Hamiltonian Walks and Hamiltonian-Connected 3-Path Graphs

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Abstract

A Hamiltonian walk in a nontrivial connected graph G is a closed walk of minimum length that contains every vertex of G. The 3-path graph $\mathcal{P}_3(G)$ of a connected graph G of order 3 or more has the set of all 3-paths (paths of order 3) of G as its vertex set and two vertices of $\mathcal{P}_3(G)$ are adjacent if they have a 2-path in common. With the aid of Hamiltonian walks in spanning trees of the 3-path graph of a graph, it is shown that if G is a connected graph with minimum degree at least 4, then $\mathcal{P}_3(G)$ is Hamiltonian-connected.

Key Words: Hamiltonian walk, line graph, 3-path graph, spanning trees, Hamiltonian graph, Hamiltonian-connected graph.

AMS Subject Classification: 05C45, 05C75

1 Introduction

Let G be a nontrivial connected graph. A *Hamiltonian walk* in G is a closed walk of minimum length that contains every vertex of G. This concept was introduced by Goodman and Hedetniemi [4]. They showed that if G is a connected graph of order n and size m, then the length of a Hamiltonian walk W in G is at least n and at most $2m$. The length of W is n if and only if G is Hamiltonian (in which case W is a Hamiltonian cycle). Every edge of G occurs at most twice in W and the length of W is $2m$ if and only if G is a tree in which case each edge of G appears exactly twice in W. Hamiltonian walks in graphs have been used to study structural properties of graphs (see [3]).

In [1] the concept of Hamiltonian walks has been applied to establish results dealing with Hamiltonian properties of certain type of derived graphs. One of the most familiar derived graphs is the line graph. The *line graph* $L(G)$ of a nonempty graph G has the set of edges in G as its vertex set where two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. Harary and Nash-Williams [5] characterized those graphs whose line graph is Hamiltonian. Their characterization primarily involved the existence of a circuit in G called a *dominating circuit* C in which every edge of G is incident with a vertex of C.

Theorem 1.1 (Harary and Nash-Williams) *Let G be a graph without isolated vertices. Then $L(G)$ is Hamiltonian if and only if G is a star $K_{1,t}$ for some $t \geq 3$ or G contains a dominating circuit.*

It is easy to give an example of a connected graph containing vertices of small degree whose line graph is not Hamiltonian. However, even a connected graph with no vertices of degree 1 or 2 need not have a Hamiltonian line graph (see [1], for example). As was stated in [2], if G is a connected graph with $\delta(G) \geq 3$, then $L(G)$ has a spanning subgraph containing an Eulerian circuit, which is then a dominating circuit of $L(G)$ and, consequently, $L(L(G))$ is Hamiltonian. While $L(L(G))$ is Hamiltonian for every connected graph G with $\delta(G) \geq 3$, the graph $L(L(G))$ need not be Hamiltonian-connected. Figure 1 shows a connected 3-regular graph G and $L(L(G))$. In this graph G, the edges of interest are labeled $1, 2, \ldots, 9$. Consequently, the corresponding vertices in $L(G)$ are labeled $1, 2, \ldots, 9$, producing edges $12, 13, 23, 14, 15, 45$, etc. in $L(G)$ and thus those vertices in $L(L(G))$. The graph $L(L(G))$ of Figure 1 is not 3-connected and therefore is not Hamiltonian-connected. For example, there is neither a 14-15 nor a 12-13 Hamiltonian path in $L(L(G))$.

The goal of this paper is to describe how the concept of Hamiltonian walks and the technique introduced in [1] can be used to show that if G is a connected graph with $\delta(G) \geq 4$, then $L(L(G))$ is Hamiltonian-connected. First, we introduce some definitions and notation. For an integer $k \geq 2$ and a graph G containing k-paths, the *k-path graph* $\mathcal{P}_k(G)$ of G has the set of k-paths of G as its vertex set where two distinct vertices of $\mathcal{P}_k(G)$ are adjacent if the corresponding k-paths of G have a $(k-1)$-path in common. Therefore, $\mathcal{P}_2(G) = L(G)$ and $\mathcal{P}_3(G) = L(L(G))$. We prove the primary

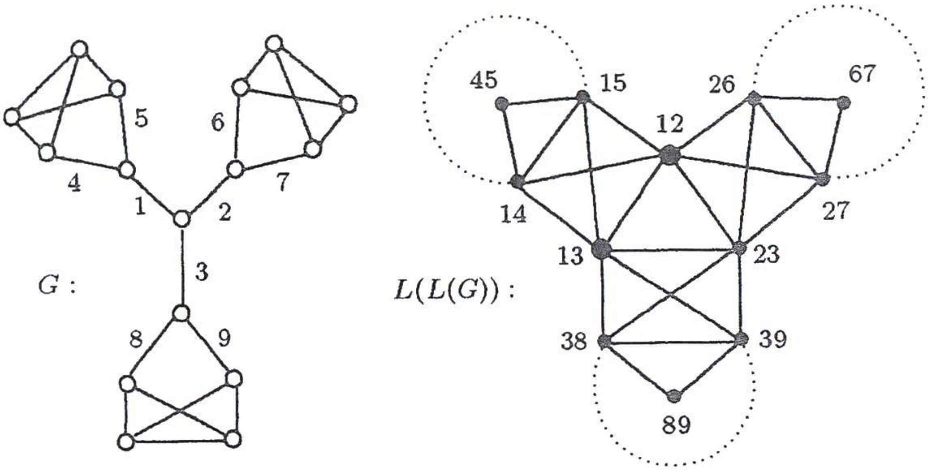


Figure 1: A connected 3-regular G and the line graph $L(L(G))$ of $L(G)$

result in this paper by observing that $L(L(G))$ is the 3-path graph $\mathcal{P}_3(G)$ of G and making use of Hamiltonian walks and certain spanning trees of a graph. Every embedding of a tree T in the plane gives rise to a Hamiltonian walk in T. For example, suppose that T is a star $K_{1,4}$ whose edges are a, b, c, d. Figures 2(a) and 2(c) show two different embeddings of T in the plane.

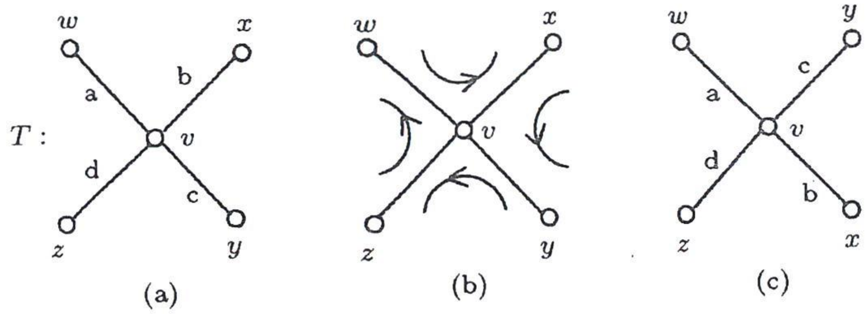


Figure 2: Two embeddings of $K_{1,4}$ in the plane

In (a), the edges a, b, c, d of T appear consecutively in clockwise order about v, while in (c), the edges a, c, b, d appear consecutively in clockwise order about v. The embedding of T in Figure 2(a) gives rise to the Hamiltonian walk $W_1 = (w, v, x, v, y, v, z, v, w)$, where the edges of T are traced in the manner shown in Figure 2(b). The embedding of T shown in Figure 2(c) gives rise to the Hamiltonian walk $W_2 = (w, v, y, v, x, v, z, v, w)$. In terms of the edges of T, these two walks can also be described as $W_1 = (a, b, b, c, c, d, d, a)$ and $W_2 = (a, c, c, b, b, d, d, a)$. While every edge of T occurs twice in both W_1 and W_2, this is not true for all 3-paths in T. For example, the 3-path $(w, v, x) = ab$ occurs in W_1 but not in W_2, while the 3-path $(w, v, y) = ac$ occurs in W_2 but not in W_1.

2 Orderings of the Edges of a Star

To prove that the 3-path graph of every connected graph G with $\delta(G) \geq 4$ is Hamiltonian-connected, it is useful to establish two lemmas, each of which describes how pairs of the edges of a star can be ordered to satisfy certain desirable conditions.

Lemma 2.1 *Let $E = \{f_1, f_2, \ldots, f_k\}$ be the edge set of a star of size $k \geq 2$. For $\ell = \binom{k}{2}$, there is a sequence $H_1, H_2, \ldots, H_\ell$ of the ℓ distinct pairs $f_i f_j$ of edges of E where*

(i) H_i and H_{i+1} have an edge in common for $i = 1, 2, \ldots, \ell - 1$ and

(ii) $H_1 = f_1 f_2$ and $H_\ell = f_1 f_k$.

Proof. We proceed by induction on k. For $k = 2$, $H_1 = f_1 f_2$ verifies the statement. For $k = 3$, the sequence $H_1 = f_1 f_2, H_2 = f_2 f_3, H_3 = f_1 f_3$ verifies the statement. Thus, the statement is true for $k = 2, 3$. Suppose that the statement is true for some integer $k \geq 3$. Let $\{f_1, f_2, \ldots, f_k, f_{k+1}\}$ be the edge set of a star of size $k+1 \geq 4$. Applying the induction hypothesis to the set $\{f_1, f_2, \ldots, f_k\}$, there is a sequence $s_0 : H_1, H_2, \ldots, H_\ell$, where $\ell = \binom{k}{2}$, consisting of the ℓ distinct ordered pairs $f_i f_j$ where $1 \leq i \neq j \leq k$ such that

(*i*) H_i and H_{i+1} have an edge in common for $i = 1, 2, \ldots, \ell - 1$ and

(*ii*) $H_1 = f_1 f_2$ and $H_\ell = f_1 f_k$.

Thus, $H_2 = f_1 f_j$ or $H_2 = f_2 f_j$ for some integer j with $3 \leq j \leq k$ (necessarily, $H_2 = f_2 f_3$ if $k = 3$). We now insert the sequence

$$s' : f_2 f_{k+1}, \ldots, f_{j-1} f_{k+1}, f_{j+1} f_{k+1}, \ldots, f_k f_{k+1}, f_j f_{k+1}$$

between $H_1 = f_1f_2$ and H_2 and add f_1f_{k+1} after f_1f_k, producing the sequence

$$s \ : \ H_1 = f_1f_2, f_2f_{k+1}, \ldots, f_{j-1}f_{k+1}, f_{j+1}f_{k+1}, \ldots, f_kf_{k+1}, f_jf_{k+1}, H_2, H_3, \ldots, H_\ell = f_1f_k, f_1f_{k+1},$$

which has the desired property. ■

Lemma 2.2 *Let $E = \{f_1, f_2, \ldots, f_k\}$ be the edge set of a star of size $k \geq 2$. For $\ell = \binom{k}{2}$, there is a sequence $H_1, H_2, \ldots, H_\ell$ consisting of the ℓ distinct pairs f_if_j of edges of E where*

(i) H_i and H_{i+1} have an edge in common for $i = 1, 2, \ldots, \ell - 1$ and

(ii) $H_1 = f_1f_2$ and $H_\ell = f_{k-1}f_k$.

Proof. We proceed by induction on k. It is straightforward to show that the statement is true for $k = 2, 3, 4$. Suppose that the statement is true for an integer $k \geq 4$. Let $\{f_1, f_2, \ldots, f_k, f_{k+1}\}$ be the edge set of a star of size $k + 1 \geq 5$. Applying the induction hypothesis to the set $\{f_1, f_2, \ldots, f_k\}$, there is a sequence $s_0 : H_1, H_2, \ldots, H_\ell$, where $\ell = \binom{k}{2}$, consisting of the ℓ distinct ordered pairs f_if_j where $1 \leq i \neq j \leq k$ such that

(*i*) H_i and H_{i+1} have an edge in common for $i = 1, 2, \ldots, \ell - 1$ and

(*ii*) $H_1 = f_1f_2$ and $H_\ell = f_{k-1}f_k$.

Thus, $H_2 = f_1f_j$ or $H_2 = f_2f_j$ for some integer j with $3 \leq j \leq k$. Let

$$s' : \ f_1f_{k+1}, f_2f_{k+1}, \ldots, f_{j-1}f_{k+1}, f_{j+1}f_{k+1}, \ldots, f_{k-1}f_{k+1}, f_jf_{k+1}.$$

We now insert s' between $H_1 = f_1f_2$ and H_2 and add f_kf_{k+1} after $f_{k-1}f_k$, producing the sequence

$$s \ : \ H_1 = f_1f_2, f_1f_{k+1}, f_2f_{k+1}, \ldots, f_{j-1}f_{k+1}, f_{j+1}f_{k+1}, \ldots, f_{k-1}f_{k+1}, f_jf_{k+1}, H_2, H_3, \ldots, f_{k-1}f_k, f_kf_{k+1},$$

which has the desired property. ■

3 The Main Theorem

We are now in a position to state and prove the primary result.

Theorem 3.1 *If G is a connected graph with $\delta(G) \geq 4$, then $\mathcal{P}_3(G)$ is Hamiltonian-connected.*

Proof. It suffices to show that for every two distinct 3-paths P and Q of G, there exists a sequence

$$\mathcal{S} : P = A_1, A_2, \ldots, A_p = Q \tag{1}$$

consisting of the distinct 3-paths A_i $(1 \leq i \leq p)$ of G that begins with P and ends with Q such that A_i and A_{i+1} have an edge in common for $i = 1, 2, \ldots, p-1$. Let there be given two distinct 3-paths P and Q of G. We consider five cases, depending on the location of P and Q in the graph G. In each case, a spanning tree T of G is constructed based on the location of P and Q. The tree T is then appropriately embedded in the plane from which a Hamiltonian walk W is constructed. The walk W then gives rise to a cyclic sequence $\mathcal{S}_1$ consisting of those 3-paths of T that lie on W. With the aid of $\mathcal{S}_1$, a sequence $\mathcal{S}$ as in (1) is constructed containing all 3-paths of G possessing the desired property, thereby showing that $\mathcal{P}_3(G)$ is Hamiltonian-connected.

Case 1. *P and Q have an edge in common.* There are three possibilities here, namely

(1) P and Q have the same interior vertex,

(2) P and Q have adjacent interior vertices and form a 4-path or

(3) P and Q have adjacent interior vertices and form a 3-cycle.

We consider these three subcases.

Subcase 1.1. *P and Q have the same interior vertex v.* Let $P = ab$ and $Q = bc$ and let d be a fourth edge incident with v. Let T be a spanning tree of G containing P and Q as well as the edge d. The tree T is embedded in the plane so that the edges c, a, b, d appear consecutively in clockwise order about v (see Figure 3). Any other edges of T incident with v lie between the edges d and c in this embedding of T.

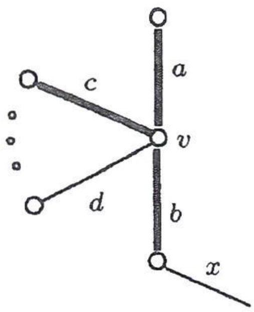


Figure 3: A planar embedding of T in Subcase 1.1

We now consider the Hamiltonian walk W in T obtained from this embedding of T that encounters the edge a before the edge b. By proceeding along W, we obtain a cyclic sequence $\mathcal{S}_1$ of all 3-paths of T lying on W. If T contains a 3-path whose interior vertex has degree 2 in T, then this 3-path occurs twice in $\mathcal{S}_1$. The sequence $\mathcal{S}_1$ contains the 3-paths $P = ab$ and bx as two consecutive terms for some edge x (where $x = d$ if b is a pendant edge). The 3-path $Q = bc$ is not a term in $\mathcal{S}_1$. We now insert Q between ab and bx to produce a sequence

$$\mathcal{S}_2 : P = ab = B_1, B_2, \ldots, B_{\ell-1} = bx, B_\ell = bc = Q$$

consisting of all 3-paths on W as well as the 3-path bc such that B_i and B_{i+1} have an edge in common for $i = 1, 2, \ldots, \ell - 1$. While any 3-path in T having an interior vertex of degree 2 occurs twice in $\mathcal{S}_2$, all other 3-paths on W occur exactly once in $\mathcal{S}_2$. We now describe additions that we make to $\mathcal{S}_2$ at each vertex u of T depending on the degree of u in T.

First, suppose that u is an end-vertex of T, where e is the edge in T that is incident with u. Let $f_1, f_2, \ldots, f_d$ be the edges of G incident with u that are not in T, where $d = \deg_G u - 1 \geq 3$. Applying Lemma 2.1 to the set $E = \{e, f_1, f_2, \ldots, f_d\}$ for $\ell = \binom{d+1}{2}$, there is a sequence $s : H_1, H_2, \ldots, H_\ell$ consisting of the ℓ distinct pairs of edges of E where (i) H_i and H_{i+1} have one edge in common for $i = 1, 2, \ldots, \ell - 1$ and (ii) $H_1 = ef_1$ and $H_\ell = ef_d$. We insert s between two consecutive terms containing e in $\mathcal{S}_2$. This is now done for each end-vertex u of T, producing a sequence $\mathcal{S}_3$ of 3-paths, consisting of all 3-paths of T lying on W and all 3-paths of G having an interior vertex of degree 1 in T such that every two consecutive terms in $\mathcal{S}_3$ have a single edge in common.

Second, suppose that T contains a vertex u of degree 2, incident w edges e and f. Then the 3-path ef occurs twice in $\mathcal{S}_3$. Since $\deg_G u \geq$ there are edges $e_1, e_2, \ldots, e_d$ $(d \geq 2)$ distinct from e and f that are incide with u and belonging to G but not to T. Since $d \geq 2$, there are 3-pat in G having the interior vertex u that do not belong to $\mathcal{S}_3$, namely, ee_i (1 $i \leq d)$, fe_i $(1 \leq i \leq d)$ and e_ie_j $(1 \leq i < j \leq d)$. In the second occurren of the 3-path ef in $\mathcal{S}_3$, there are three consecutive terms he, ef, fg in for some edges h and g. In this case, we replace the 3-path ef here by th sequence $ee_d, ee_{d-1}, \ldots, ee_1, \mathcal{S}'', fe_d, fe_{d-1}, \ldots, fe_1$ of 3-paths, where $\mathcal{S}''$ a sequence of distinct 3-paths e_ie_j $(1 \leq i < j \leq d)$ beginning with e_1e and ending with $e_{d-1}e_d$ such that consecutive 3-paths in $\mathcal{S}''$ have an edg in common. By Lemma 2.2, such a sequence $\mathcal{S}''$ exists. We do this fo each vertex u of degree 2 in T, producing a sequence $\mathcal{S}_4$ of distinct 3-path (having ab and bc are two consecutive terms), consisting of all 3-paths of T lying on W and all 3-paths of G having an interior vertex of degree 1 or 2 in T, where every two consecutive terms have an edge in common.

Next, suppose that T contains a vertex u of degree 3 in T. Then every 3-path of T having interior vertex u occurs exactly once in both W and $\mathcal{S}_4$. Let e_1, e_2, e_3 be the three edges of T incident with u. We may assume that these three edges appear in counter-clockwise order about u as e_1, e_2, e_3 in T. Then xe_1, e_1e_2, e_2y are three consecutive terms in $\mathcal{S}_4$ for some edges x and y. Let $f_1, f_2, \ldots, f_d$ be the edges of G that are incident with u but are not in T, where $d = \deg_G u - 3 \geq 1$. Applying Lemma 2.1 to the set $E = \{e_1, f_1, f_2, \ldots, f_d, e_2\}$, for $\ell = \binom{d+2}{2}$, there is a sequence $s : H_1, H_2, \ldots, H_\ell$ consisting of the ℓ distinct pairs of edges of E where (i) H_i and H_{i+1} have exactly one edge in common for $i = 1, 2, \ldots, \ell - 1$ and (ii) $H_1 = e_1f_1$ and $H_\ell = e_1e_2$. We now delete e_1e_2 from $\mathcal{S}_4$ and insert s between xe_1 and e_2y. Furthermore, insert the sequence $e_3f_1, e_3f_2, \ldots, e_3f_d$ between two consecutive terms in $\mathcal{S}_4$ containing e_3. We do this for each vertex u of degree 3 in T, producing a sequence $\mathcal{S}_5$ of distinct 3-paths (having ab and bc as consecutive terms) consisting of all 3-paths of T lying on W, the 3-path bc and all 3-paths of G having an interior vertex of degree 1, 2 or 3 in T, where every two consecutive terms have an edge in common.

Finally, let u be a vertex of degree 4 or more in T. First, suppose that every edge incident with u belongs to T, say $e_1, e_2, \ldots, e_d$ are the edges incident with u where $d = \deg_T u = \deg_G u = d \geq 4$. We may assume that the edges of T incident with u appear consecutively in T in counter-clockwise order about u as $e_1, e_2, \ldots, e_d$. Thus, $e_1e_2, e_2e_3, \ldots, e_{d-1}e_d, e_de_1$ are 3-paths in W. Consequently, there are $\binom{d}{2} - d$ (≥ 2) 3-paths of G with the interior vertex u that do not lie on W. Let X be the set of 3-paths whose interior vertex is u that do not appear in S_5. For each integer i with $1 \leq i \leq d-2$, let $X_i = \{e_ie_j \in X : i+1 < j\}$ and let s_i be any ordering of the 3-paths in X_i. For $1 \leq i \leq d-1$, insert the 3-paths in X_i in the order s_i between two consecutive terms containing e_i in S_5. We do this for every vertex u of degree 4 or more, each of whose incident edges belongs to T.

Next, suppose that there are edges of G incident with u that do not belong to T. Let $e_1, e_2, \ldots, e_d$ be the edges incident with u that belong to T and let $f_1, f_2, \ldots, f_{d'}$ be the edges incident with u that do not belong to T. Then $d \geq 4$ and $d' \geq 1$ and $d+d' = \deg_G u \geq 5$. We may assume that the edges of T incident with u appear consecutively in T in counter-clockwise order about u as $e_1, e_2, \ldots, e_d$. Thus, $e_1e_2, e_2e_3, \ldots, e_{d-1}e_d, e_de_1$ are 3-paths in W. Then xe_1, e_1e_2, e_2y are three consecutive terms in S_5 for some edges x and y. Applying Lemma 2.1 to the set $E = \{e_1, f_1, f_2, \ldots, f_{d'}, e_2\}$, for $\ell = \binom{d'+2}{2}$, there is a sequence $s : H_1, H_2, \ldots, H_\ell$ consisting of the ℓ distinct pairs of edges of E where (i) H_i and H_{i+1} have exactly one edge in common for $i = 1, 2, \ldots, \ell-1$ and (ii) $H_1 = e_1f_1$ and $H_\ell = e_1e_2$. We now delete e_1e_2 from S_5 and insert s between xe_1 and e_2y. Let Y be the set of 3-paths whose interior vertex is u, at least one of whose edges is in T that are not in S_5. For $1 \leq i \leq d$, let $Y_i = \{e_ie_j : i+1 < j\} \cup \{e_if_j : 1 \leq j \leq d'\} \subseteq Y$ for $1 \leq i \leq d$, where $\{e_ie_j : i+1 < j\} = \emptyset$ if $i = d-1, d$ and let s_i be any ordering of the 3-paths in Y_i. For $1 \leq i \leq d$, insert the 3-paths in Y_i in the order s_i between two consecutive terms containing e_i in S_5. We do this for every vertex u of degree 4 or more in G, producing the sequence S with the desired properties as described in (1).

Subcase 1.2. P and Q have adjacent interior vertices and form a 4-path. Let $P = ab$ and $Q = bc$ where v is the interior vertex of P, shown in Figure 4. Let T be a spanning tree of G containing the 4-path a, b, c as well as an edge w incident with v distinct from a and b (see Figure 4).

There exists a Hamiltonian walk W of T and a cyclic sequence S_1 of the 3-paths of T on W such that xa, ab, bc, cy are four consecutive terms in S_1 for some edges x and y in T (where possibly $x = w$ if a is a pendant edge

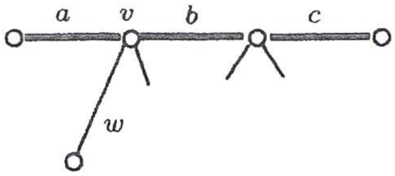


Figure 4: A planar embedding of T in Subcase 1.2

of T), that is, $\mathcal{S}_2 : P = ab = B_1, B_2, \ldots, B_\ell = bc = Q$ is a sequence of all 3-paths of T on W where B_i and B_{i+1} have a single edge in common for $i = 1, 2, \ldots, \ell - 1$. We now proceed as in Subcase 1.1 to produce a sequence $\mathcal{S}$ with the desired properties as described in (1).

Subcase 1.3. *P and Q have adjacent interior vertices and form a 3-cycle.* Let $P = ab$ and $Q = bc$ and let v be the interior vertex of P. Let T be a spanning tree of G containing the 3-path P (but not the edge c), an edge w incident with v distinct from a and b as well as an edge z incident with the interior vertex of Q different from b and c such that w and z are not adjacent (see Figure 5).

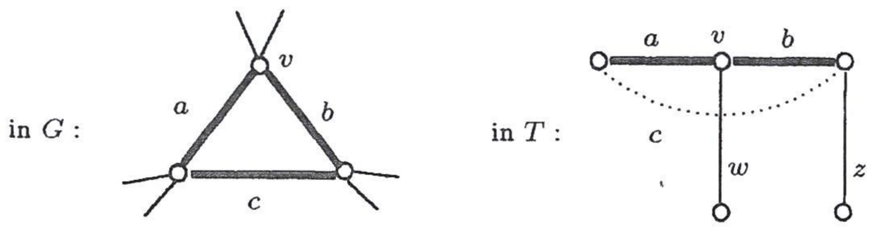


Figure 5: A planar embedding of T in Subcase 1.3

There exists a Hamiltonian walk W of T and a cyclic sequence $\mathcal{S}_1$ of the 3-paths of T lying on W such that ab, bz are two consecutive terms in $\mathcal{S}_1$ but bc is not a term of $\mathcal{S}_1$. We insert bc between ab and bz producing the sequence $\mathcal{S}_2 : P = ab = B_1, B_2, \ldots, bz, B_\ell = bc = Q$, which consists of bc and all 3-paths of T lying on W where B_i and B_{i+1} have an edge in common for $i = 1, 2, \ldots, \ell - 1$. We now proceed as in Subcase 1.1 to produce a sequence $\mathcal{S}$ with the desired properties as described in (1).

The remaining four cases deal with situations in which P and Q do not have an edge in common. In these cases, $P = ab$ and $Q = cd$, where then a, b, c, d are four distinct edges of G.

Case 2. *P and Q do not have an edge in common and there is a path* *in G containing P and Q*. Let R be a shortest path in G containing P and Q. There are two possibilities here. Either P and Q have a vertex in common or P and Q are vertex-disjoint (see Figure 6). We consider these subcases.

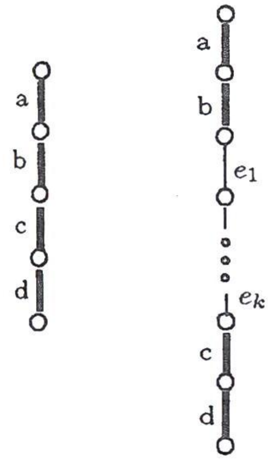


Figure 6: The 3-paths P and Q in Case 2

Subcase 2.1. *P and Q have a vertex in common and so* $R = (a, b, c, d)$. Let T be a spanning tree of G containing R. Then there is a Hamiltonian walk W of T such that R is a path in W, resulting in a cyclic sequence $\mathcal{S}_1$ of 3-paths of T occurring in the order they are encountered on W. We may assume that T is embedded in the plane so that ab, bc, cd are three consecutive terms in $\mathcal{S}_1$. Since each edge of T is encountered twice in W, each edge of T that is not a pendant edge of T occurs in two consecutive 3-paths twice in $\mathcal{S}_1$. Thus, in addition to ab, bc, there is another pair of consecutive 3-paths in $\mathcal{S}_1$ containing b, say xb and by (where possibly $x = c$ and/or $y = a$). If the 3-path bc occurs twice in $\mathcal{S}_1$, then we remove bc from its first occurrence (between ab and cd). Otherwise, we remove bc from $\mathcal{S}_1$ and insert bc between xb and by, producing a sequence

$$\mathcal{S}_2 : P = ab = B_1, B_2, \ldots, B_{\ell-1}, B_\ell = cd = Q \qquad (2)$$

consisting of all 3-paths on W where B_i and B_{i+1} have an edge in common for $i = 1, 2, \ldots, \ell - 1$. We now proceed as in Case 1 to place all 3-paths in G that are not in $\mathcal{S}_2$ to produce a sequence $\mathcal{S}$ that begins with P and ends with Q consisting of all distinct 3-paths of G such that consecutive 3-paths in $\mathcal{S}$ have an edge in common, as described in (1).

Subcase 2.2. *P and Q are vertex-disjoint.* Let R=(a, b, e_1, e_2, ..., e_k,

c, d), $k \geq 1$, be a shortest path in G containing P and Q. Let T be a spanning tree of G containing R and embedded in the plane so that there is a Hamiltonian walk W of T such that R is a path in W. This results in a cyclic sequence $\mathcal{S}_1$ of 3-paths of T occurring in the order they are encountered on W. Thus,

$$xa, ab, \underline{be_1, e_1e_2, \ldots, e_kc,} \; cd, dy \tag{3}$$

are consecutive terms in $\mathcal{S}_1$ for some edges x and y. Each of the edges b, c, e_i ($1 \leq i \leq k$) appears between consecutive terms only once in (3) and there is another pair of consecutive terms in $\mathcal{S}_1$ containing each such edge. We can now delete each of the 3-paths $be_1, e_1e_2, \ldots, e_kc$ in (3) whose interior vertex has degree 2 and move every other such 3-path to an appropriate position in $\mathcal{S}_1$ where the interior vertex of the 3-path is encountered on W. (For example, we can insert be_1 between consecutive 3-paths in $\mathcal{S}_1$ containing e_1, insert e_1e_2 between consecutive 3-paths in $\mathcal{S}_1$ containing e_1 or containing e_2 and so on.) This creates a new sequence $\mathcal{S}_2$ as in (2). We then proceed as in Case 1 to produce a sequence $\mathcal{S}$ with the desired properties as described in (1).

Case 3. $P = ab$ and $Q = cd$ do not have an edge in common but have two vertices in common. There are two possibilities here, as shown in Figures 7(a) and 7(b).

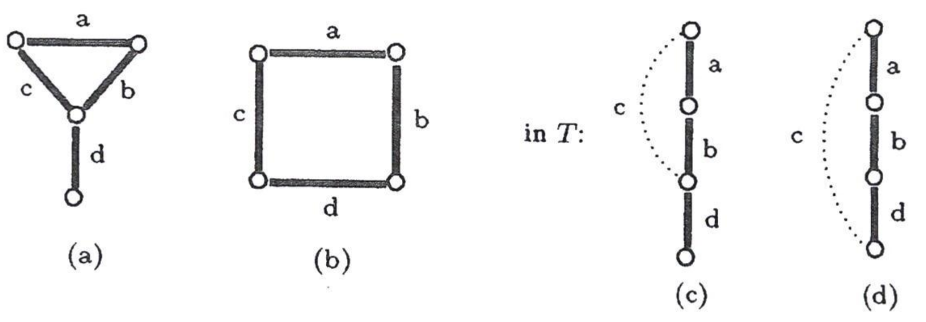
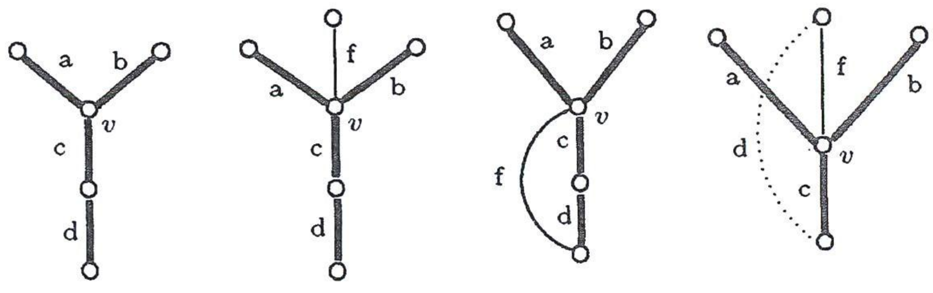


Figure 7: The 3-paths P and Q in Case 3

Let $R = (a, b, d)$ be the 4-path of G in the graphs shown in both Figures 7(a) and 7(b). Let T be a spanning tree of G containing R but not the edge c. See Figures 7(c) and 7(d). There is an embedding of T in the plane so that the resulting Hamiltonian walk W of T contains the 4-path $R = (a, b, d)$. This, in turn, results in a cyclic sequence $\mathcal{S}_1$ of those 3-paths in T occurring in the order they are encountered on W. Thus, ab, bd are consecutive terms in S_1 and the 3-path cd does not occur in W and so not in S_1 either. We then insert cd between ab and bd in S_1, resulting in a sequence of 3-paths of G that begins with $P = ab$ and ends with $Q = cd$ consisting of all distinct 3-paths of S_1, together with cd, where consecutive 3-paths have an edge in common. We then proceed as Case 1 to add all 3-paths in G not in this sequence and produce a sequence S that begins with P and ends with Q consisting of all distinct 3-paths of G such that consecutive 3-paths have an edge in common, as described in (1).

Case 4. *$P = ab$ and $Q = cd$ do not lie on a common path and have exactly one vertex v in common.* Here, P and Q produce one of the two trees of order 5 that is not a path. We consider these two possibilities.

Subcase 4.1. *$P = ab$ and $Q = cd$ form the tree of order* 5 *containing a vertex v of degree* 3 *in Figure* 8(*a*). Let f be an edge incident with v that is distinct from a, b and c. The edges f and d may or may not be adjacent.



(a) (b) (c) (d)

Figure 8: The 3-paths P and Q in Subcase 4.1

First, suppose that f and d are not adjacent. Let T be a spanning tree of G containing P, Q and f, which is embedded in the plane so that a, f, b, c appear consecutively in clockwise order about v as shown in Figure 8(b). Then there is a Hamiltonian walk W of T and a cyclic sequence S_1 consisting of certain 3-paths of T occurring in the order they are encountered on W. Thus, S_1 does not contain ab but does contain bc and cd as consecutive terms. We then insert ab between bc and cd in S_1, resulting in a sequence of 3-paths of G that begins with $P = ab$ and ends with $Q = cd$ consisting of all distinct 3-paths of S_1, together with ab, where consecutive 3-paths have an edge in common. We then proceed as in Case 1 to add all 3-paths in G not in this sequence to produce a sequence S that begins with P and ends with Q with the desired properties as described in (1).

Next, suppose that f and d are adjacent as shown in Figure 8(c). Let T be a spanning tree of G containing a, f, b and c (but not d), which is embedded in the plane so that a, f, b, c appear consecutively in clockwise order about v as shown in Figure 8(d). Then there is a Hamiltonian walk W of T and a cyclic sequence $\mathcal{S}_1$ consisting of certain 3-paths of T occurring in the order they are encountered on W. Thus, $\mathcal{S}_1$ contains neither ab nor cd but contains xb, bc as consecutive terms for some edge x (where possibly $x = f$). We now insert ab, cd between xb and bc in $\mathcal{S}_1$, resulting in a sequence of 3-paths of G that begins with $P = ab$ and ends with $Q = cd$ consisting of all distinct 3-paths of $\mathcal{S}_1$, together with ab, cd, where consecutive 3-paths have an edge in common. We then proceed as in Case 1 to add all 3-paths in G not in this sequence to produce a sequence $\mathcal{S}$ that begins with P and ends with Q with the desired properties as described in (1).

Subcase 4.2. *P and Q form the star $K_{1,4}$.* Let T be a spanning tree of G containing P and Q, which is embedded in the plane so that a, c, b, d appear consecutively in clockwise order about v. See Figure 9.

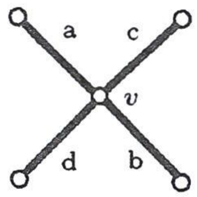


Figure 9: The 3-paths P and Q in Subcase 4.2

Then there is a Hamiltonian walk W of T and a cyclic sequence $\mathcal{S}_1$ consisting of 3-paths of T occurring in the order they are encountered on W. Thus, $\mathcal{S}_1$ contains neither ab nor cd but contains xb, bd as consecutive terms for some edge x, where possibly $x = c$. Then we insert ab, cd between xb and bd in $\mathcal{S}_1$, resulting in a sequence of 3-paths of G that begins with $P = ab$ and ends with $Q = cd$ consisting of all distinct 3-paths of $\mathcal{S}_1$, together with ab, cd, where consecutive 3-paths have an edge in common. We then proceed as in Case 1 to produce a sequence $\mathcal{S}$ with the desired properties as described in (1).

Case 5. *P and Q do not lie on a common path and are vertex-disjoint.* There are two possibilities, shown in Figures 10(a) and 10(c).

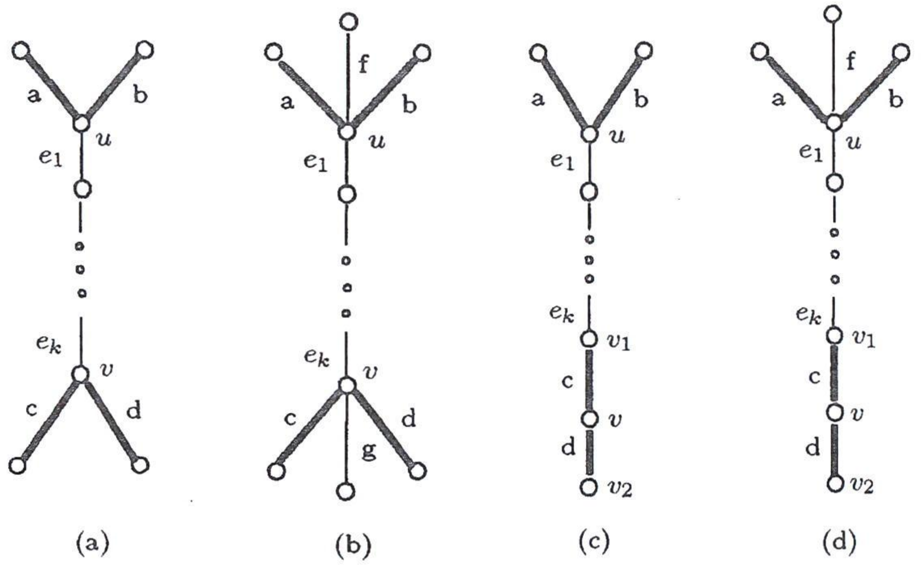


Figure 10: The 3-paths P and Q in Case 5

Subcase 5.1. *There is no path in* G *containing one of* P *and* Q *and one edge of the other.* Necessarily, there is a path in G containing one edge of each of P and Q. Let u be the interior vertex of P and v the interior vertex of Q and let R be a shortest $u-v$ path in G, say $R=(e_1,e_2,\ldots,e_k)$ for some positive integer k. See Figure 10(a). We consider two subcases.

Subcase 5.1.1. *There is an edge* f *incident with* u *distinct from* a,b,e_1 *and an edge* g *incident with* v *distinct from* c,d,e_k *such that* f *and* g *are not adjacent.* See Figure 10(b). Since R is a shortest $u-v$ path in G and there is no path in G containing one of P and Q and one edge of the other, f is not adjacent to any of the edges $e_2,e_3,\ldots,e_k,c,d,g$ and g is not adjacent to any of $a,b,f,e_1,e_2,\ldots,e_{k-1}$. Let T be a spanning tree of G containing P, Q, f, g and R, which is embedded in the plane as shown in Figure 10(b). Then there is a Hamiltonian walk W of T and a cyclic sequence S_1 consisting of those 3-paths of T occurring in the order they are encountered on W. Thus, S_1 contains neither ab nor cd. If $k=1$, then S_1 contains be_1,e_1d as consecutive terms; while if $k\geq 2$, then S_1 contains

$$be_1,\underline{e_1e_2,\ldots,e_{k-1}e_k},e_kd \tag{4}$$

as consecutive terms. If $k=1$, we insert ab,cd between be_1 and e_1d, creating a new sequence beginning at $P=ab$ and ending at $Q=cd$. If $k\geq 2$, then we insert ab,cd between be_1 and e_1e_2 and delete each of the 3-paths $e_1e_2,\ldots,e_{k-1}e_k$ in (4) whose interior vertex has degree 2 and move every other such 3-path from the sequence in (4) to another appropriate position, creating a new sequence that begins at $P = ab$ and ends at $Q = cd$. We then proceed as in Case 1 to produce a sequence S with the desired properties as described in (1).

Subcase 5.1.2. *Subcase* 5.1.1 *does not occur.* Hence, if f is an edge incident with u distinct from a, b, e_1 and g is an edge incident with v distinct from c, d, e_k, then f and g are adjacent. Then $\deg_G u = \deg_G v = 4$ and either $k = 1$ or $k = 2$. See Figures 11(a) and 11(c). Let T be a spanning tree of G containing P, Q, e_1 (if $k = 1$) or e_1, e_2 (if $k = 2$) and the edge f but not g, which is embedded in the plane as shown in Figures 11(b) and 11(d).

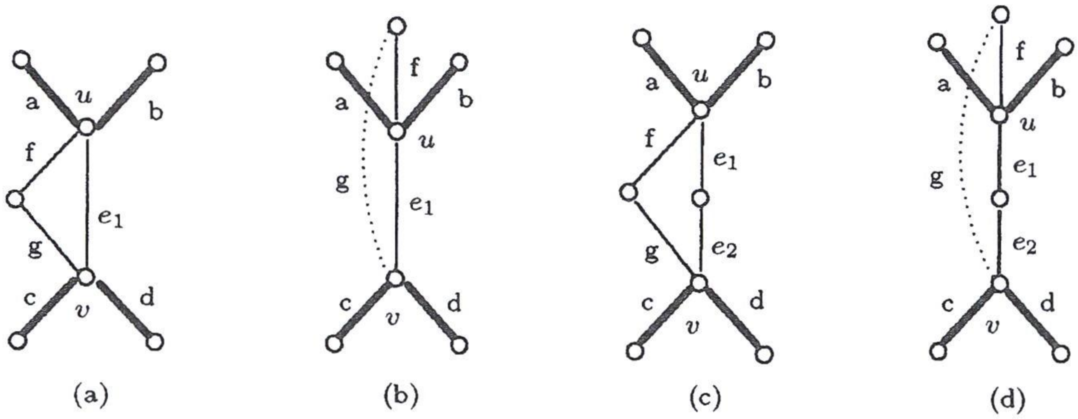


Figure 11: The 3-paths P and Q in Subcase 5.1.2

We only consider the case when $k = 2$ since the argument for the case when $k = 1$ is similar. From this planar embedding of T, a Hamiltonian walk W of T is produced as well as a cyclic sequence S_1 of 3-paths that contains

$$be_1, \underline{e_1e_2, e_2d, dx, \ldots, yd}, dc, cz \qquad (5)$$

as consecutive terms for some edges x, y and z of G. Note that S_1 contains neither ab nor the three 3-paths dg, e_2g, cg of G whose interior vertex is v but S_1 does contain the 3-path cd on W. We insert ab between be_1 and e_1e_2 and insert dg, e_2g, cg between dc and cz so that dc, dg, e_2g, cg, cz are consecutive terms. We move the terms $e_1e_2, e_2d, dx, \ldots, yd$ in (5) and insert them between dg and e_2g such that $dg, yd, \ldots, dx, e_2d, e_1e_2, e_2g$ are consecutive terms and then delete each 3-path in the resulting sequence whose interior vertex has degree 2. This produces a sequence $P = ab, be_1, \ldots, gd, dc = Q$

consisting of all 3-paths of W together with the three 3-paths dg, e_2g, cg of G whose interior vertex is v. We then proceed as in Case 1 (but excluding the vertex v) to produce a sequence S with the desired properties as described in (1).

Subcase 5.2. *There is a path in G containing one of P and Q and one edge of the other.* See Figure 10(c). Let R be shortest such path, say R contains b and Q, where $R = (b, e_1, e_2, \ldots, e_k, c, d)$ for some positive integer k. We consider two subcases.

Subcase 5.2.1. *There is an edge f incident with u distinct from a, b, e_1 that is not adjacent to d.* Let T be a spanning tree of G containing P, Q, f and R, which is embedded in the plane as shown in Figure 10(d). Then there is a Hamiltonian walk W of T and a cyclic sequence S_1 consisting of 3-paths of T occurring in the order they are encountered on W. Thus, S_1 does not contain ab but contains

$$be_1, \underline{e_1e_2, \ldots, e_kc}, \ cd \tag{6}$$

as consecutive terms. We insert ab between be_1 and e_1e_2 in S_1, delete each of those 3-paths $e_1e_2, \ldots, e_kc$ in the sequence (6) having an interior vertex of degree 2 and move every other such 3-path in (6) to another appropriate position in the sequence. This creates a new sequence that begins at $P = ab$ and ends at $Q = cd$. We then proceed as in Case 1 to produce a sequence S with the desired properties as described in (1).

Subcase 5.2.2. *Every edge f incident with u distinct from a, b, e_1 is adjacent to d.* First, suppose that f is incident with v. See Figure 12(a). Let T_1 be the tree as shown in Figure 12(b) and let T be a spanning tree of G containing T_1 but not the edge c. The tree T is embedded as shown in Figure 12(b). Then there is a Hamiltonian walk W of T and a cyclic sequence S_1 consisting of 3-paths of T occurring in the order they are encountered on W such that bf, fd are consecutive terms in S_1 but ab and cd are not in S_1. We now insert ab, cd between bf and fd, resulting in a sequence of 3-paths of G that begins with $P = ab$ and ends with $Q = cd$. We then proceed as in Case 1 to produce a sequence S with the desired properties as described in (1).

Next, suppose that f is incident with w. By the defining property of R, it follows that $k = 1$ and $R = e_1$. See Figure 12(c). Let T_2 be the tree

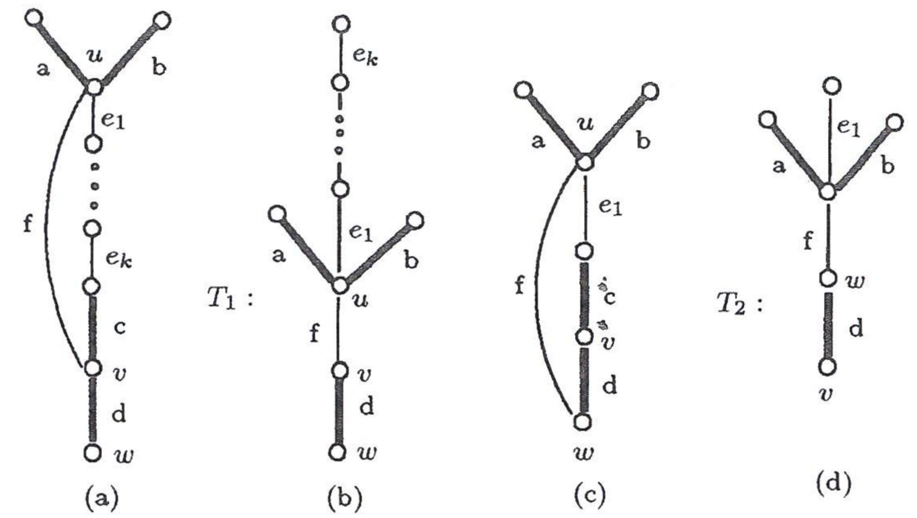


Figure 12: The 3-paths P and Q in Subcase 5.2.2

as shown in Figure 12(d) and let T be a spanning tree of G containing T_2 which is embedded in the plane as shown in Figure 12(d). Again, there is a Hamiltonian walk W of T and a cyclic sequence $\mathcal{S}_1$ consisting of 3-paths of T such that bf, fd are consecutive terms in $\mathcal{S}_1$ but ab and cd are not in $\mathcal{S}_1$. We then insert ab, cd between bf and fd and proceed as in Subcase 5.2.1. ■

A connected graph G of order $n \geq 3$ is called *k-tree-connected* (or *k-leaf-connected*) for an integer k with $2 \leq k \leq n-1$, if for every set S of k distinct vertices of G, there exists a spanning tree T of G whose set of end-vertices is S. Thus, a 2-tree-connected graph is Hamiltonian-connected. By Theorem 3.1, if G is a connected graph with $\delta(G) \geq 4$, then $\mathcal{P}_3(G)$ is 2-tree-connected. The following were shown in [1].

Theorem 3.2 *If T is a tree of order at least 6 containing no vertices of degree 2, 3 or 4, then $\mathcal{P}_3(T)$ is 3-tree-connected.*

Theorem 3.3 *If G is k-tree-connected for some integer $k \geq 2$, then G is $(k+1)$-connected.*

Since $S = \{ab, ac, ad\}$ is a vertex-cut in the 3-path graph $\mathcal{P}_3(G)$ of the graph G in Figure 13, it follows that $\mathcal{P}_3(G)$ is not 4-connected and so $\mathcal{P}_3(G)$ is not 3-tree-connected by Theorem 3.3. Hence, if G is a connected graph with $\delta(G) \geq 4$, then $\mathcal{P}_3(G)$ need not be 3-tree-connected. However, no connected graph G is known such that $\delta(G) \geq 5$ and $\mathcal{P}_3(G)$ is not 3-tree-connected. Therefore, we conclude with the following conjecture.

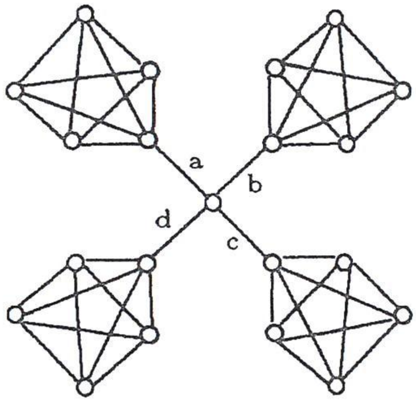


Figure 13: A graph G whose 3-path graph is not 3-tree-connected

Conjecture 3.4 *If G is a connected graph with $\delta(G) \geq 5$, then $\mathcal{P}_3(G)$ is 3-tree-connected.*

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References

[1] A. Byers, G. Chartrand, D. Olejniczak and P. Zhang, Trees and Hamiltonicity. *J. Combin. Math. Combin. Comput.* Special Issue In Memory of Peter Slater. **104** (2018) 187-204.

[2] G. Chartrand and C. E. Wall, On the Hamiltonian index of a graph. *Studia Sci. Math. Hungar.* **8** (1973) 43-48.

[3] F. Fujie and P. Zhang, *Covering Walks in Graphs.* Springer, New York (2014).

[4] S. E. Goodman and S. T. Hedetniemi, On Hamiltonian walks in graphs. *SIAM J. Comput.* **3** (1974) 214-221.

[5] F. Harary and C. St. J. A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs. *Canad. Math. Bull.* **8** (1965) 701-709.