Maximizing algebraic connectivity for quasi-tree graphs with given matching number*

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Abstract

A connected graph $G = (V, E)$ is called a quasi-tree graph if there exists a vertex $v_0 \in V(G)$ such that $G - v_0$ is a tree. In this paper, we determine the largest algebraic connectivity together with the corresponding extremal graphs among all quasi-tree graphs of order n with given matching number.

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1 Introduction

Let G be a simple undirected graph with vertex set $V = V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E(G)$. For a graph G, $A(G)$ is its adjacency matrix and $D(G)$ is the diagonal matrix of its degrees. The matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G. The Laplacian characteristic polynomial of G, denoted by $\Phi(G; x)$, is just $\det(xI - L(G))$. As usual, we shall index the eigenvalues of $L(G)$ in nonincreasing order, denote them as $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$. The second smallest eigenvalue $\mu_{n-1}(G)$ of $L(G)$ is called the algebraic connectivity of G, denoted by $a(G)$.

The investigation on the algebraic connectivity of a graph is an important topic in the theory of graph spectra because it features many interesting properties and has a lot of applications in theoretical chemistry, control theory, combinatorial optimization, etc. Much work has been done concerning the algebraic connectivity of a graph, see e. g. [1, 2]. **Given a set of graphs, which graphs maximize the algebraic connectivity?** This problem arises in many diverse areas (see [15]) and has been studied extensively. For example, maximizing the algebraic connectivity has been discussed by Fiedler [7] and Zhang [24] for trees, Fallat et. al [6] and He et. al [11] for unicyclic graphs, Fallat et. al [5] and Wang et. al [22] for graphs with given diameter, Molitierno [18] and Barriere et. al [4] for planar graphs, Molitierno [19] for outerplanar graphs, Kirkland [13] and [14] subject to the number of cut-points, Lu et. al [17] in terms of the domination number, Lal et. al [16] subject to the number of pendant vertices, Zhu [25] subject to matching number. Moreover, the reader may be referred to [2, 21, 23] and the references therein.

A connected graph $G = (V, E)$ is called a quasi-tree graph if there exists a vertex $v_0 \in V(G)$ such that $G - v_0$ is a tree. In this paper, we determine the largest algebraic connectivity together with the corresponding extremal graphs among all quasi-tree graphs of order n with given matching number.

The rest of the paper is organized as follows. In Section 2, we recall some basic notions and lemmas used further, and prove some new lemmas. In Section 3, we give our main results.

2 Preliminaries

Let $G-v$ denote the graph obtained from a graph G by deleting the vertex $v \in V(G)$ and all the edges incident with v, and $G-uv$ denote the graph obtained from a graph G by deleting the edge $uv \in E(G)$. We denote the minimum degree of the vertices of G by $\delta = \delta(G)$, and the complete bipartite graph with two parts of sizes s and t by $K_{s,t}$. Denote by $\kappa(G)$ and $\varepsilon(G)$ the vertex connectivity and the edge connectivity of G respectively. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs with $V_1 \cap V_2 = \emptyset$, their union is $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The join $G_1 \vee G_2$ is the graph G obtained from $G_1 + G_2$ by joining each vertex of G_1 to each vertex of G_2.

Lemma 2.1. ([7]) *For a connected graph G and all $v \in V(G)$,*

$$a(G) \leq a(G-v) + 1.$$

Lemma 2.2. ([8]) *If T is a tree with diameter d, then*

$$a(T) \leq 2(1 - cos\frac{\pi}{d+1}).$$

Lemma 2.3. ([12]) *Let G be a graph of order n, e be an edge of G and $G' = G - e$. Then*

$$\mu_1(G) \geq \mu_1(G') \geq \mu_2(G) \geq \mu_2(G') \geq \cdots \geq \mu_{n-1}(G) \geq \mu_{n-1}(G')$$

$$\geq \mu_n(G) = \mu_n(G') = 0.$$

Lemma 2.4. ([7]) *Let G be a non-complete graph. Then*

$$a(G) \leq \kappa(G) \leq \varepsilon(G) \leq \delta(G).$$

If $v \in V(G)$, let $L_v(G)$ be the principal submatrix of $L(G)$ formed by deleting the row and column corresponding to the vertex v. Similarly, if H is a subgraph of G, let $L_H(G)$ be the principal submatrix of $L(G)$ formed by deleting the rows and columns corresponding to all vertices of $V(H)$.

The following lemmas display the relations between the characteristic polynomial of $L(G)$ and the polynomial of $L_v(G)$.

Lemma 2.5. ([10]) *Let v be a vertex of a graph G, let $\varphi(v)$ be the collection of cycles containing v. Then the Laplacian characetristic polynomial $\Phi(L(G))$ satisfies*

$$\Phi(L(G)) = (x - d(v))\Phi(L_v(G)) - \Sigma_w(\Phi(L_{vw}(G)))$$

$$- 2\Sigma_{Z\in\varphi(v)}(-1)^{|Z|}\Phi(L_Z(G)),$$

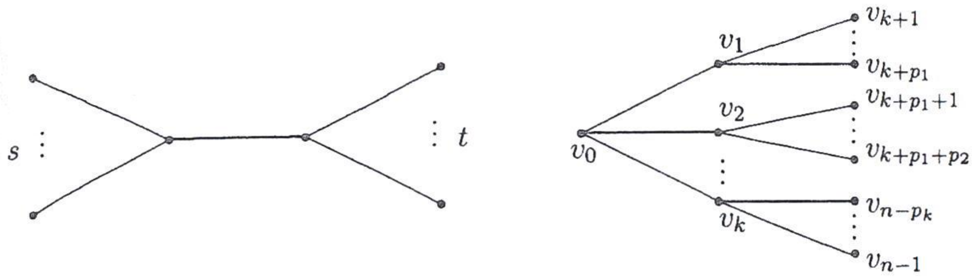
where the first summation extends over those vertices w adjacent to v, the second summation extends over all $Z \in \varphi(v)$, and $|Z|$ denotes the length of Z.

Lemma 2.6. ([10]) *Let H be a proper subgraph of G, and let v be a vertex of G such that $v \notin V(H)$. Then we have*

$$\Phi(L_H(G)) = (x - d(v))\Phi(L_{H,v}(G)) - \Sigma_{uv\in E(G);u\notin V(H)}(\Phi(L_{H,uv}(G)))$$

$$- 2\Sigma_{Z\in\varphi(v);V(z)\cap V(H)=\emptyset}(-1)^{|Z|}\Phi(L_{H,Z}(G)).$$

Denote by $T(n, k; p_1, \ldots, p_k)$ the tree of order n, shown in Fig. 2.1, obtained from star graphs $K_{1,k}, K_{1,p_1}, K_{1,p_2}, \ldots, K_{1,p_k}$ by identifying the pendent vertices of $K_{1,k}$ and the centers of $K_{1,p_1}, \ldots, K_{1,p_k}$, respectively, where $p_1 \geq p_2 \geq \cdots \geq p_k \geq 0$, $p_1 \geq p_2 > 0$, $k \geq 2$, $1 + k + p_1 + p_2 + \cdots + p_k = n$.



$S(s,t)$ $\qquad$ $T(n,k;p_1,\ldots,p_k)$

Fig. 2.1 $\quad S(s,t)$, $T(n,k;p_1,\ldots,p_k)$

Lemma 2.7. ([24]) *Let $T(n,k;p_1,\ldots,p_k)$ be a tree of order n with diameter 4, where $k \geq 2, p_1 \geq \cdots \geq p_k \geq 0$. Then $a(n,k;p_1,\ldots,p_k) \leq \frac{3-\sqrt{5}}{2}$ with equality if and only if $k \geq 2$ and $p_1 = p_2 = 1$, i.e., $T(n,k;p_1,\ldots,p_k)$ is $T(n,k;1,\ldots,1,0,\ldots,0)$.*

Lemma 2.8. ([20]) *Let G and H be two connected graphs of order r and s, respectively. If $\sigma(G) = (\mu_1(G), \mu_2(G), \ldots, \mu_r(G))$ and $\sigma(H) = (\mu_1(H), \mu_2(H), \ldots, \mu_s(H))$, then*

$$\sigma(G \vee H) = (r+s, \mu_1(G)+s, \ldots, \mu_{r-1}(G)+s, \mu_1(H)+r, \ldots,$$
$$\mu_{s-1}(H)+r, 0).$$

where $\sigma(G)$ and $\sigma(H)$ denote the Laplacian spectrum of G and H respectively.

A double star $S(s,t)$, shown in Fig. 2.1, is a tree obtained from two stars $K_{1,s}$ and $K_{1,t}$ by adding an edge between the centers of the two stars. Without loss of generality, we may assume that $s+t+2=n$ and $1 \leq s \leq t \leq n-3$.

Lemma 2.9. ([9]) *The algebraic connectivity of the double star $S(s,t)$ is a strictly decreasing function of s for $1 \leq s \leq \frac{n}{2}-1$.*

Lemma 2.10. ([3]) *Let f be real-valued and continuous on a connected subset S of R^n. If f takes on two different values in S, say a and b, then for each real c between a and b there exists a point x in S such that $f(x) = c$.*

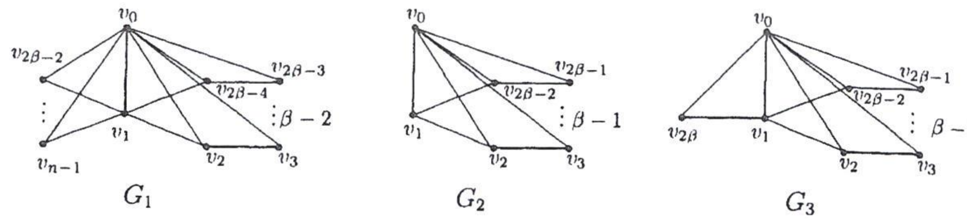


Fig. 2.2 G_1, G_2, G_3

Lemma 2.11. *Let G_1 be the graph, shown in Fig. 2.2, of orde n with matching number β. If $\beta \geq 4$, then $a(G_1) = \frac{5-\sqrt{5}}{2}$.*

Proof. Let $T_1 = G_1 - v_0$. Then $T_1 = T(n, n-\beta; 1, \ldots, 1, 0, \ldots, 0)$ By Lemmas 2.7 and 2.8, we obtain $a(G_1) = \frac{5-\sqrt{5}}{2}$. □

Lemma 2.12. *If $\beta \geq 4$, then $a(G_1 - v_0v_1) = \frac{5-\sqrt{5}}{2}$.*

Proof. Since $\beta \geq 4$, it follows that $n \geq 8$. For $n = 8$ and 9, by computation with computer, we can check that $a(G_1 - v_0v_1) = \frac{5-\sqrt{5}}{2}$. Next we assume $n \geq 10$. By Lemmas 2.5 and 2.6, we have

$$\Phi(G_1 - v_0v_1; x) = x(x-2)^{n-2\beta+1}(x^2 - 5x + 5)^{\beta-3} f_1(x),$$

where

$$\begin{aligned} f_1(x) &= x^4 - (2n - \beta + 5)x^3 + (n^2 - \beta n + 10n - 4\beta + 3)x^2 \\ &\quad -(5n^2 - 5\beta n + 10n - 10)x + 5n^2 - 4\beta n - 2n. \end{aligned}$$

Taking the first and second derivatives of $f_1(x)$ with respect to x, we obtain that

$$\begin{aligned} f_1'(x) &= 4x^3 - 3(2n - \beta + 5)x^2 + 2(n^2 + 10n - n\beta - 4\beta + 3)x \\ &\quad -5n^2 - 10n + 5n\beta + 10, \\ f_1''(x) &= 12x^2 - 6(2n - \beta + 5)x + 2(n^2 + 10n - n\beta - 4\beta + 3). \end{aligned}$$

Since $n \geq 2\beta$ and $\beta \geq 4$, it follows that

$$\begin{aligned} f_1''(x) &= 12x^2 - 6(2n - \beta + 5)x + 2(n^2 + 10n - n\beta - 4\beta + 3) \\ &> -12(2n - \beta + 5) + 2(n^2 + 10n - n\beta - 4\beta + 3) \\ &= 2n(n - \beta - 2) + 4\beta - 54 \\ &\geq 2n(\beta - 2) + 4\beta - 54 \geq 4(n + \beta) - 54 > 0 \end{aligned}$$

when $0 \le x \le 2$. This implies that $f_1'(x)$ is strictly increasing on the interval $[0, 2]$. Since $f_1'(\frac{3}{2}) = -2n(n-\beta-\frac{13}{4}) - \frac{21}{4}\beta - \frac{5}{4} \le -2n(\beta-\frac{13}{4}) - \frac{21}{4}\beta - \frac{5}{4} < 0$, it follows that $f_1'(x) < 0$ for $0 \le x \le \frac{3}{2}$. This implies that $f_1(x)$ is strictly decreasing on the interval $[0, \frac{3}{2}]$. Noting that

$$f_1(\frac{5-\sqrt{5}}{2}) = n\beta - 2n - 5\beta + 10 \ge 4(n-5) - 2n + 10 = 2(n-5) > 0,$$

we have that the least root of $f_1(x)$ is greater than $\frac{5-\sqrt{5}}{2}$. It follows from the characteristic polynomial $\Phi(G_1 - v_0v_1; x)$ that $a(G_1 - v_0v_1) = \frac{5-\sqrt{5}}{2}$. □

Lemma 2.13. *If $\beta \ge 4$, then $a(G_1 - v_0v_2) < \frac{5-\sqrt{5}}{2}$.*

Proof. By Lemmas 2.5 and 2.6, we have

$$\Phi(G_1 - v_0v_2; x) = x(x-2)^{n-2\beta+1}(x^2 - 5x + 5)^{\beta-4} f_2(x),$$

where

$$\begin{aligned} f_2(x) \;=\; & x^6 - (2n - \beta + 10)x^5 + (n^2 + 19n - n\beta - 8\beta + 35)x^4 \\ & -(9n^2 + 65n - 9n\beta - 18\beta + 44)x^3 + (28n^2 + 97n - \\ & 27n\beta - 6\beta - 2)x^2 - (35n^2 + 63n - 31n\beta + 8\beta - 24)x \\ & +15n^2 + 16n - 12n\beta. \end{aligned}$$

It follows that $f_2(1) = 2n - 3\beta + 4 > 0$,

$$\begin{aligned} 5^6 f_2(\frac{6}{5}) &= -2475n^2 + 34090n - 3150n\beta - 19320\beta + 8856 \\ &\le -2475n^2 + 21490n - 68424 < 0. \end{aligned}$$

By Lemma 2.10, there exist at least one root of $f_2(x)$ in the interval $(1, \frac{6}{5})$. This implies that $a(G_1 - v_0v_2) < \frac{6}{5} < \frac{5-\sqrt{5}}{2}$. □

Lemma 2.14. *Let G_2 be the graph, shown in Fig.2.2, of order n with matching number β. If $\beta \ge 4$, then $a(G_2) = \frac{5-\sqrt{5}}{2}$.*

Proof. Let $T_2 = G_2 - v_0$. Then $T_2 = T(n, \beta - 1; 1, 1, \ldots, 1)$. By Lemmas 2.7 and 2.8, we obtain $a(G_2) = \frac{5-\sqrt{5}}{2}$. $\square$

Lemma 2.15. *If $\beta \geq 4$, then $a(G_2 - v_0v_1) = \frac{5-\sqrt{5}}{2}$.*

Proof. By Lemmas 2.5 and 2.6, we have

$$\begin{aligned}\Phi(G_2 - v_0v_1; x) = x(x^2 - 5x + 5)^{\beta-2}(x^3 - (3\beta + 2)x^2 \\ + (2\beta^2 + 8\beta - 5)x - 6\beta^2 + 6\beta).\end{aligned}$$

Let $f_3(x) = x^3 - (3\beta + 2)x^2 + (2\beta^2 + 8\beta - 5)x - 6\beta^2 + 6\beta$. Since $\beta \geq 4$, it follows that

$$\begin{aligned} f_3'(x) &= 3x^2 - (6\beta + 4)x + 2\beta^2 + 8\beta - 5 \\ &> -2(6\beta + 4) + 2\beta^2 + 8\beta - 5 \\ &= 2\beta^2 - 4\beta - 13 > 0 \end{aligned}$$

when $0 \leq x \leq 2$. This implies that $f_3(x)$ is strictly increasing on the interval $[0, 2]$. Noting that $f_3(\frac{5-\sqrt{5}}{2}) = -\frac{\sqrt{5}+1}{2}(2\beta^2 - 7\beta + 5) < 0$, we have that the least root of $f_3(x)$ is greater than $\frac{5-\sqrt{5}}{2}$. It follows from the characteristic polynomial $\phi(G_2 - v_0v_1; x)$ that $a(G_2 - v_0v_1) = \frac{5-\sqrt{5}}{2}$. $\square$

Lemma 2.16. *If $\beta \geq 4$, then $a(G_2 - v_0v_2) < \frac{5-\sqrt{5}}{2}$.*

Proof. By Lemmas 2.5 and 2.6, we have

$$\Phi(G_2 - v_0v_2; x) = x(x^2 - 5x + 5)^{\beta-3} f_4(x),$$

where

$$\begin{aligned} f_4(x) &= x^5 - (3\beta + 7)x^4 + (2\beta^2 + 22\beta + 13)x^3 \\ &\quad -(14\beta^2 + 50\beta)x^2 + (30\beta^2 + 34\beta - 8)x - (18\beta^2 + 4\beta). \end{aligned}$$

Since $\beta \geq 4$, it follows that $f_4(0) = -18\beta^2 - 4\beta < 0$,

$$5^5 f_4(\frac{6}{5}) = 4050\beta^2 - 10640\beta + 2616 > 0.$$

By Lemma 2.10, there exist at least one root of $f_4(x)$ in the interval $(0, \frac{6}{5})$. This implies that $a(G_2 - v_0v_2) < \frac{6}{5} < \frac{5-\sqrt{5}}{2}$. $\square$

Lemma 2.17. *Let G_3 be the graph, shown in Fig.2.2, of order n with matching number β. If $\beta \geq 4$, then $a(G_3) = \frac{5-\sqrt{5}}{2}$.*

Proof. Let $T_3 = G_3 - v_0$. Then $T_3 = T(n, \beta; 1, 1, \ldots, 1, 0)$. By Lemmas 2.7 and 2.8, we obtain $a(G_3) = \frac{5-\sqrt{5}}{2}$. $\square$

Lemma 2.18. *If $\beta \geq 4$, then $a(G_3 - v_0v_1) = \frac{5-\sqrt{5}}{2}$.*

Proof. By Lemmas 2.5 and 2.6, we have

$$\Phi(G_3 - v_0v_1; x) = x(x^2 - 5x + 5)^{\beta-2}(x^2 - (\beta+3)x + 2\beta + 1)$$
$$(x^2 - (2\beta+3)x + 6\beta - 1).$$

Let $g_1(x) = x^2 - (\beta+3)x + 2\beta + 1$ and $g_2(x) = x^2 - (2\beta + 3)x + 6\beta - 1$. The axis of symmetry of $g_1(x)$ is $x = \frac{\beta+3}{2} > \frac{5-\sqrt{5}}{2}$ when $\beta \geq 4$. Noting that $g_1(\frac{5-\sqrt{5}}{2}) = \frac{(\beta-2)(\sqrt{5}-1)}{2} > 0$, we have $\frac{\beta+3-\sqrt{\beta^2-2\beta+5}}{2} > \frac{5-\sqrt{5}}{2}$. Similarly, we can prove that $\frac{2\beta+3-\sqrt{4\beta^2-12\beta+13}}{2} > \frac{5-\sqrt{5}}{2}$. It follows from the characteristic polynomial $\phi(G_3 - v_0v_1; x)$ that $a(G_3 - v_0v_1) = \frac{5-\sqrt{5}}{2}$. $\square$

Lemma 2.19. *If $\beta \geq 4$, then $a(G_3 - v_0v_2) < \frac{5-\sqrt{5}}{2}$.*

Proof. By Lemmas 2.5 and 2.6, we have

$$\Phi(G_3 - v_0v_2; x) = x(x^2 - 5x + 5)^{\beta-3} f_5(x),$$

where

$$\begin{aligned} f_5(x) \;=\; & x^6 - (3\beta + 11)x^5 + (2\beta^2 + 31\beta + 46)x^4 \\ & -(18\beta^2 + 121\beta + 91)x^3 + (58\beta^2 + 219\beta + 90)x^2 \\ & -(78\beta^2 + 181\beta + 51)x + 36\beta^2 + 56\beta + 19. \end{aligned}$$

Since $\beta \geq 4$, it follows that $f_5(0) = 36\beta^2 + 56\beta + 19 > 0$,

$$5^6 f_5(\frac{6}{5}) = -16200\beta^2 + 29510\beta + 18001 < 0.$$

By Lemma 2.10, there exist at least one root of $f_5(x)$ in the interval $(0, \frac{6}{5})$. This implies that $a(G_3 - v_0v_2) < \frac{6}{5} < \frac{5-\sqrt{5}}{2}$. $\square$

3 Main results

Let $\mathcal{Q}_t(n,\beta)$ denote the set of all quasi-tree graphs with n vertices and matching number β. If $\beta = 1$, then $\mathcal{Q}_t(n,\beta) = \{K_{1,n-1}\}$. It is known that $a(K_{1,n-1}) = 1$. For $\beta \geq 2$, we obtain the following theorems.

Theorem 3.1. *Let $G \in \mathcal{Q}_t(n,\beta)$ and $G_i\,(i = 1,2,3)$ be the graphs shown in Fig. 2.2. If $\beta \geq 4$, then*

$$a(G) \leq \frac{5-\sqrt{5}}{2},$$

and the equality holds if and only if $G \in \{G_1, G_2, G_1 - v_0v_1, G_2 - v_0v_1\}$ for $n = 2\beta$; $G \in \{G_1, G_3, G_1 - v_0v_1, G_3 - v_0v_1\}$ for $n = 2\beta + 1$; $G \in \{G_1, G_1 - v_0v_1\}$ for $n \geq 2\beta + 2$.

Proof. Let $G \in \mathcal{Q}_t(n,\beta)$ and $v_0 \in V(G)$ such that $G - v_0$ is a tree. Denote by d the diameter of $G - v_0$. Since $\beta \geq 4$, it follows that $d \geq 4$. If G has a pendant vertex, by Lemma 2.4, we have $a(G) \leq 1 < \frac{5-\sqrt{5}}{2}$. If $d > 4$, by Lemmas 2.1 and 2.2, we have

$$a(G) \leq a(G-v_0)+1 \leq 2(1-cos\frac{\pi}{d+1})+1 \leq 3-\sqrt{3} < \frac{5-\sqrt{5}}{2}.$$

If $d = 4$, then $G - v_0 = T(n,k;p_1,\ldots,p_k)$, where $k \geq 2$ and $p_1 \geq p_2 \geq 1$. Namely G is a spanning subgraph of $K_1 \vee T(n,k;p_1,\ldots,p_k)$. If $G - v_0 \neq T(n,k;1,\ldots,1,0,\ldots,0)$, by Lemma 2.7, we have $a(G - v_0) < \frac{3-\sqrt{5}}{2}$. By Lemma 2.8, we have

$$a(G) \leq a(K_1 \vee T(n,k;p_1,\ldots,p_k)) < \frac{5-\sqrt{5}}{2}.$$

In what follows, we assume that $G - v_0 = T(n, n-\beta; 1,\ldots,1,0,\ldots,0)$ and G has no pendant vertices.

Case 1. $n = 2\beta$. If $G \notin \{G_1, G_2, G_1 - v_0v_1, G_2 - v_0v_1\}$, then G is a spanning subgraph of either $G_1 - v_0v_2$ or $G_2 - v_0v_2$. By Lemmas 2.3, 2.13 and 2.16, we have

$$a(G) \leq \max\{a(G_1 - v_0v_2),\, a(G_2 - v_0v_2)\} < \frac{5-\sqrt{5}}{2}.$$

If $G \in \{G_1, G_2, G_1 - v_0v_1, G_2 - v_0v_1\}$, by Lemmas 2.11, 2.12, 2.14 and 2.15, we have $a(G) = \frac{5-\sqrt{5}}{2}$.

Case 2. $n = 2\beta + 1$. If $G \notin \{G_1, G_3, G_1 - v_0v_1, G_3 - v_0v_1\}$, then G is a spanning subgraph of either $G_1 - v_0v_2$ or $G_3 - v_0v_2$. By Lemmas 2.3, 2.13 and 2.19, we have

$$a(G) \leq \max\{a(G_1 - v_0v_2),\, a(G_3 - v_0v_2)\} < \frac{5-\sqrt{5}}{2}.$$

If $G \in \{G_1, G_3, G_1 - v_0v_1, G_3 - v_0v_1\}$, by Lemmas 2.11, 2.12, 2.17 and 2.18, we have $a(G) = \frac{5-\sqrt{5}}{2}$.

Case 3. $n \geq 2\beta + 2$. If $G \notin \{G_1, G_1 - v_0v_1\}$, then G is a spanning subgraph of $G_1 - v_0v_2$. By Lemmas 2.3 and 2.13, we have

$$a(G) \leq a(G_1 - v_0v_2) < \frac{5-\sqrt{5}}{2}.$$

If $G \in \{G_1, G_1 - v_0v_1\}$, by Lemmas 2.11 and 2.12, we have $a(G) = \frac{5-\sqrt{5}}{2}$.

Combining the above arguments, we have the proof. □

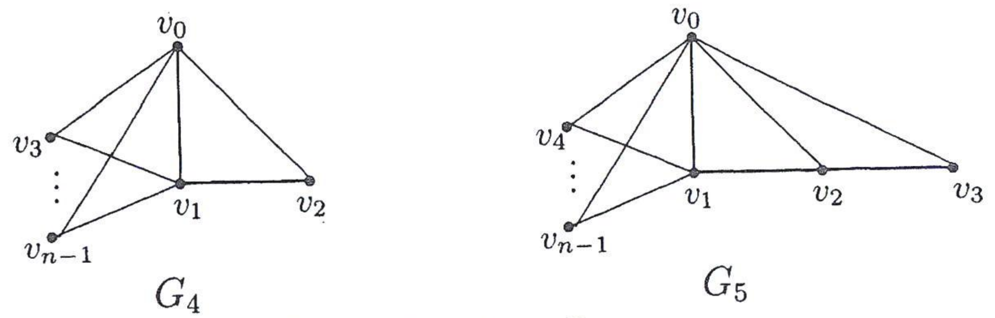


Fig. 3.1 G_4, G_5

Theorem 3.2. *Let $G \in \mathcal{Q}_t(n,2)$ and G_4 be the graph shown i Fig. 3.1. Then $a(G) \le 2$ and the equality holds if and only i $G = G_4$ or $G_4 - v_0v_1$.*

Proof. Let $G \in \mathcal{Q}_t(n,2)$ and $v_0 \in V(G)$ such that $G - v_($ is a tree. If G has a pendant vertex, by Lemma 2.4, we have $a(G) \le 1 < 2$. Next, we assume that G has no pendant vertices.

Since $\beta = 2$, it follows that $G - v_0$ is the star $K_{1,n-2}$. This implies that $G = G_4$ or $G_4 - v_0v_1$. Noting $G_4 = K_1 \vee K_{1,n-2}$, by Lemma 2.8, we have $a(G_4) = 2$. By direct computation, we have

$$\Phi(G_4 - v_0v_1; x) = x(x-n)(x-n+2)(x-2)^{n-3}.$$

This implies that $a(G_4 - v_0v_1) = 2$.

Combining the above arguments, we have the proof. □

Theorem 3.3. *Let $G \in \mathcal{Q}_t(n,3)$ and G_5 be the graph shown in Fig. 3.1. Then $a(G) \le a(G_5)$, where the equality holds if and only if $G = G_5$, and $a(G_5) - 1$ is the least root of the equation $x^3 - (n+1)x^2 + (3n-5)x - n + 1 = 0$.*

Proof. Clearly, $G_5 \in \mathcal{Q}_t(n,3)$ and $G_5 = K_1 \vee S(1, n-4)$. By direct computation, we have

$$\Phi(S(1,n-4);x) = x(x-1)^{n-5}(x^3-(n+1)x^2+(3n-5)x-n+1).$$

Let $f_6(x) = x^3 - (n+1)x^2 + (3n-5)x - n + 1$. Taking the first and second derivatives of $f_6(x)$ with respect to x, we obtain that

$$f_6'(x) = 3x^2 - (2n+2)x + 3n - 5, \quad f_6''(x) = 6x - 2n - 2.$$

Since $f_6''(x) < 0$ for $x \le 1$, it follows that $f_6'(x)$ is strictly decreasing on the interval $(-\infty, 1]$. Noting that $f_6'(1) = n - 4 > 0$, we have $f_6'(x) > 0$ for $x \le 1$. This implies that $f_6(x)$ is strictly increasing on the interval $(-\infty, 1]$. Since $f_6(\frac{3-\sqrt{5}}{2}) = -1 < 0$

and $f_6(1) = n - 4 > 0$, it follows that the least root of $f_6(x)$ is less than 1 and greater than $\frac{3-\sqrt{5}}{2}$. By Lemma 2.8, we have $\frac{5-\sqrt{5}}{2} < a(G_5) < 2$ and $a(G_5) - 1$ is the least root of $f_6(x)$.

Let $G \in \mathcal{Q}_t(n, 3)$ and $v_0 \in V(G)$ such that $G - v_0$ is a tree. Denote by d the diameter of $G - v_0$. Since $\beta = 3$, it follows that $d \geq 3$. If G has a pendant vertex, by Lemma 2.4, we have $a(G) \leq 1 < a(G_5)$. If $d > 4$, by Lemmas 2.1 and 2.2, we have

$$a(G) \leq a(G - v_0) + 1 \leq 2(1 - cos\frac{\pi}{d+1}) + 1 \leq \frac{5-\sqrt{5}}{2} < a(G_5).$$

Next, we assume that $d = 3$ and G has no pendant vertices. In this case, G is a spanning subgraph of $K_1 \vee S(s,t)$, where $1 \leq s \leq t \leq n - 4$, $s + t = n - 3$. By Lemma 2.9, we have $a(S(s,t)) \leq a(S(1, n-4))$ and the equality holds if and only if $s = 1$. It follows that $a(K_1 \vee S(s,t)) \leq a(K_1 \vee S(1, n-4))$ and the equality holds if and only if $s = 1$.

If G is a spanning subgraph of $K_1 \vee S(s,t)$ $(s \geq 2)$, by Lemma 2.3, we have

$$a(G) \leq a(K_1 \vee S(s,t)) < a(K_1 \vee S(1, n-4)) = a(G_5).$$

If G is a spanning subgraph of $K_1 \vee S(1, n-4)$, by Lemma 2.3, we have $a(G) \leq a(G_6)$ or $a(G) \leq a(G_7)$, where $G_6 = G_5 - v_0v_2$ and $G_7 = G_5 - v_0v_1$. Again by Lemma 2.3, we have $a(G_6) \leq a(G_5)$ and $a(G_7) \leq a(G_5)$. Now we show $a(G_6) < a(G_5)$ and $a(G_7) < a(G_5)$.

Let $a = a(G_6)$ and $X = (x_0, x_1, x_2, \ldots, x_{n-1})^T$ be an eigenvector corresponding to a. Then $a = a(G_6) \leq a(G_5) < 2$, By the eigenvalue equation $L(G_6)X = aX$, we have $x_4 = \cdots = x_{n-1}$,

$$\begin{aligned}
(a - n + 2)x_0 + x_1 + x_3 + (n-4)x_4 &= 0, \\
x_0 + (a - n + 2)x_1 + x_2 + (n-4)x_4 &= 0, \\
x_1 + (a-2)x_2 + x_3 &= 0, \\
x_0 + x_2 + (a-2)x_3 &= 0, \\
x_0 + x_1 + (a-2)x_4 &= 0.
\end{aligned}$$

Since X is an eigenvector, it follows that

$$\begin{vmatrix} a-n+2 & 1 & 0 & 1 & n-4 \\ 1 & a-n+2 & 1 & 0 & n-4 \\ 0 & 1 & a-2 & 1 & 0 \\ 1 & 0 & 1 & a-2 & 0 \\ 1 & 1 & 0 & 0 & a-2 \end{vmatrix} = 0.$$

This implies that a is the least positive root of the following equation

$$\begin{vmatrix} x-n+2 & 1 & 0 & 1 & n-4 \\ 1 & x-n+2 & 1 & 0 & n-4 \\ 0 & 1 & x-2 & 1 & 0 \\ 1 & 0 & 1 & x-2 & 0 \\ 1 & 1 & 0 & 0 & x-2 \end{vmatrix} = 0.$$

Denote the left hand side of the above equation by $\varphi(x)$. By a computation, we have

$$\varphi(x) = x(x^2 - nx + n)(x^2 - (n+2)x + 3n - 4).$$

This implies that $a = \frac{n-\sqrt{n^2-4n}}{2}$. Since $\frac{n-\sqrt{n^2-4n}}{2}$ is strictly decreasing with respect to n, it follows that

$$a = \frac{n-\sqrt{n^2-4n}}{2} < \frac{6-\sqrt{6^2-24}}{2} = 3-\sqrt{3} < \frac{5-\sqrt{5}}{2} < a(G_5).$$

Noting that $G_7 = G_5 - v_0v_1$, by a similar reasoning as the proof of $a(G_6)$, we have $a(G_5)$ and $a(G_7)$ are the least positive roots of the following polynomials

$$\begin{aligned} \varphi_{G_5}(x) &= x(x-n)(x^3 - (n+4)x^2 + 5nx - 5n + 4), \\ \varphi_{G_7}(x) &= x(x^4 - (2n+2)x^3 + (n^2+7n-9)x^2 \\ &\quad -(5n^2 - 5n - 10)x + 5n^2 - 14n), \end{aligned}$$

respectively. Let

$$\begin{aligned} g(x) &= x^3-(n+4)x^2+5nx-5n+4, \\ h(x) &= x^4-(2n+2)x^3+(n^2+7n-9)x^2 \\ &\quad -(5n^2-5n-10)x+5n^2-14n. \end{aligned}$$

Then $a(G_5)$ and $a(G_7)$ are the least positive roots of $g(x)$ and $h(x)$ respectively. It is easy to see that $h(x) = g(x)(x-n+2) - x^2 + 6x - 8$, namely

$$g(x)(x-n+2) - h(x) = x^2 - 6x + 8 = (x-2)(x-4) > 0$$

when $0 \le x < 2$. Let $a_5 = a(G_5)$ and $a_7 = a(G_7)$. Since $a_7 < 2$, it follows that $g(a_7)(a_7 - n + 2) - h(a_7) > 0$. This implies that $0 = h(a_7) < g(a_7)(a_7 - n + 2)$. Noting that $a_7 - n + 2 < 0$, we have $g(a_7) < 0$. Since $g(a_5) = 0$, it follows that $a_7 \neq a_5$. Recalling that $a_7 \le a_5$, we conclude $a(G_7) < a(G_5)$.

Combining the above arguments, we have the proof. □

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