Double Italian and double Roman domination in digraphs

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Abstract

A double Italian dominating function on a digraph D with vertex set $V(D)$ is defined as a function $f : V(D) \to \{0,1,2,3\}$ such that each vertex $u \in V(D)$ with $f(u) \in \{0,1\}$ has the property that $\sum_{x \in N^-[u]} f(x) \geq 3$, where $N^-[u]$ is the closed in-neighborhood of u. The weight of a double Italian dominating function is the sum $\sum_{v \in V(D)} f(v)$, and the minimum weight of a double Italian dominating function f is the double Italian domination number, denoted by $\gamma_{dI}(D)$. We initiate the study of the double Italian domination number for digraphs, and we present different sharp bounds on $\gamma_{dI}(D)$. In addition, several relations between the double Italian domination number and other domination parameters such as double Roman domination number, Italian domination number and domination number, are established.

Keywords: Digraph, double Italian dominating function, double Italian domination number, double Roman domination number

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1 Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [10]. Specifically, let D be a finite digraph with neither loops nor multiple arcs (but pairs of opposite arcs are allowed) with vertex set $V(D) = V$ and arc set $A(D) = A$. The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the *order* and the *size* of the digraph D, respectively. For two different vertices $u, v \in V(D)$, we use $\imath$ to denote the arc with tail u and head v, and we also call v an *out-neighb.* of u and u an *in-neighbor* of v. For $v \in V(D)$, the *out-neighborhood* ar *in-neighborhood* of v, denoted by $N_D^+(v) = N^+(v)$ and $N_D^-(v) = N^-(v$ are the sets of out-neighbors and in-neighbors of v, respectively. The *clos* *out-neighborhood* and *closed in-neighborhood* of a vertex $v \in V(D)$ are tl sets $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N_D^-[v] = N^-[v] = N^-(v)$ $\{v\}$, respectively. In general, for a set $X \subseteq V(D)$, we define $N_D^+(X)$: $N^+(X) = \bigcup_{v \in X} N^+(v)$ and $N_D^-(X) = N^-(X) = \bigcup_{v \in X} N^-(v)$. The *ou* *degree* and *in-degree* of a vertex v are defined by $d_D^+(v) = d^+(v) = |N^+(v$ and $d_D^-(v) = d^-(v) = |N^-(v)|$. The *maximum out-degree*, *maximum in* *degree*, *minimum out-degree* and *minimum in-degree* of a digraph D ar denoted by $\Delta^+(D) = \Delta^+$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\delta^-(D) = \delta^-$ respectively. The *associated digraph* $D(G)$ of a graph G is the digrapl obtained when each edge e of G is replaced by two oppositely oriented arc with the same ends as e. The *underlying graph* of a digraph D is that grapl obtained by replacing each arc uv or symmetric pairs uv, vu of arcs by th edge uv. A digraph D is *bipartite* if its underlying graph is bipartite. If X is a nonempty subset of the vertex set $V(D)$ of a digraph D, then $D[X$ is the subdigraph of D *induced* by X. Let K_n^* be the complete digraph o order n and $K_{p,q}^*$ the complete bipartite digraph with partite sets X and Y, where $|X| = p$ and $|Y| = q$.

A set $S \subseteq V(D)$ of a digraph D is a *dominating set* of D if $N^+[S] = V(D)$. The *domination number* $\gamma(D)$ of a digraph D is the minimum cardinality of a dominating set of D. The domination number of a digraph was introduced by Fu [8].

A *double Roman dominating function* (DRDF) on a digraph D is defined in [9] as a function $f : V(D) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two in-neighbors assigned 2 under f or one in-neighbor assigned 3, while if $f(v) = 1$, then the vertex v must have at least one in-neighbor assigned 2 or 3. The weight of a double Roman dominating function is the sum $\sum_{v \in V(D)} f(v)$, and the minimum weight of a double Roman dominating function f is the *double Roman domination number*, denoted by $\gamma_{dR}(D)$. A DRDF of D with weight $\gamma_{dR}(D)$ is called a *$\gamma_{dR}(D)$-function* of D.

An *Italian dominating function* on a digraph D is defined in [14] as a function $f : V(D) \to \{0, 1, 2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under f or one in-neighbor assigned 2. The weight of an Italian dominating function is the sum $\sum_{v \in V(D)} f(v)$, and the minimum weight of an Italian dominating function f is the *Italian domination number*, denoted by $\gamma_I(D)$. An Italian dominating function of D with weight $\gamma_I(D)$ is called a *$\gamma_I(D)$-function* of D.

An Italian dominating function f of a digraph D can be represented by the ordered partition (V_0, V_1, V_2) of $V(D)$, where $V_i = \{v \in V(D) \mid f(v) = i\}$ for $i \in \{0, 1, 2\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$.

In this paper we continue the study of Roman and Italian dominating functions in graphs and digraphs (see, for example, [1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 13, 14]). Inspired by an idea of the work [3], we define the double Italian domination number of a digraph as follows. A *double Italian dominating function* (DIDF) on a digraph D is defined as a function $f : V(D) \to \{0, 1, 2, 3\}$ such that each vertex $u \in V(D)$ with $f(u) \in \{0, 1\}$ has the property that $\sum_{x \in N^-[u]} f(x) \geq 3$. The weight of a DIDF is the sum $\sum_{v \in V(D)} f(v)$, and the minimum weight of a DIDF f is the *double Italian domination number*, denoted by $\gamma_{dI}(D)$. A double Italian dominating function of D with weight $\gamma_{dI}(D)$ is called a $\gamma_{dI}(D)$-*function* of D. A double Italian dominating function f of a digraph D can be represented by the ordered partition (V_0, V_1, V_2, V_3) of $V(D)$, where $V_i = \{v \in V(D) \mid f(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2| + 3|V_3|$. Clearly, every double Roman dominating function is a double Italian dominating function of D and thus $\gamma_{dI}(D) \leq \gamma_{dR}(D)$. Therefore all the upper bounds for $\gamma_{dR}(D)$ in [9] are also upper bounds for $\gamma_{dI}(D)$.

Our purpose in this paper is to initiate the study of the double Italian domination number for digraphs. Several relations between the double Italian domination number and other domination parameters such as double Roman domination number, Italian domination number and domination number, are established. Furthermore, we present different sharp bounds on $\gamma_{dI}(D)$. Finally, we improve a lower bound of the double Roman domination number of Hao, Chen and Volkmann [9].

2 Relations to other domination parameters

In this section, we shall relate the double Italian domination number to other domination parameters such as Italian domination number, double Roman domination number and domination number.

Proposition 1. Let D be a digraph and let $f = (V_0, V_1, V_2)$ be a $\gamma_I(D)$-function. Then $\gamma_{dI}(D) \leq \gamma_{dR}(D) \leq 2|V_1| + 3|V_2|$.

Proof. Define the function $g : V(D) \to \{0, 1, 2, 3\}$ by $g(v) = 0$ for $v \in V_0$, $g(v) = 2$ for $v \in V_1$ and $g(v) = 3$ for $v \in V_2$. Then g is a DRDF on D and thus $\gamma_{dI}(D) \leq \gamma_{dR}(D) \leq 2|V_1| + 3|V_2|$. □

Theorem 2. If D is a digraph, then $\gamma_{dI}(D) \leq \gamma_{dR}(D) \leq 2\gamma_I(D)$.

If $\gamma_{dI}(D) = 2\gamma_I(D)$ and $f = (V_0, V_1, V_2)$ is a $\gamma_I(D)$-function, then $|V_2| = 0$ and the subdigraph $D[V_1]$ is empty.

Proof. If $f = (V_0, V_1, V_2)$ is a $\gamma_I(D)$-function, then Proposition 1 implies

$$\begin{aligned}\gamma_{dI}(D) &\leq \gamma_{dR}(D) \leq 2|V_1| + 3|V_2| = 2(|V_1| + 2|V_2|) - |V_2| \\ &= 2\gamma_I(D) - |V_2| \leq 2\gamma_I(D),\end{aligned}$$

and this is the desired bound.

Now let $\gamma_{dI}(D) = 2\gamma_I(D)$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_I(D)$-function. Then the inequality chain above shows that $|V_2| = 0$. Suppose that there exists an arc uv in the subdigraph $D[V_1]$. Define the function $g : V(D) \to \{0, 1, 2, 3\}$ by $g(x) = 0$ for $x \in V_0$, $g(v) = 1$ and $g(y) = 2$ for $y \in V_1 \setminus \{v\}$. Since v has the in-neighbor u of weight 2 and $g(N^-[x]) \geq 3$ for each vertex $x \in V_0$, g is a DIDF on D of weight $2\gamma_I(D) - 1$. This is a contradiction, and thus $D[V_1]$ is empty. □

Example 3. Let H be the digraph consisting of an arbitrary digraph Q with vertex set $V(Q) = \{v_1, v_2, \ldots, v_k\}$ and a further vertex set $V_1 = \{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$ such that $x_iv_i, y_iv_i \in A(H)$ for $1 \leq i \leq k$. It is easy to see that $\gamma_{dI}(H) = 2\gamma_I(H) = 4k$. This example demonstrates that Theorem 2 is sharp.

Theorem 4. If D is a digraph, then $\gamma_{dI}(D) \geq \gamma_I(D) + 1$.

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{dI}(D)$-function. If $|V_3| \geq 1$, then every vertex in V_3 can be reassigned the value 2, and the resulting function is an IDF on D. This implies

$$\gamma_{dI}(D) = |V_1| + 2|V_2| + 3|V_3| \geq \gamma_I(D) + |V_3| \geq \gamma_I(D) + 1,$$

and this is the desired bound. Next suppose that $|V_3| = 0$ and $|V_2| \geq 1$. Let $w \in V_2$, and define the function $g : V(D) \to \{0, 1, 2\}$ by $g(v) = 0$ for $v \in V_0$, $g(v) = 1$ for $v \in V_1$, $g(w) = 1$ and $g(x) = 2$ for $x \in V_2 \setminus \{w\}$. Then g is an IDF on D of weight $\gamma_{dI}(D) - 1$ and thus $\gamma_{dI}(D) \geq \gamma_I(D) + 1$.

Finally, let $|V_3| = |V_2| = 0$. Let $w \in V_1$, and define the function $h : V(D) \to \{0, 1, 2\}$ by $h(v) = 0$ for $v \in V_0$, $h(w) = 0$ and $h(x) = 1$ for $x \in V_1 \setminus \{w\}$. Since w has at least two in-neighbors in V_1, the function h is an IDF on D of weight $\gamma_{dI}(D) - 1$ and so $\gamma_{dI}(D) \geq \gamma_I(D) + 1$. □

We now establish relations between the double Italian domination number and the domination number.

Theorem 5. If D is a digraph, then $\gamma_{dI}(D) \leq 3\gamma(D)$, with equality if and only if there exists a $\gamma_{dI}(D)$-function $f = (V_0, V_1, V_2, V_3)$ with $|V_1| = |V_2| = 0$.

Proof. Let S be a dominating set such that $|S| = \gamma(D)$. Define $f = (V_0 = V(D) \setminus S, \emptyset, \emptyset, V_3 = S)$. Then f is a DIDF on D and thus $\gamma_{dI}(D) \leq 3|S| = 3\gamma(D)$.

Now suppose that $\gamma_{dI}(D) = 3\gamma(D)$, and let S be a dominating set such that $|S| = \gamma(D)$. Define the function $g : V(D) \rightarrow \{0,1,2,3\}$ by $g(v) = 3$ for $v \in S$ and $g(v) = 0$ for $v \in V(D) \setminus S$. Then g is a desired $\gamma_{dI}(D)$-function.

Conversely, assume that $|V_1| = |V_2| = 0$. Then V_3 is a dominating set of D and so $|V_3| \geq \gamma(D)$. Therefore $\gamma_{dI}(D) = 3|V_3| \geq 3\gamma(D)$. Since $\gamma_{dI}(D) \leq 3\gamma(D)$, we obtain $\gamma_{dI}(D) = 3\gamma(D)$. □

Theorem 6. If D is a digraph of order $n \geq 2$, then $\gamma(D) + 2 \leq \gamma_{dI}(D)$.

Proof. Let $g = (V_0, V_1, V_2, V_3)$ be a $\gamma_{dI}(D)$-function. We distinguish two cases.

Case 1. Let $|V_2| \geq 2$ or $|V_3| \geq 1$. Then

$$\gamma(D) \leq |V_1| + |V_2| + |V_3| \leq |V_1| + 2|V_2| + 3|V_3| - 2 = \gamma_{dI}(D) - 2.$$

Case 2. Let $|V_3| = 0$ or $|V_2| \leq 1$. If $|V_1| = 0$, then $n = 1$, a contradiction to $n \geq 2$. If $|V_1| = 1$, then $|V_2| = 1$ and V_2 is a dominating set of D. Thus $\gamma(D) = 1 = 3 - 2 = \gamma_{dI}(D) - 2$. Let now $|V_1| \geq 2$. If $u, v \in V_1$ are two distinct vertices, then $V_2 \cup (V_1 \setminus \{u, v\})$ is a dominating set of D and therefore $\gamma(D) \leq \gamma_{dI}(D) - 2$. □

Example 7. Let C_n be a cycle of order n, and let $D(C_n)$ be its associated digraph. Let H be the digraph consisting of $D(C_n)$ with $n \geq 3$ and a vertex set V_0 of $\binom{n}{3}$ further vertices. Let each vertex of V_0 have exactly three in-neighbors in $V(D(C_n))$ such that the in-neighborhoods of every two different vertices of V_0 are distinct.

The function $f : V(H) \rightarrow \{0,1,2,3\}$ with $f(x) = 1$ for $x \in V(D(C_n))$ and $f(x) = 0$ for $x \in V_0$ is a $\gamma_{dI}(H)$-function and thus $\gamma_{dI}(H) = n$.

If u and v are two distinct vertices of $D(C_n)$, then it is easy to see that $V(D(C_n)) \setminus \{u, v\}$ is a dominating set of H of cardinality $\gamma(H)$ and so $\gamma(H) = n - 2$. Consequently, $\gamma_{dI}(H) = n = (n-2) + 2 = \gamma(H) + 2$. This example shows that Theorem 6 is sharp.

Theorem 8. If D is a digraph, then $\gamma_{dR}(D) \leq 2\gamma_{dI}(D) - 2$.

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{dI}(D)$-function. If $V_1 = \emptyset$, then f is also a DRDF on D and so $\gamma_{dR}(D) = \gamma_{dI}(D) \leq 2\gamma_{dI}(D) - 2$. Now let $V_1 = \{v_1, v_2, \ldots, v_t\} \neq \emptyset$. Define the function $g : V(D) \rightarrow \{0,1,2,3\}$ by $g(v) = 0$ for $v \in V_0$ and $g(v_1) = 0$, $g(v_2) = g(v_3) = \ldots = g(v_t) = 2$ and $g(v) = 3$ for $v \in V_2 \cup V_3$. Since v_1 has at least one in-neighbor in $V_2 \cup V_3$

or two in-neighbors in V_1, we note that g is a DRDF on D such that

$$\begin{aligned}\gamma_{dR}(D) &\leq 2|V_1|-2+3|V_2|+3|V_3| \\ &= 2(|V_1|+2|V_2|+3|V_3|)-2-|V_2|-3|V_3| \\ &= 2\gamma_{dI}(D)-2-|V_2|-3|V_3| \leq 2\gamma_{dI}(D)-2.\end{aligned}$$

This is the desired bound, and the proof is complete. □

Let $Q = K^*_{n_1,n_2,\ldots,n_r}$ be the complete r-partite digraph with $r \geq 4$ and $3 \leq n_1 \leq n_2 \leq \ldots \leq n_r$. Then $\gamma_{dI}(Q) = 4$ and $\gamma_{dR}(Q) = 6$ and thus $6 = \gamma_{dR}(Q) = 2 \cdot 4 - 2 = 2\gamma_{dI}(Q) - 2$. This example shows that Theorem 8 is sharp.

3 Upper and lowers bounds

In this section we present upper and lower bounds on the double Italian domination number in terms of its order, maximum out-degree and minimum in-degree.

Proposition 9. If D is a digraph of order n, then $\gamma_{dI}(D) \leq 2n$ with equality if and only if D is empty.

Proof. Define the function $f : V(D) \to \{0,1,2,3\}$ by $f(x) = 2$ for each $x \in V(D)$. Then D is a DIDF on D and thus $\gamma_{dI}(D) \leq 2n$. If D is empty, then obviously $\gamma_{dI}(D) = 2n$. Now assume that $\gamma_{dI}(D) \leq 2n$, and suppose to the contrary that D contains an arc uv. Define the function $g : V(D) \to \{0,1,2,3\}$ by $g(v) = 0$, $g(u) = 3$ and $g(x) = 2$ for $x \in V(D) \setminus \{u, v\}$. Them g is a DIDF of weight $2n - 1$, a contradiction. □

Proposition 10. If D is a bipartite digraph of order n with $\delta^-(D) \geq 1$, then $\gamma_{dI}(D) \leq \gamma_{dR}(D) \leq (3n)/2$.

Proof. Let D be a bipartite digraph with the partite sets X and Y. Assume, without loss of generality, that $|X| \leq |Y|$. Define the function $f : V(D) \to \{0,1,2,3\}$ by $f(x) = 3$ for $x \in X$ and $f(y) = 0$ for $y \in Y$. Since $\delta^-(D) \geq 1$, every vertex $y \in Y$ has an in-neighbor in X, and hence f is DRDF on D of weight $3|X|$. Consequently,

$$\gamma_{dI}(D) \leq \gamma_{dR}(D) \leq 3|X| \leq \frac{3}{2}(|X|+|Y|) = \frac{3}{2}n.$$

□

If C_n^* is an oriented cycle of even length, then $\gamma_{dI}C_n^*) = \gamma_{dR}(C_n^*) = (3n)/2$, and therefore Proposition 10 is sharp. If C_n^* is an oriented cycle of odd length, then $\gamma_{dI}(C_n^*) = \gamma_{dR}(C_n^*) = (3(n-1))/2 + 2$, and thus Proposition 10 is not valid in general.

Theorem 11. If D is a digraph with $\delta^-(D) \geq 2$, then

$$\gamma_{dI}(D) \leq |V(D)| + 2 - \delta^-(D).$$

Proof. Let $V(D) = \{u_1, u_2, \dots, u_n\}$, and let $U = \{u_1, u_2, \dots, u_{n+2-\delta^-(D)}\}$. Define the function $f : V(D) \to \{0,1,2,3\}$ by $f(x) = 1$ for $x \in U$ and $f(x) = 0$ for $x \in V(D) \setminus U$. Then $\sum_{x \in N^-(u)} f(x) \geq 2$ for $u \in U$ and $\sum_{x \in N^-(v)} f(x) \geq 3$ for $v \in V(D) \setminus U$. Hence f is a DIDF on D of weight $n + 2 - \delta^-(D)$ and so $\gamma_{dI}(D) \leq |V(D)| + 2 - \delta^-(D)$. □

We observe that $\gamma_{dI}(K_n^*) = 3 = |V(K_n^*)| + 2 - \delta^-(K_n^*)$ for $n \geq 3$.

If $K^*_{n_1,n_2,\dots,n_r}$ is the complete r-partite digraph with the property that $n_1 = n_2 = \dots = n_r = 2$ and $r \geq 2$, then

$$\gamma_{dI}(K^*_{n_1,n_2,\dots,n_r}) = 4 = |V(K^*_{n_1,n_2,\dots,n_r})| + 2 - \delta^-(K^*_{n_1,n_2,\dots,n_r}).$$

$\gamma_{dI}(K^*_{3,3}) = 5 = |V(K^*_{3,3})| + 2 + \delta^-(K^*_{3,3})$,
$\gamma_{dI}(K^*_{4,4}) = 6 = |V(K^*_{4,4})| + 2 + \delta^-(K^*_{4,4})$,
$\gamma_{dI}(K^*_{3,3,3}) = 5 = |V(K^*_{3,3,3})| + 2 + \delta^-(K^*_{3,3,3})$.
All these digraphs demonstrate that Theorem 11 is sharp.

Theorem 12. If D is a digraph of order n with maximum out-degree $\Delta^+(D) = \Delta^+$, then

$$\gamma_{dI}(D) \geq \min\left\{\frac{2n + 2\Delta^+ + 6}{\Delta^+ + 2}, \frac{2n + \Delta^+}{\Delta^+ + 1}\right\}.$$

Proof. If $\Delta^+ = 0$, then D is empty, and therefore $\gamma_{dI}(D) = 2n$ by Proposition 9. Since $(2n + \Delta^+)/(\Delta^+ + 1) = 2n$, the lower bound is valid in this case. Let now $\Delta^+ \geq 1$, and let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{dI}(D)$-function. Define $V_0' = \{x \in V_0 : N^-(x) \cap V_3 \neq \emptyset\}$ and $V_0'' = V_0 \setminus V_0'$. We distinguish three cases.

Case 1. Assume that $|V_2|, |V_3| \geq 1$. Since every vertex of V_3 has at most Δ^+ out-neigbors in V_0', we note that $|V_0'| \leq \Delta^+|V_3|$. As every vertex of V_0'' has at least two in-neighbors in $V_1 \cup V_2$, we observe that $2|V_0''| \leq \Delta^+(|V_1| + |V_2|)$ and hence

$$|V_0| = |V_0'| + |V_0''| \leq \Delta^+|V_3| + \frac{\Delta^+}{2}(|V_1| + |V_2|).$$

This leads to

$$\begin{aligned}
\frac{\Delta^+ + 2}{2}\gamma_{dI}(D) &= \frac{\Delta^+ + 2}{2}(|V_1| + 2|V_2| + 3|V_3|) \\
&= \frac{\Delta^+}{2}|V_1| + \Delta^+|V_2| + \frac{3\Delta^+}{2}|V_3| + |V_1| + 2|V_2| + 3|V_3| \\
&= |V_1| + |V_2| + |V_3| + \Delta^+|V_3| + \frac{\Delta^+}{2}(|V_1| + |V_2|) \\
&+ \frac{\Delta^+}{2}(|V_2| + |V_3|) + |V_2| + 2|V_3| \\
&\geq |V_1| + |V_2| + |V_3| + |V_0| \\
&+ \frac{\Delta^+}{2}(|V_2| + |V_3|) + |V_2| + 2|V_3| \\
&= n + \frac{\Delta^+}{2}(|V_2| + |V_3|) + |V_2| + 2|V_3| \geq n + \Delta^+ + 3
\end{aligned}$$

and thus $\gamma_{dI}(D) \geq (2n + 2\Delta^+ + 6)/(\Delta^+ + 2)$.

Case 2. Assume that $|V_2| \geq 1$ and $|V_3| = 0$. Since every vertex of V_1 has at least one in-neigbor in $V_1 \cup V_2$, and every vertex of V_0 has at least two in-neighbors in $V_1 \cup V_2$, we deduce that

$$2|V_0| \leq \Delta^+(|V_1| + |V_2|) - |V_1|.$$

It follows that

$$\begin{aligned}
\frac{\Delta^+ + 1}{2}\gamma_{dI}(D) &= \frac{\Delta^+ + 1}{2}(|V_1| + 2|V_2|) \\
&= \frac{\Delta^+}{2}|V_1| + \Delta^+|V_2| + \frac{|V_1|}{2} + |V_2| \\
&= |V_1| + |V_2| + \frac{\Delta^+}{2}(|V_1| + |V_2|) - \frac{|V_1|}{2} + \frac{\Delta^+}{2}|V_2| \\
&\geq |V_1| + |V_2| + |V_0| + \frac{\Delta^+}{2}|V_2| \\
&= n + \frac{\Delta^+}{2}|V_2| \geq n + \frac{\Delta^+}{2} = \frac{2n + \Delta^+}{2}
\end{aligned}$$

and so $\gamma_{dI}(D) \geq (2n + \Delta^+)/(\Delta^+ + 1)$.

Case 3. Assume that $|V_2| = 0$. Assume that exactly $t \leq |V_1|$ vertices of V_1 have an in-neighbor in V_3. Then $|V_1| - t$ vertices of V_1 have at least two in-neighbors in V_1. This implies that $|V_0'| \leq \Delta^+|V_3| - t$ and $3|V_0''| \leq \Delta^+|V_1| - 2(|V_1| - t)$ and therefore

$$\begin{aligned}
|V_0| &= |V_0'| + |V_0''| \leq \Delta^+|V_3| - t + \frac{\Delta^+|V_1|}{3} - \frac{2}{3}|V_1| + \frac{2}{3}t \\
&\leq \Delta^+|V_3| + \frac{\Delta^+|V_1|}{3} - \frac{2}{3}|V_1|.
\end{aligned}$$

Using this bound, we find that

$$
\begin{aligned}
\frac{\Delta^+ + 1}{3}\gamma_{dI}(D) &= \frac{\Delta^+ + 1}{3}(|V_1| + 3|V_3|) \\
&= \frac{\Delta^+}{3}|V_1| + \Delta^+|V_3| + \frac{|V_1|}{3} + |V_3| \\
&= \frac{\Delta^+}{3}|V_1| + \Delta^+|V_3| - \frac{2}{3}|V_1| + |V_1| + |V_3| \\
&\geq |V_0| + |V_1| + |V_3| = n
\end{aligned}
$$

and hence $\gamma_{dI}(D) \geq (3n)/(\Delta^+ + 1) \geq (2n + \Delta^+)(\Delta^+ + 1)$. Combining the Cases 1, 2 and 3, we obtain the desired lower bound. □

Corollary 13. Let D be a digraph of order $n \geq 2$. Then $\gamma_{dI}(D) = 3$ if and only if $\Delta^+(D) = n - 1$.

Proof. Clearly, $\gamma_{dI}(D) \geq 3$ by the definition. Let now $\Delta^+(D) = n-1$, and let w be a vertex of maximumn out-degree $\Delta^+(D)$. Define the function $f : V(D) \to \{0, 1, 2, 3\}$ by $f(w) = 3$ and $f(x) = 0$ for $x \in V(D) \setminus \{w\}$. Then f is a DIDF on D of weight 3. Therefore $\gamma_{dI}(D) \leq 3$ and thus $\gamma_{dI}(D) = 3$.

Conversely, assume that $\gamma_{dI}(D) = 3$. If $\Delta^+ = \Delta^+(D) \leq n-2$, then Theorem 12 leads to the contradiction

$$
\begin{aligned}
\gamma_{dI}(D) &\geq \min\left\{\left\lceil \frac{2n + 2\Delta^+ + 6}{\Delta^+ + 2} \right\rceil, \left\lceil \frac{2n + \Delta^+}{\Delta^+ + 1} \right\rceil\right\} \\
&\geq \min\left\{\left\lceil \frac{4n+2}{n} \right\rceil, \left\lceil \frac{3n-2}{n-1} \right\rceil\right\} \geq 4,
\end{aligned}
$$

and the proof is complete. □

4 A new lower bound on $\gamma_{dR}(D)$

In [9], we presented the following lower bound on $\gamma_{dR}(D)$.

Theorem 14. [9] If D is a connected digraph of order $n \geq 4$, then

$$
\gamma_{dR}(D) \geq \left\lceil \frac{6n+3}{2\Delta^+(D) + 3} \right\rceil.
$$

For $\Delta^+(D) \geq 2$, we improve this bound

Theorem 15. If D is a digraph of order n with $\Delta^+(D) \geq 2$, then

$$\gamma_{dR}(D) \geq \left\lceil \frac{3n}{\Delta^+(D)+1} \right\rceil.$$

Proof. Let $\Delta^+ = \Delta^+(D)$, and let f be $\gamma_{dR}(D)$-function. According to [9] (see Proposition 1), we can assume, without loss of generality, that $f(x) \in \{0,2,3\}$ for each vertex $x \in V(D)$. If V_i is the set of vertices assigned i by the function f, then $\gamma_{dR}(D) = 2|V_2| + 3|V_3|$ and $n = |V_0| + |V_2| + |V_3|$. Let now $V_0' = \{x \in V_0 : N^-(x) \cap V_3 \neq \emptyset\}$ and $V_0'' = V_0 \setminus V_1'$. Since every vertex of V_3 has at most Δ^+ out-neigbors in V_0', we note that $|V_0'| \leq \Delta^+|V_3|$. As every vertex of V_0'' has at least two in-neighbors in V_2, we observe that $2|V_0''| \leq \Delta^+|V_2|$ and hence

$$|V_0| = |V_0'| + |V_0''| \leq \Delta^+|V_3| + \frac{\Delta^+}{2}|V_2|.$$

This bound and the hypothesis $\Delta^+ \geq 2$ lead to

$$\begin{aligned}
\frac{\Delta^+ + 1}{3}\gamma_{dR}(D) &= \frac{\Delta^+ + 1}{3}(2|V_2| + 3|V_3|) \\
&= \frac{2\Delta^+}{3}|V_2| + \Delta^+|V_3| + \frac{2}{3}|V_2| + |V_3| \\
&= \frac{\Delta^+}{2}|V_2| + \Delta^+|V_3| + \frac{2}{3}|V_2| + \frac{\Delta^+}{6}|V_2| + |V_3| \\
&\geq |V_0| + \frac{2}{3}|V_2| + \frac{2}{6}|V_2| + |V_3| = |V_0| + |V_2| + |V_3| = n
\end{aligned}$$

and so $\gamma_{dR}(D) \geq \lceil (3n)/(\Delta^+ + 1) \rceil$. □

Since

$$\frac{3n}{\Delta^+(D)+1} \geq \frac{6n+3}{2\Delta^+(D)+3},$$

Theorem 15 is an improvement of Theorem 14 for $\Delta^+(D) \geq 2$. Clearly, if $\Delta^+(D) = 0$, then D is empty and thus $\gamma_{dR}(D) = 2n$. Therefore Theorem 15 is not valid when $\Delta^+(D) = 0$. The next example will show that Theorem 15 is not valid for $\Delta^+(D) = 1$, in general.

Example 16. Let the digraph H consisting of an oriented cycle $C_k^* = v_1v_2 \dots v_k$ of order $k \geq 3$ and $2k$ further vertices $x_1, y_1, x_2, y_2, \dots, x_k, y_k$ such that $x_iv_i, y_iv_i \in E(H)$ for $1 \leq i \leq k$. Then $n(H) = 3k$ and $\Delta^+(H) = 1$. Define the function $f : V(H) \to \{0,1,2,3\}$ by $f(x_i) = f(y_i) = 2$ and $f(v_i) = 0$ for $1 \leq i \leq k$. Clearly, f is $\gamma_{dR}(H)$-function of weight $4k$. However,

$$\frac{3n(H)}{\Delta^+(H)+1} = \frac{9k}{2} > 4k = \gamma_{dR}(H).$$

orollary 17. [13] If G is a graph of order n with maximum degree $\Delta(G) \geq$ then

$$\gamma_{dR}(G) \geq \left\lceil \frac{3n}{\Delta(G)+1} \right\rceil.$$

'roof. If $\Delta(G) \geq 2$, then let $D(G)$ be its associated digraph. We observe ıat $\gamma_{dR}(G) = \gamma_{dR}(D(G))$, $\Delta^+(D(G)) = \Delta(G)$ and $n(D(G)) = n(G)$. 'herefore the desired lower bound follows from Theorem 15.

If $\Delta(G) = 1$, then $G = pK_2 \cup tK_1$ with $p \geq 1$. Obviously, $n(G) = 2p+t$ nd $\gamma_{dR}(G) = 3p + 2t$. Thus

$$\gamma_{dR}(G) = 3p + 2t \geq \frac{6p+3t}{2} = \frac{3n(G)}{\Delta(G)+1},$$

ınd the proof is complete. □

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