New Small Trivalent Graphs for Girths 17, 18 and 20

Geoffrey Exoo
Department of Mathematics and Computer Science
Indiana State University
Terre Haute, IN 47809
gexoo@indstate.edu

July 18, 2007

Abstract

Constructions of the smallest known trivalent graph for girths 17, 18 and 20 are given. All three graphs are voltage graphs. Their orders are 2176, 2560, and 5376, respectively, improving the previous values of 2408, 2640, and 6048.

AMS Subject Classifications: 05C25, 05C35

Introduction

ıe cage problem asks for the construction of regular graphs with specified gree and girth. In [2], Biggs provided a short history of the construction pect of the problem for trivalent (cubic) graphs. More recent data is ailable from Royle's web site [11]. From these sources, we learn that :fore the graphs described in this paper were discovered, the best known nstructions for girths 17, 18, and 20 had orders 2408, 2640, and 6048.

Since the constructions presented here use them, we begin with a brief view of *voltage graphs* [9]. Given a graph G, we denote the set of arcs of G $D(G)$. In this set, each edge is represented twice, once in each direction.

A voltage graph is constructed from a *base graph* G, a *voltage group* Γ, ıd a *voltage assignment* α, where

$$\alpha : D(G) \to \Gamma$$

such that

$$\alpha(uv) = \alpha(vu)^{-1}.$$

The voltage graph, denoted G^α is the graph with vertex set

$$V(G^\alpha) = V(G) \times \Gamma$$

wherein (u, g) is adjacent to (v, h) whenever

$$g \cdot \alpha(uv) = h.$$

We will have occasion to refer to the trivalent multigraph of order 2, which we denote by G_2. It consists of two vertices joined by three edges. A graph which is a subdivision of G_2 is called a *θ-graph.* So a θ-graph consists of two vertices, joined by three independent paths. The *total length* of such a graph is taken to be the sum of the lengths of the three paths.

Voltage graphs have previously been used to obtain constructions for the closely related degree/diameter problem [1, 3, 4]. They were also used in [6] to construct the smallest known $(3, 16)$-graph. The latter graph is a lift of the Petersen graph by an abelian group. As indicated in [6], this is the largest possible girth for a lift of the Petersen graph by an abelian group. The reason for this is that the Petersen graph contains a θ-subgraph of total length 8, and in any θ-graph it is possible to construct a closed walk that traverses each edge twice, once in each direction. In view of this, the fact that the smallest $(3, 16)$-graph known is a voltage graph produced by an abelian group seemed somewhat surprising. But it did motive us to look at groups that are "almost" abelian. So we consider groups whose commutator subgroups are abelian. All three of the constructions presented in this paper use nonabelian groups with abelian commutator subgroups. The relation between girth limits on voltage graphs and the derived length of the voltage group is studied in [7], where it is shown that the maximum girth of a lift of G_2 using a group whose commutator subgroup is abelian is 22. The girth 20 graph constructed below is in fact a lift of G_2 by such a group.

A second apparently important property of voltage groups that are useful for the cage problem was suggested by the results in [5] where lifts by nonabelian pq-groups were studied. There we constructed a graph that was (briefly) the smallest known $(3, 17)$-graph using the nonabelian group of order $301 = 7 \times 43$. Such pq-groups were also used to find some of the smallest known girth 5 graphs for degrees in the range 10 to 20.

A third useful property seems to be the presence of relatively large Sylow 2-subgroups with small exponent. To summarize, we investigated groups with the following three properties.

1. Abelian commutator subgroups.

2. An odd order nonabelian pq-subgroup (for p, q prime).

3. Relatively large nonabelian Sylow 2-subgroups with small exponents.

It should be noted that attempts to attack the cage problem using Cayley graphs has also been met with success [2]. In this case, the groups that seemed most useful were groups that were simple or contained large simple subgroups. However, this may not be the case for voltage graphs, as our experience with the $(3, 20)$-graph described below suggests. Since the construction in this case was the most difficult, we describe it first, and in greatest detail. The other two constructions are briefly presented at end of the paper. All three graphs, along with a variety of supporting data, can be found at `http://ginger.indstate.edu/ge/CAGES`.

2 A Graph of Degree 3, Girth 20 and Order 5376

At the time of the Biggs survey [2], the smallest known trivalent graph of girth 20 had been constructed by J. Bray, C. Parker and P. Rowley [10]. Their graph had 8096 vertices. In subsequent unpublished improvements, Parker and Rowley found a graph of order 6072 vertices, and this writer found one on 6048 vertices. In this note, the bound is improved to 5376 using voltage graphs.

We began by looking for values of n of the form $2^k \times p \times (2p + 1)$ that could be used with small base graphs. We started by looking at the smallest possible base graph: the θ-multigraph with two vertices. Since the goal was to improve the old bound of 6048, groups whose orders were less that 3000 we considered. With these criterion in mind, the obvious candidate for n was $2688 = 2^7 \times 3 \times 7$.

For groups of small order (up to 2000) one can consult the GAP [8] catalog of small groups and obtain representations of all groups of a given order. If the orders of the group and the base graph are small enough, one can do complete searches for the graph. Indeed, this was done for our two smaller examples. For larger groups one needs to have some idea of the structure of the group beforehand. Initially, we tried to find an extension of $PSL(3, 2)$ (of order $168 = 2^3 \times 3 \times 7$) to use as the voltage group (in violation of our first criterion above). However, this effort was not successful. When our attachment to $PSL(3, 2)$ was overcome, the group we now describe was found rather quickly.

Let Q denote the quaternion group of order 8. The choice of the quaternion was not random. This group has been a useful building block in similar constructions for the degree diameter problem (see [3]). Recall that Q has elements $\{1, -1, i, j, k, -i, -j, -k\}$ such that $i^2 = j^2 = k^2 = 1$, $ij = k$, $jk = i$, $ki = j$. In particular, Q is generated by i and j.

Now let K be the direct product $Z_2 \times Z_2 \times Q$. We wish to form a semidirect product of K by the cyclic group Z_4. Consider the automorphism

$$\phi : K \to K$$

defined as follows.

$$\begin{aligned} \phi(1,0,0) &= (0,1,0) \\ \phi(0,1,0) &= (1,0,0) \\ \phi(0,0,i) &= (0,0,j) \\ \phi(0,0,j) &= (0,1,i) \end{aligned}$$

It is straightforward to check that $\phi \in Aut(K)$. If we define $\psi : Z_4 \to Aut(K)$ by mapping a generator of Z_4 to ϕ we obtain a semidirect product $H = K \times_\psi Z_4$. The order of H is 128. We note that there are 576 choices for ϕ that produce this group, and that there are 25 (pairwise nonisomorphic) groups that can be described as semidirect products of K by Z_4.

Finally, let T be the nonabelian group of order 21. Then our voltage group is $H \times T$. The elements of the group can be written in the form (w, x, y, z, u, v) where $w \in Z_2$, $x \in Z_2$, $y \in Q$, and $z \in Z_4$. Elements of K can be written in the form $t_1^u t_2^v$ where t_1 and t_2 are generators of T of orders 3 and 7, respectively, such that $t_1 t_2 t_1^{-1} = t_2^4$.

The base graph used in this construction is G_2, defined above. Let a_1, a_2, and a_3 be the arcs from one of the vertices to the other. It remains to define the voltage assignment α.

Without loss of generality, we can assign the identity to one of the arcs. So define α as follows.

$$\begin{aligned} \alpha(a_1) &= (0,0,0,0,0,0) \equiv 1 \\ \alpha(a_2) &= (0,0,0,1,1,0) \equiv g \\ \alpha(a_3) &= (0,0,i,2,2,1) \equiv h \end{aligned}$$

Actually, there are 133,062 distinct pairs of elements from this group that could be assigned to a_2 and a_3 to produce the same graph. In other words, if two elements are chosen at random, the probability that they work is roughly 0.03685.

Finally, we show that the lift has girth 20. Observe that cycles in the lift correspond to alternating sequences of elements taken from the sets $\{1, g, h\}$ and $\{1, g^{-1}, h^{-1}\}$ such that no element in the sequence is followed by its inverse. Since the graph is evidently bipartite, we need to show that there are no cycles of lengths $2m$ for $2 \leq m \leq 9$. This can be checked easily with any of the popular symbolic mathematics programs. We present a solution using GAP [8].

The first step is to construct the voltage group. This can be done with the GAP code given below, in which we have chosen transparency over efficiency

```
Quat  := SmallGroup(8, 4);
Z2    := CyclicGroup(2);
Klein := DirectProduct(Z2, Z2);
K     := DirectProduct(Klein, Quat);
A     := AutomorphismGroup(K);
H     := CyclicGroup(4);

gen   := [K.1, K.2, K.3, K.4, K.5];
img   := [K.2, K.1, K.4, K.2*K.3, K.5];

phi   := GroupHomomorphismByImages(K, K, gen, img);
psi   := GroupHomomorphismByImages(H, A, [H.1], [phi]);

S     := SemidirectProduct(H,psi,K);
T     := SmallGroup(21, 1);
G     := DirectProduct(S, T);
```

Next, we create the elements to be used for voltage assignments.

```
g  := G.1*G.8;
h  := G.2*G.5*G.8*G.8*G.9;
i  := Identity(G);
```

The idea behind the rest of the code is to create a list of all group elements that are at distance d from the origin, for d from 1 to 10. The `list` variable consists of pairs, while a pair consists of a group element and the generator used to reach that element. It is initialized to reflect the situation at distance one.

The variable `volt` is a list of voltage assignments that can be used to get to the next distance, and its content alternates between $[1, g, h]$ and $[1, g^{-1}, h^{-1}]$.

Ss long as the graph contains no cycles of length $2d$,

the size of the list will double after each iteration. When this fails to happen, we have found the girth of the graph.

The variable `elem` is a list of elements reached at the current distance, duplicating some of the information in `list`, and is used as a convenience.

```
volt := [i,Inverse(g),Inverse(h)];
list := [[i,i],[g,g],[h,h]];

for d in [2..10] do
  newlist := [];
  elem := [];
  for s in list do
    for x in volt do
      if x <> Inverse(s[2]) then
        y := s[1]*x;
        Add(newlist,[y,x]);
        if not y in elem then
          Add(elem,y);
        fi;
      fi;
    od;
  od;
  list := newlist;
  Apply(volt, Inverse);
  Print(d," ",Size(elem),"\n");
od;
```

The group described above can also be described as an extension of $Z_2 \times Z_2 \times Z_4 \times Z_7$ by $Z_2 \times Z_4 \times Z_3$. However, this was observed after the fact and did not help in the construction.

3 A Graph of Degree 3, Girth 18 and Order 2560

The (3,18)-graph has 2560 vertices. It is a lift of the multigraph in Figure 1. The voltage group has order 320, and is `SmallGroup(320,696)` in the GAP [8] catalog of small groups.

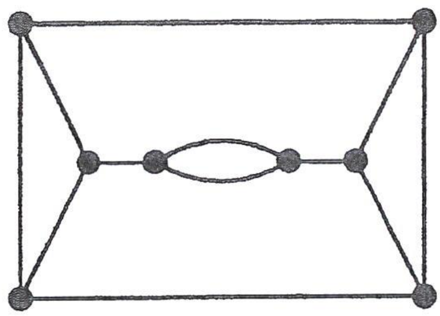

Figure 1: Base Graph for Girth 18

4 A Graph of Degree 3, Girth 17 and Order 2176

The (3,17)-graph of order 2176 is a lift of the multigraph shown below in Figure 2. The lifting group is a group of order 272 (`SmallGroup(272,28)` in the GAP [8] catalog).

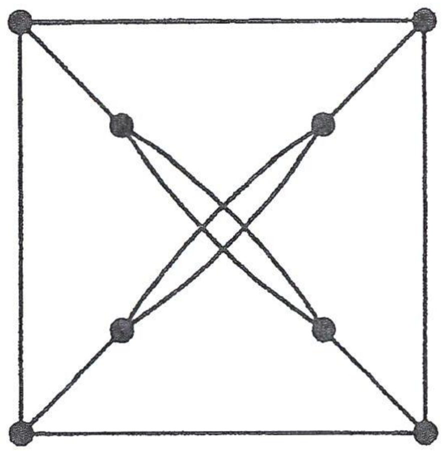

Figure 2: Base Graph for Girth 17

References

[1] M. Miller B. D. McKay and J. Siran. A note on large graphs of diameter two and given maximum degree. *J. Combinatorial Theory - Series B*, 74:110–118, 1998.

[2] N. L. Biggs. Constructions for cubic graphs of large girth. *Electron. J. Combin.*, 5:A1, 1998.

[3] F. Comellas. The degree/diameter problem for graphs. World Wide Web Document, 2005. (`http://www-mat.upc.es/grup_de_grafs/table_g.html`).

[4] G. Exoo. A family of graphs and the degree/diameter problem. *J. Graph Theory*, 37:118–124, 2001.

[5] G. Exoo. Some applications of pq-groups in graph theory. *Discussiones Mathematicae Graph Theory*, 24:109–114, 2004.

[6] G. Exoo. Voltage graphs, group presentations and cages. *Electron. J. Combin.*, 11(1):N2, 2004.

[7] G. Exoo and R. Jajcay. Voltage graph girth and the derived length of voltage groups. submitted.

[8] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2005. (`http://www.gap-system.org`).

[9] J. L. Gross. Voltage graphs. *Discrete Math*, 9:239–246, 1974.

[10] C. Parker J. Bray and P. Rowley. Cayley type graphs and cubic graphs of large girth. *Discrete Math*, 214:113–121, 2000.

[11] G. Royle. Cubic cages. World Wide Web Document, 2006. (`http://people.csse.uwa.edu.au/gordon/remote/cages/index.html`)