top right subnetworks is $2(2^3(3)) + 2$. A similar

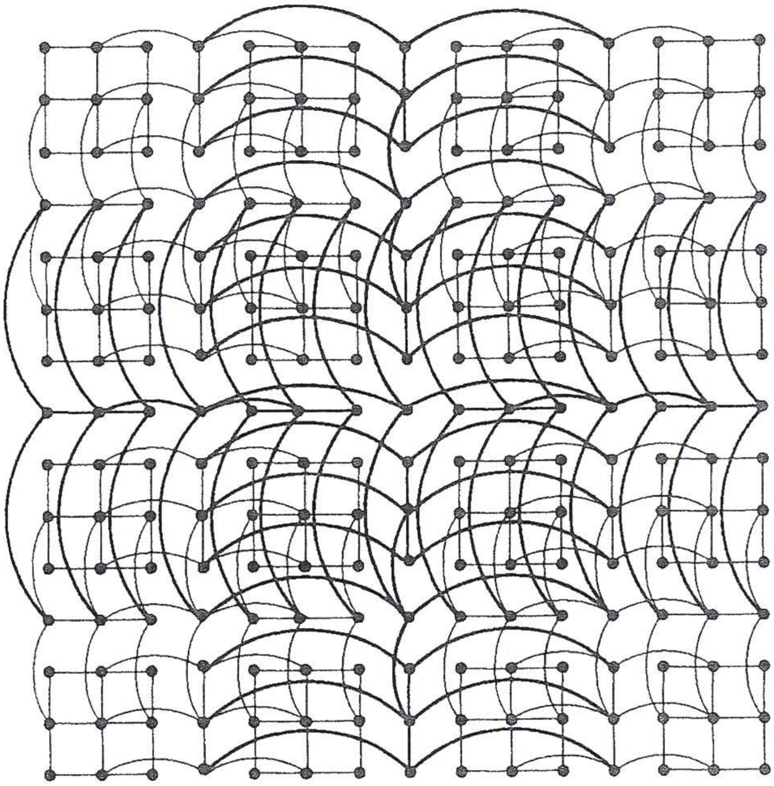

Figure 2: $M(4)$

count is obtained in bottom left (BL) and bottom right (BR) subnetworks. Thus $Cr_D\left(ARE, M(4)\right) = 4\left(2^3(3)\right) + 2(2)$. Similarly $Cr_D\left(ACE, M(4)\right) = 4\left(2^3(3)\right) + 2(2)$.

The next count is the number of crossings of the additional row edges with additional column edges. In TL this count is 3(3). The count in TR is 3(3) along with 3 extra crossings. In BL the count is 4(3) and in BR the count is 4(3) along with 3 extra crossings. Thus

$$\begin{aligned} Cr_D\left(ARE, ACE\right) &= (2 \cdot 3(3) + 3) + (2 \cdot 4(3) + 3) \\ &= 16(3) \\ &= 2^4 \cdot 3 \end{aligned}$$

The number of crossings of the additional middle row or additional middle column trees with additional column edges or additional row edges of $M(4)$ is 5 each. Thus $Cr_D\left(AME, M(4)\right) = 4(5)$. Consequently

$$
\begin{aligned}
Cr_D\,[M(4)] &= 4Cr_D\,(M(3)) + Cr_D\,(ARE, M(4)) + Cr_D\,(ACE, M(4)) \\
&\quad + Cr_D\,(ARE, ACE) + Cr_D\,(AME, M(4)) \\
&= 4(28) + 4\left(2^3(3)\right) + 2(2) + 4\left(2^3(3)\right) + 2(2) + 2^4 \cdot 3 + 4(5) \\
&= 380
\end{aligned}
$$

proving the result for $h = 4$. Assume that the theorem is true for $h = k, k \geq 4$. We prove that it is true for $h = k+1$. It is clear that $M(k+1)$ contains 4 copies of $M(k)$. As before using symmetry of the network we begin counting the number of crossings of the additional row and column trees with the edges of the top left (TL) subnetwork. There are $(2^k - 2)$ additional row edges of $M(k+1)$ contributing to the crossing number of D. Each of a collection of 2^{k-1} additional row edges crosses $(2^{k-1} - 1)$ mesh edges. Further the additional row edges cross the non-mesh edges of $M(k)$, $(k-2)2^{k-1}(2^{k-1}-1)$ number of times. This is because of the count follows the pattern $1 \cdot 2^2(3), 2 \cdot 2^3(7), 3 \cdot 2^4(15)$, etc. Hence the number of crossings of the additional row edges of $M(k+1)$ with the top left (TL) subnetwork is $2^{k-1}(2^{k-1} - 1) + (k-2)2^{k-1}(2^{k-1} - 1) = (k-1)2^{k-1}(2^{k-1} - 1)$. In the case of the number of crossings of additional row edges with the edges of TR, the count is the same but with $(k-2)^3 + (k-2)$ extra crossings. This is because the extra crossings in TR follow the pattern $2 = 1^3 + 1$, $10 = 2^3 + 2$, $30 = 3^3 + 3$ for $k = 3, 4$ etc. Hence

$$
Cr_D\,(ARE, M(k+1)) = 4(k-1)2^{k-1}(2^{k-1} - 1) + 2\left((k-2)^3 + (k-2)\right)
$$

Similarly the number of crossings of the additional column trees with the edges of $M(k+1)$ is given by

$$
Cr_D\,(ACE, M(k+1)) = 4(k-1)2^{k-1}(2^{k-1} - 1) + 2\left((k-2)^3 + (k-2)\right)
$$

The next count is the number of crossings of the additional row edges with additional column edges. In TL this count is $(2^{k-1} - 1)(2^{k-1} - 1)$. The count in TR is $(2^{k-1} - 1)^2$ along with $(2^{k-1} - 1)$ extra crossings. In BL the count is $2^{k-1}(2^{k-1} - 1)$ and in BR the count is $2^{k-1}(2^{k-1} - 1)$ along with $(2^{k-1} - 1)$ extra crossings. Thus

$$
\begin{aligned}
Cr_D\,(ARE, ACE) &= 2\left(2^{k-1}-1\right)^2 + \left(2^{k-1}-1\right) + 2\cdot 2^{k-1}\left(2^{k-1}-1\right) \\
&\quad + \left(2^{k-1}-1\right) \\
&= \left(2^{k-1}-1\right)2^{k+1}
\end{aligned}
$$

he number of crossings of the additional middle row or additional mid-le column trees with additional column edges or additional row edges f $M(k+1)$ is $(k-2)2^{k-1}+1$ each. Hence $Cr_D\,(AME, M(k+1)) =$ $[(k-2)2^{k-1}+1]$

hus

$r_D\,[ARE, M(k+1)] + Cr_D\,[ACE, M(k+1)] + Cr_D\,[ARE, ACE]$
$+Cr_D\,[AME, M(k+1)]$

$$
\begin{aligned}
&= 4(k-1)2^{k-1}(2^{k-1}-1) + 2((k-2)^3 + (k-2)) \\
&+ 4(k-1)2^{k-1}(2^{k-1}-1) + 2((k-2)^3 + (k-2)) + (2^{k-1}-1)2^{k+1} \\
&+ 4((k-2)2^{k-1}+1) \\
&= 8(k-1)2^{k-1}(2^{k-1}-1) + 4((k-2)^3 + (k-2)) + (2^{k-1}-1)2^{k+1} \\
&+ 4((k-2)2^{k-1}+1) \\
&= 4[2(k-1)2^{k-1}(2^{k-1}-1) + (k-2)^3 + (k-2) + (2^{k-1}-1)2^{k-1} \\
&+ (k-2)2^{k-1}+1] \\
&= 4[(k-1)2^{k}(2^{k-1}-1) + (k-2)^3 + (k-2) + (2^{k-1}-1)2^{k-1} \\
&+ (k-2)2^{k-1}+1]
\end{aligned}
$$

We now proceed to the main proof. By induction hypothesis

$$
\begin{aligned}
Cr\,(M(k)) = \sum_{i=0}^{k-3} 4^{k-2-i}[&(i+1)2^{i+2}(2^{i+1}-1) + (i^3+i) \\
&+ 2^{i+1}(2^{i+1}-1) + (i\cdot 2^{i+1}+1)]
\end{aligned}
$$

Now,
$Cr(M(k+1)) = 4(CrM(k))$ + Additional crossings in $M(k+1)$

$$= 4[\sum_{i=0}^{k-3} 4^{k-2-i}[(i+1)2^{i+2}(2^{i+1}-1) + (i^3+i) + 2^{i+1}(2^{i+1}-1)$$
$$+ (i \cdot 2^{i+1} + 1)] + 4[((k-1)2^k 2^{k-1} - 1) + (k-2)^3 + (k-2)$$
$$+ (2^{k-1}-1)2^{k-1} + (k-2)2^{k-1} + 1]$$
$$= \sum_{i=0}^{k-3} 4^{k-1-i}[(i+1)2^{i+2}(2^{i+1}-1) + (i^3+i) + 2^{i+1}(2^{i+1}-1)$$
$$+ (i \cdot 2^{i+1} + 1)] + 4[(k-1)2^k(2^{k-1} - 1) + (k-2)^3 + (k-2)$$
$$+ 2^{k-1}(2^{k-1}-1) + (k-2)2^{k-1} + 1]$$
$$= \sum_{i=0}^{k-2} 4^{k-1-i}[(i+1)2^{i+2}(2^{i+1}-1) + (i^3+i) + 2^{i+1}(2^{i+1}-1)$$
$$+ (i \cdot 2^{i+1} + 1)]$$

□

3 Proposed new drawing of mesh connected trees

In this section we propose a new drawing of the mesh connected trees $M(h)$. We describe the drawing for $M(3)$ and use it to construct $M(h)$. The row trees and column trees in $M(3)$ are divided into two types each. The row tree in Fig.3(a) is called a row tree of type(1) and a reflection about a horizontal line is called a row tree of type(2). The column tree in Fig.3(b) is called a column tree of type(1) and a reflection about a vertical line is called a column tree of type(2). This applies only for drawing. The mesh connected tree $M(3)$ contains seven row trees and seven column trees. The row trees in $M(3)$ will be as follows: two row trees of type(1), one row tree of type(2), two row trees of type(1) followed by two row trees of type(2). The rotation through 90^o of the row trees in the anticlockwise direction gives the column trees. The middle row tree is a row tree of type(1) and the middle column tree is a column tree of type(2).

As in the previous section, there are four subnetworks $M(h-1)$ in $M(h)$ called top left (TL), top right (TR), bottom left (BL) and bottom right (BR) subnetworks. In the construction of $M(h), h > 3$ the roots of the row trees and column trees of $M(h-1)$ are connected by paths of length two (trees on three vertices). The new vertices are connected suitably to form

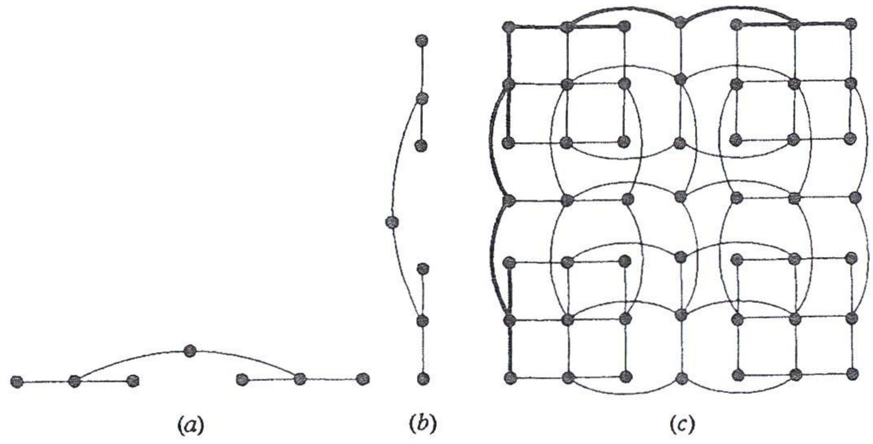

Figure 3: (a) A row tree of type(1), (b) A column tree of type(1), (c) $M(3)$

middle trees in $M(h)$. These additional trees (row or column) continue to be of types(1) or (2) depending upon their types in lower dimension. The additional edges are termed as additional row edges (ARE), additional column edges (ACE) and additional middle edges (AME).

These additional edges cross the four subnetworks $M(h-1)$ and contribute to the crossing number of $M(h)$.
In what follows we count $Cr_D\,(ARE, M(h))\,, Cr_D\,(ACE, M(h))\,,$ $Cr_D\,(ARE, ACE)\,, Cr_D\,(AME, M(h))$. We have the following result from Fig.3(c).

Lemma 3.1. *Let G be $M(3)$. Then $Cr(G) \leq 16$.*

Theorem 3.2. *Let G be $M(h), h \geq 4$. Then*

$$Cr(G) \leq 2^{2h-2} + \sum_{i=1}^{h-3} 4^{h-2-i}[6\,(2^{i+1}-1)\,(2^{i+1}-3) + (2^{i+1}) + (i^3+i) + 2^{i+1}(2^{i+1}-1) + (i \cdot 2^{i+1}+1)]$$

Proof. We use induction on h. By Lemma 3.1, the theorem is true for $h = 3$. We first prove that it is true for $h = 4$. Let D be a good drawing of $M(4)$ as in Fig.4. There are four copies of $M(3)$ in $M(4)$ along with additional edges. To estimate $Cr_D\,(M(4))$ we use the bound for $Cr_D\,(M(3))$ from Lemma 3.1 and using symmetry of the network we begin counting the number of crossings of the additional row edges with the edges of top left (TL) subnetwork.

We observe that the top left subnetwork is $M(3)$ and it contains four copies of $M(2)$. The additional row edges does not cross the edges of $M(2)$.

The additional row edges of $M(4)$ cross the top left subnetwork of $M(3)$ edges 3(3)+2 number of times.

In the case of the number of crossings of additional row edges with the edges of top right subnetwork (TR) it is the same count together with 2 extra crossings. Hence the number of crossings of additional row edges with top left and top right subnetworks is 2(3(3)+2)+2. A similar count is observed in bottom left (BL) and bottom right (BR) subnetworks. Thus $Cr_D\,(ARE, M(4)) = 4\,(3(3)+2)+2(2)$. Similarly $Cr_D\,(ACE, M(4)) = 4\,(3(3)+2)+2(2)$.

The next count is the number of crossings of the additional row edges with additional column edges. In TL this count is 3(3). The count in TR is 3(3) along with 3 extra crossings. In BL the count is 4(3) and in BR the count is 4(3) along with 3 extra crossings. Thus

$$\begin{aligned} Cr_D\,(ARE, ACE) &= (2\cdot 3(3)+3)+(2\cdot 4(3)+3) \\ &= 16(3) \\ &= 2^4\cdot 3 \end{aligned}$$

The number of crossings of the additional middle row or additional middle column trees with additional column edges or additional row edges of $M(4)$ is 5 each. Thus $Cr_D\,(AME, M(4)) = 4(5)$. Consequently

$$\begin{aligned} Cr_D\,[M(4)] &= 4Cr_D\,(M(3)) + Cr_D\,(ARE, M(4)) + Cr_D\,(ACE, M(4)) \\ &\quad + Cr_D\,(ARE, ACE) + Cr_D\,(AME, M(4)) \\ &= 4(16)+4\,(3(3)+2)+2(2)+2(2)+4\,(3(3)+2)+2(2) \\ &\quad + 2^4\cdot 3+4(5) \\ &= 228 \end{aligned}$$

proving the result for $h=4$. Assume that the theorem is true for $h=k, k\geq 4$. We prove that it is true for $h=k+1$. It is clear that $M(k+1)$ contains 4 copies of $M(k)$. As before using symmetry of the network we begin counting the number of crossings of the additional row and column trees with the edges of the top left (TL) subnetwork. There are (2^k-2) additional row edges of $M(k+1)$ contributing to the crossing number of D. The additional row edges cross the non-mesh edges of $M(k)$, $3(2^{k-1}-1)(2^{k-1}-3)+2^{k-2}$ number of times. This is because of the count follows the pattern 3(3)+2, 15(7)+4, 39(15)+8, etc. Hence the number of crossings of the additional row edges of $M(k+1)$ with the top left (TL)

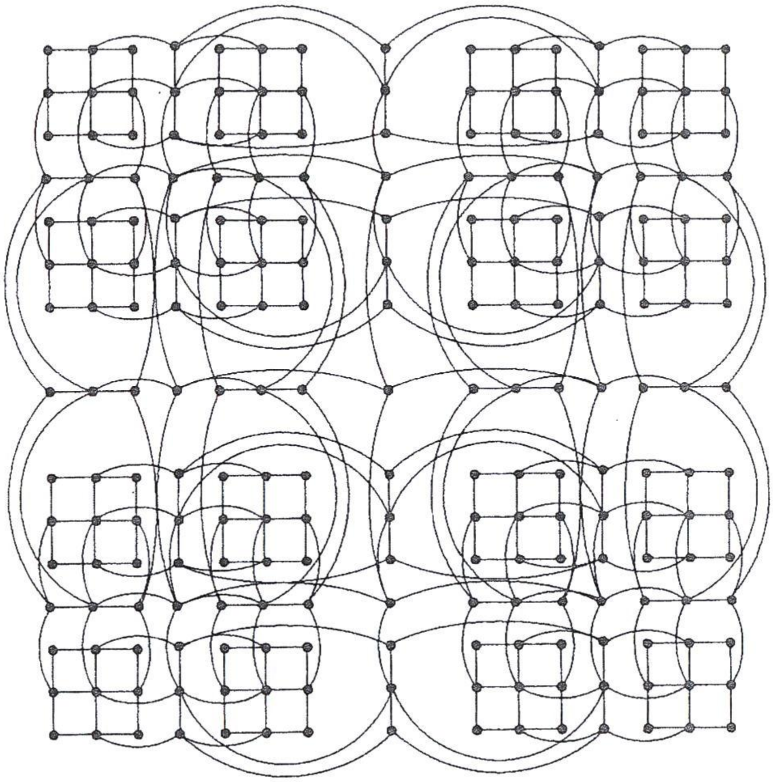

Figure 4: $M(4)$

subnetwork is $3(2^{k-1}-1)(2^{k-1}-3)+2^{k-2}$. In the case of the number of crossings of additional row edges with the edges of TR, the count is the same but with $(k-2)^3+(k-2)$ extra crossings. This is because the extra crossings in TR follow the pattern $2=2^1$, $4=2^2$, $8=2^3$ for $k=3,4$ etc. Hence

$$Cr_D\,(ARE, M(k+1)) = 4(3(2^{k-1}-1)(2^{k-1}-3)+2^{k-2}) \\ +2\left((k-2)^3+(k-2)\right)$$

Similarly the number of crossings of the additional column trees with the edges of $M(k+1)$ is given by

$$Cr_D\,(ACE, M(k+1)) = 4(3(2^{k-1}-1)(2^{k-1}-3)+2^{k-2}) \\ +2\left((k-2)^3+(k-2)\right)$$

The next count is the number of crossings of the additional row edges with additional column edges. In TL this count is $(2^{k-1}-1)(2^{k-1}-1)$. The count in TR is $(2^{k-1}-1)^2$ along with $(2^{k-1}-1)$ extra crossings. In BL the count is $2^{k-1}(2^{k-1}-1)$ and in BR the count is $2^{k-1}(2^{k-1}-1)$ along with $(2^{k-1}-1)$ extra crossings. Thus

$$\begin{aligned} Cr_D\,(ARE, ACE) &= 2\left(2^{k-1}-1\right)^2 + \left(2^{k-1}-1\right) + 2\cdot 2^{k-1}\left(2^{k-1}-1\right) \\ &\quad + \left(2^{k-1}-1\right) \\ &= \left(2^{k-1}-1\right)2^{k+1} \end{aligned}$$

The number of crossings of the additional middle row trees with additional row edges and with additional column edges of $M(k+1)$ is $(k-2)2^{k-1}+1$ each. Hence $Cr_D\,(AME, M(k+1)) = 4\left[(k-2)2^{k-1}+1\right]$.
Thus $Cr_D\,[ARE, M(k+1)] + Cr_D\,[ACE, M(k+1)] + Cr_D\,[ARE, ACE] + Cr_D\,[AME, M(k+1)]$

$$\begin{aligned} &= 4(3(2^{k-1}-1)(2^{k-1}-3)+2^{k-2}) + 2((k-2)^3+(k-2)) \\ &+ 4(3(2^{k-1}-1)(2^{k-1}-3)+2^{k-2}) + 2((k-2)^3+(k-2)) \\ &+ (2^{k-1}-1)2^{k+1} + 4((k-2)2^{k-1}+1) \\ &= 8(3(2^{k-1}-1)(2^{k-1}-3)+2^{k-2}) + 4((k-2)^3+(k-2)) \\ &+ (2^{k-1}-1)2^{k+1} + 4((k-2)2^{k-1}+1) \\ &= 4[(6(2^{k-1}-1)(2^{k-1}-3)+2^{k-1}) + (k-2)^3 + (k-2) \\ &+ (2^{k-1}-1)2^{k-1} + (k-2)2^{k-1} + 1] \end{aligned}$$

We now proceed to the main proof. By induction hypothesis

$$\begin{aligned} Cr\,(M(k)) = 2^{2k-2} + \sum_{i=1}^{k-3} 4^{k-2-i}[6\left(2^{i+1}-1\right)\left(2^{i+1}-3\right) + (2^{i+1}) \\ + (i^3+i) + 2^{i+1}(2^{i+1}-1) + (i\cdot 2^{i+1}+1)] \end{aligned}$$

Now,

$$
\begin{aligned}
Cr(M(k+1)) &= 4(CrM(k)) + \text{Additional crossings in } M(k+1) \\
&= 4(2^{2k-2}) + 4[\sum_{i=0}^{k-3} 4^{k-2-i}[6\,(2^{i+1}-1)\,(2^{i+1}-3) \\
&+ (i^3+i) + 2^{i+1}(2^{i+1}-1) + (i(2^{i+1})+1)] \\
&+ 4\left[(6(2^{k-1}-1)(2^{k-1}-3) + 2^{k-1}) + (k-2)^3 + (k-2)\right] \\
&+ 4\left[(2^{k-1}-1)2^{k-1} + (k-2)2^{k-1} + 1\right] \\
&= (2^{2k}) + \sum_{i=0}^{k-3} 4^{k-1-i}[6\,(2^{i+1}-1)\,(2^{i+1}-3) \\
&+ (i^3+i) + 2^{i+1}(2^{i+1}-1) + (i(2^{i+1})+1)] \\
&+ 4[6(2^{k-1}-1)(2^{k-1}-3) + 2^{k-1}) + (k-2)^3 + (k-2) \\
&+ 2^{k-1}(2^{k-1}-1) + (k-2)2^{k-1} + 1] \\
&= 2^{2k} + \sum_{i=0}^{k-2} 4^{k-1-i}[6\,(2^{i+1}-1)\,(2^{i+1}-3) \\
&+ (i^3+i) + 2^{i+1}(2^{i+1}-1) + (i(2^{i+1})+1)]
\end{aligned}
$$

□

The table in Fig.5 gives the number of crossings of the mesh connected tree $M(h)$ in the given form and in the new drawing. The comparison is illustrated by a graph drawn in Microsoft Excel. See Fig.5.

4 Mesh of trees

The 2-dimensional mesh of trees has a very natural and regular structure. Let $N = 2^n$. The $N \times N$ mesh of trees M_n, is constructed from an $N \times N$ grid of vertices by adding vertices and edges to form a complete binary tree in each row and each column; see Fig.6(a). The leaves of the tree are precisely the original vertices of the grid and added vertices are precisely the internal vertices of the trees [7]. This network has $3N^2 - 3N$ vertices. The leaf and root vertices have degree 2 and all other vertices have degree 3.

Let G be an $N \times N$ mesh of trees. This graph is modified by adding new edges to G so that the modified graph is 3-regular. This graph is called a 3-regular mesh of trees and is denoted by $MT(n)$ [8].

Cr(D)	h=3	h=4	h=5	h=6
M(h) given	28	380	3196	21740
M(h) proposed	16	228	2116	13972

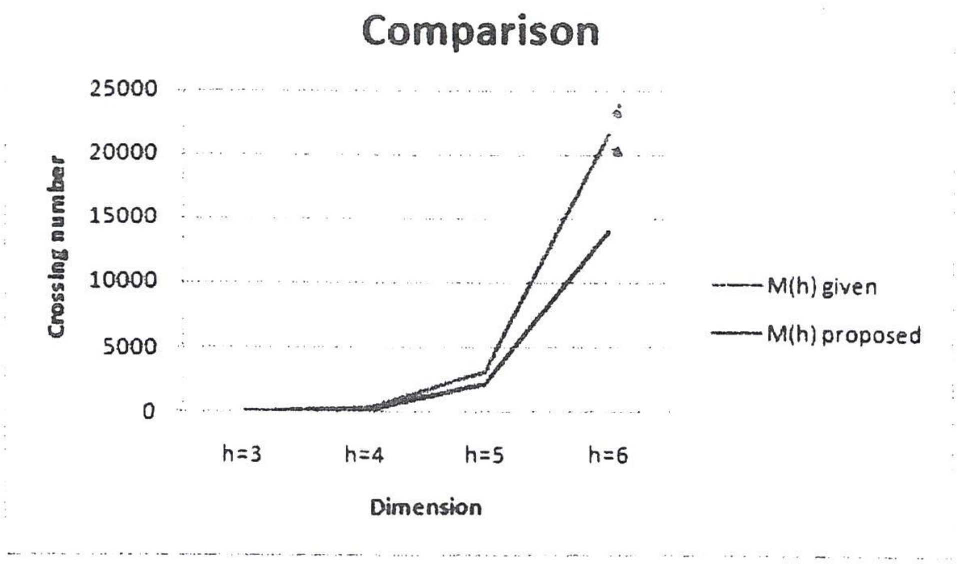

Figure 5: Comparison of crossing numbers of the two diagrams of $M(h)$

The graph $MT(n)$ contains 2^n row trees R_i and 2^n column trees C_j, $1 \leq i,j \leq 2^n$. The R_i's are listed from top to bottom and the C_j's are listed from left to right.

The edges of a complete binary tree between levels $i-1$ and i are called level(i) edges, or L_i edges, $1 \leq i \leq n$. The graph $MT(n)$ is constructed from four copies of $MT(n-1)$ by including additional edges which form paths of length 2 and these are level(1) edges of the complete binary tree of height n, the middle vertex of each path of length 2 being the root of the complete binary tree. Further consecutive roots are joined by an edge to make the graph 3-regular. Fig.6(b) depicts a 3-regular mesh of trees $MT(2)$.

In what follows the notation $L_i(R_j)$ would stand for the set of level(i) edges of the jth row tree. Similarly $L_i(C_j)$ is defined.

Lemma 4.1. *Let G be $MT(2)$. Then $Cr(G) \leq 8$.*

Proof. Let D be a good drawing of G as in Fig.6(b). No two row (column) trees intersect. The second and the third row trees cross the second and the third column trees. This gives a count of 4. Further each pair of chords

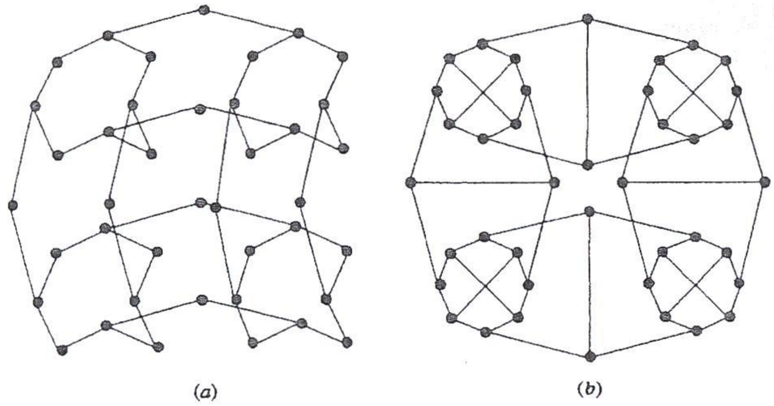

Figure 6: (a) The M_4 (b) 3-Regular $MT(2)$

in each 8-cycle contributes 1 to the crossing number of D. □

Lemma 4.2. *Let G be $MT(3)$. Then $Cr(G) \leq 80$.*

Proof. Let D be a good drawing of G. There are four copies of $MT(2)$ in $MT(3)$. In view of symmetry we need only to count the number of crossings of the additional edges with the edges of, say, the top left copy of $MT(2)$. The row (column) trees R_1 and R_8 (C_1 and C_8) do not contribute for crossing in $MT(3)$. The L_1 edges of R_3 and R_4 cross the L_1 edges of C_3 and C_4. This count is 4. L_1 edges of R_2 and R_3 cross L_2 edges of C_3 and C_4. This count is 4. Similarly L_1 edges of C_2 and C_3 cross L_2 edges of R_3 and R_4, the count being 4. Hence
$Cr_D\,(MT(3)) = 4Cr_D\,(MT(2))$ + Additional crossings in $MT(3)$

$$\begin{aligned} &= (4 \times 8) + 4\,(4+4+4) \\ &= 32 + 48 \\ &= 80 \end{aligned}$$

□

Theorem 4.3. *Let G be $MT(n)$. Then $Cr(G) \leq 2^{2n-2}\{(n-1)^2+1\}$*

Proof. We prove the above theorem by induction on n, where $N = 2^n$. We assume that the theorem is true for $n = k, k \geq 3$. Let D be a good drawing of $M(k+1)$. By induction hypothesis, $Cr_D[MT(k)] \leq 2^{2k-2}\{(k-1)^2+1\}$. Now

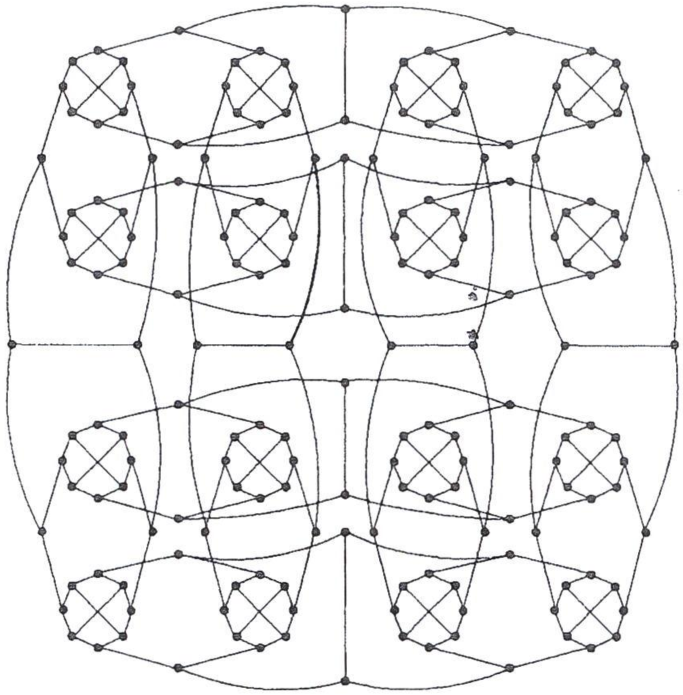

Figure 7: 3-Regular $MT(3)$

$Cr_D[MT(k+1)] = 4Cr_D[MT(k)] +$ Additional crossings in $MT(k+1)$
We first compute the additional crossings in $MT(k+1)$. In the view of symmetry we count the number of crossings of the row trees and column trees in a quadrant, say, the top left copy of $MT(k-1)$. There are $L_1, L_2, L_3, \ldots, L_{k-1}$ level in $MT(k-1)$. The L_1 edges of R_1 and C_1 do not contribute to crossing. The L_1 edges of $R_2, R_3, R_4, R_5, \ldots, R_{2^{k-1}}$ and $C_2, C_3, C_4, C_5, \ldots, C_{2^{k-1}}$ cross $L_{(k-1)}, L_{(k-2)}, L_{(k-3)}, L_{(k-4)}, \ldots, L_2$ edges of columns $C_{2^{k-2}+1}$ to $C_{2^{k-1}}$ and rows $R_{2^{k-2}+1}$ to $R_{2^{k-1}}$ edges in the following order.
L_1 edges of $R_{2^{k-2}+1}$ to $R_{2^{k-1}}$ cross the L_1 edges of $C_{2^{k-2}+1}$ to $C_{2^{k-1}}$, the count being $2^{k-2}(2^{k-2})$.
Now the L_1 edges of $R_{2^{k-3}+1}$ to $R_{3.2^{k-3}}$ cross the L_2 edges of $C_{2^{k-2}+1}$ to $C_{2^{k-1}}$, the count being $2^{k-2}(2^{k-2})$. Similarly the L_1 edges of $C_{2^{k-3}+1}$ to $C_{3.2^{k-3}}$ cross the L_2 edges of $R_{2^{k-2}+1}$ to $R_{2^{k-1}}$, count being $2^{k-2}(2^{k-2})$.
Hence the number of crossings of L_1 edges and L_2 edges in the top left corner of $MT(k)$ is given by $2 \cdot 2^{k-2}(2^{k-2})$.
A closer look at the crossings in $MT(k)$ yields the following observations regarding

1. L_{k-1} edges:
   L_1 edges of $R_{2^{j-1}-2}$ and $R_{2^{j-1}-1}, j = 3, 4, 5, \ldots, k$ cross the L_{k-1}

edges of $C_{2^{k-2}+1}$ to $C_{2^{k-1}}$. The 2^{k-2} rows are grouped into 2-subsets numbering 2^{k-3}.

2. L_{k-2} edges:
   L_1 edges of $R_{2^{j-1}-5}$ to $R_{2^{j-1}-2}$, $j = 4, 5, 6, \ldots, k$ cross the L_{k-2} edges of $C_{2^{k-2}+1}$ to $C_{2^{k-1}}$. The 2^{k-2} rows are grouped into 4-subsets numbering 2^{k-4}.
   This process continues ... ,

3. L_4 edges:
   L_1 edges of $R_{2^{k-5}+1}$ to $R_{3.2^{k-5}}$, $R_{5.2^{k-5}+1}$ to $R_{7.2^{k-5}}$, $R_{9.2^{k-5}+1}$ to $R_{11.2^{k-5}}$, $R_{13.2^{k-5}+1}$ to $R_{15.2^{k-5}}$ cross the L_4 edges of $C_{2^{k-2}+1}$ to $C_{2^{k-1}}$. The 2^{k-2} rows are grouped into 2^{k-4}-subsets numbering 4.

4. L_3 edges:
   L_1 edges of $R_{2^{k-4}+1}$ to $R_{3.2^{k-4}}$, $R_{5.2^{k-4}+1}$ to $R_{7.2^{k-4}}$, cross L_3 edges of $C_{2^{k-2}+1}$ to $C_{2^{k-1}}$. The 2^{k-2} rows are grouped into 2^{k-3}-subsets numbering 2.

Thus in each of the above $(k-2)$ cases all the 2^{k-2} rows cross all the 2^{k-2} columns, giving a count of $(k-2)2^{k-2}(2^{k-2})$. The same pattern is observed in column trees also. Hence the additional crossings in the first quadrant is $2^{k-2}(2^{k-2}) + 2(k-2)2^{k-2}(2^{k-2})$
The additional crossings in $MT(k)$ is $4[2^{k-2}(2^{k-2}) + 2(k-2)2^{k-2}(2^{k-2})] = 2^{2k-2} + (k-2)(2^{2k-1})$.
$Cr_D[MT(k+1)] = 4Cr_D[MT(k)]$ + Additional crossings in $MT(k+1)$

$$
\begin{aligned}
&= 4\left(2^{2k-2}\{(k-1)^2+1\}\right) + (k+1-2)(2^{2(k+1)-1}) + 2^{2(k+1)-2} \\
&= 2^{2k}\{(k^2+1-2k)+1\} + (k-1)2^{2k+1} + 2^{2k} \\
&= 2^{2(k+1)-2}\{((k+1)-1)^2+1\}
\end{aligned}
$$

□

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